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The calculation of multifractal properties of directed random walks on hierarchic trees with continuous branching

David B Saakian[1,2,3]

[1] Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan
[2] A I Alikhanyan National Science Laboratory (Yerevan Physics Institute) Foundation, 2 Alikhanian Brothers Street, Yerevan 375036, Armenia
[3] National Center for Theoretical Sciences (North): Physics Division, National Taiwan University, Taipei 10617, Taiwan
E-mail: saakian@yerphi.am

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Abstract. We consider the hierarchic tree random energy model with continuous branching and calculate the moments of the corresponding partition function. We establish the multifractal properties of those moments. We derive formulas for the normal distribution of random variables, as well as for the general case. We compare our results for the moments of the partition function with corresponding results of logarithmic 1d REM and conjecture a specific power law tail for the partition function distribution in the high-temperature phase. Our results establish a connection between reaction–diffusion equations and multi-scaling.

Keywords: scaling in socio-economic systems, disordered systems (theory)

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1. Introduction

Random energy models on hierarchic trees are rather well known and much investigated objects of statistical physics [1]–[4]. The tree is constructed via a deterministic branching process, starting from the root node and adding q branches from every node of the tree, so that after K steps of the procedure there are q^K nodes of the last generation (the endpoints of the tree). One then associates random energy variables to every branch of the tree, and further attributes the random energy to every node of the last generation by adding up all the energy variables along the unique path connecting the given endpoint to the root. At the last step of the procedure one builds up the partition function [1]–[3]. In the context of statistical physics the model has been identified with directed polymers on the trees, and has been also related to the spin-glass model REM [5, 6]. Independently, similar models have been applied and extensively investigated also in the context of financial mathematics and turbulence [7, 8, 33, 10, 11]. They are also intimately connected to 2d conformal models [12, 13]. The hierarchic structure of the tree naturally induces recursive relations for the partition function, with the branching number q being a parameter in the recursion. While by construction the branching number q is an integer, one can formally consider the recursive equations [14, 1] in the $q \to 1$ limit, simultaneously allowing $K \gg 1$ and keeping the number of endpoints q^K fixed. Such an idea has been suggested in [15], and later worked out in more detail in [16]. Along that line an exact renormalization equation for continuous tree models has been derived in [17] for the general distribution of random variables.

The hierarchic tree models belong to the type of random energy models (REMs) which all share the leading term in the free energy with that of the simplest Derrida model. The investigation of REM-like models in finite dimensions was started in [18] and gained serious impetus from recent solution of 1d models with logarithmic correlation of energies at different sites [19, 20]. That solution essentially used the generalization of Selberg integrals [21, 22] to a complex number of integrations (see the rigorous mathematical justification in [23]). The 1d models are intimately connected with the model [24, 25]. Recently [26] we have developed a statistical physics approach to related dynamical

Markov switching multi-fractal (MSM) models [27]. While all three types of REM model: hierarchic tree [1], 1d logarithmic REM [20] and MSM [27] have exactly the same mean free energy as the standard REM, they have different distributions for the free energy and the partition function. Most essentially, in the Derrida REM the probability distribution of the partition function has no fat tails in the high-temperature phase [6], while such tails are present in 1d models [19, 20]. In this work we give an indication that such tails exist in the hierarchic model, and also reveal the multifractal properties of the latter, which are important for applications [7]–[10]. For a recent discussion of multifractal properties of REMs, and other models with logarithmic correlations see [28, 29] and references therein.

In the multifractal approach [30, 31] one considers the moments of a random variable (partition function) Z at some scale l defined with respect to the maximal scale L:

$$\langle Z^n \rangle = \mathrm{e}^{\xi(n)\ln(l/L)} (A(L))^n \hat{Z}_n, \tag{1}$$

where the exponent $\xi(n)$ defines the multi-scaling.

Knowing the $\xi(n)$ and the coefficients $\hat{Z}_n \sim \mathrm{O}(1)$ in the limit of large L, one can reconstruct the probability distribution of the partition function. In this way such a distribution has been explicitly derived for 1d logarithmic REMs [19, 20].

In this paper we will calculate $\xi(n)$ for hierarchic trees, and derive recursive relations to define $\hat{Z}_n$, which, in principle, could be applied to calculate the probability distribution of Z within some accuracy. We shall see indications of the fact that the divergence of moments is essentially the same as for the 1d case, thus the two types of model must share the same fat tails.

2. The calculation of the moments

2.1. The model with normal distribution of random variables

Let us define the model outlined in the introduction following the papers [15, 17]. Starting with the hierarchic tree with integer branching number q and q^K endpoints, we consider two such endpoints w_i and w_j. The two paths connecting these points to the root coincide up to a level m, counting from the root. Accordingly, we can define the hierarchic distance between the two points as

$$v(w_i, w_j) = \frac{mV_0}{K}, \tag{2}$$

where V_0 is defined as

$$q^K \equiv \mathrm{e}^{V_0} \equiv L. \tag{3}$$

Associated with every branch of the tree is a Gaussian random variable ϵ distributed according to the law

$$\rho\left(\frac{V_0}{K}, \epsilon\right) = \sqrt{\frac{K}{2V_0\pi}} \exp\left[-\frac{K\epsilon^2}{2V_0}\right]. \tag{4}$$

We define the energy y_i at the endpoint w_i as a sum of corresponding variables ϵ sampled along the unique path connecting the point w_i to the end of the tree.

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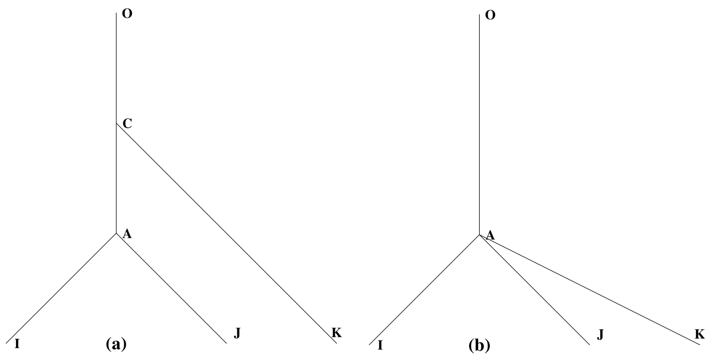

Figure 1. The sum is over three marked endpoints I, J, K of the hierarchic tree. Only the (a) contributes to the leading order. The locations of I, J, K given by (b) yield negligible $\mathrm{O}(q-1)$ contribution to the partition function in equation (8).

We further define the partition function

$$Z = \sum_i \mathrm{e}^{-\beta y_i - V_0(\beta^2/2)}, \tag{5}$$

where the sum is over $\mathrm{e}^V \equiv l$ endpoints.

Then we obtain

$$\langle Z(\mathrm{e}^V, b)^n \rangle = \sum_{i_1,\ldots,i_n} \langle \mathrm{e}^{-\beta y_{i_1} \cdots - \beta y_{i_n}} \rangle. \tag{6}$$

In equation (6) the sum is over all endpoints of the tree separated by the hierarchic distance V from a chosen point.

Now we consider the limit

$$q \to 1 \tag{7}$$

while keeping the value of V_0 fixed and large. Recall that $V_0 = K \log q$. In such a limit there are e^v endpoints at the hierarchic distance v. While calculating the $\langle Z^n \rangle$, we may discard the contributions given by figure 1(b).

We formally replace the sum over the tree with integration over a measure $\mathrm{d}w$ [15, 16],

$$\langle Z(\mathrm{e}^V, b)^n \rangle = \prod_{i=1}^{n} \int \mathrm{d}w_i \, \langle \mathrm{e}^{-\beta \sum_i y(w_i)} \rangle. \tag{8}$$

We are able to calculate $\langle Z^n \rangle$ for the integration range going over the maximal distance V. When calculating the correlations in equation (8) we use the following trick: the part of the trajectory of w_i with hierarchic length v which has no overlap with trajectories from other points yields a factor $\mathrm{e}^{\beta^2 v/2}$, while the part of the trajectory common to n such trajectories and with hierarchic length v yields a factor $\mathrm{e}^{n^2 \beta^2 v/2}$.

Consider $n = 2$ for simplicity, with summation going up to the maximal distance V. We then have e^{V-v} positions for the top level of hierarchy. Then, summing (or 'integrating') over the number of two points at the distance e^{2v} we get

$$\langle Z(\mathrm{e}^V,b)^2\rangle = \int_0^V \mathrm{d}v\, \mathrm{e}^{V+v}\mathrm{e}^{2(V_0-v)\beta^2+v\beta^2-V_0\beta^2} = \frac{1}{1-\beta^2}\mathrm{e}^{V(2-\beta^2)+V_0\beta^2} = \frac{1}{1+b}\mathrm{e}^{V(2+b)-bV_0}, \tag{9}$$

where we have introduced the parameter $b=-\beta^2$.

Consider now the integration in equation (8) going up to the maximal hierarchic distance V. We assume the following ansatz:

$$\langle Z(\mathrm{e}^V,b)^n\rangle \equiv \mathrm{e}^{(V-V_0)(n+bn(n-1)/2)}\mathrm{e}^{nV_0}\hat{Z}_n, \tag{10}$$

where we have introduced dimensionless coefficients $\hat{Z}_n$. The results of our calculation support the ansatz in equation (10), see equations (13)–(14).

Actually the ansatz equation (10) is correct only when

$$(n-1) - n\frac{b}{2} > -\frac{n^2 b}{2}, \tag{11}$$

otherwise there is a phase transition to a different phase [6], which manifests itself via diverging integration in equation (8).

We will explicitly consider the case of positive b (imaginary β) while calculating some eventual expression (there is not any restriction on n-like equation (11)), like the analytical approximation of the fractional moment

$$\langle Z^\alpha(V,b)\rangle, \tag{12}$$

and then continue analytically the resulting expressions to the realistic case of negative $b<0$. The analytical continuation gives the wrong results in SG, when we continue the expressions for moments to the other statistical physics phase. Hopefully in our case we are interested in continuing to expressions of moments in the same phase.

Considering now the general case we calculate Z_n recursively. At the highest hierarchy level, n points are split into two groups with m and $(n-m)$ points, $1\le m<n$ accordingly, see figure 2. Having the expressions of Z_m, Z_{n-m} at our disposal, we can calculate Z_n. We assume that the minimum hierarchy level where all the paths from the root to n points meet is given by v. There are e^{V-v} such locations on the tree. Integrating over the positions of all points with a given v we arrive at

$$Z^n(\mathrm{e}^V,b) = \sum_{1\le m<n}\int_0^V \mathrm{d}v\,[Z_m(v)Z_{n-m}(v)]\mathrm{e}^{V-v+bvm(n-m)}. \tag{13}$$

Using the scaling ansatz equation (10) gives

$$\hat{Z}_n(b) = \sum_{1\le m<n}\frac{\hat{Z}_m(b)\hat{Z}_{n-m}}{n-1+(b/2)(m(m-1)+(n-m)(n-m-1)-2m(n-m))}$$
$$= \frac{\sum_{1\le m<n}\hat{Z}_m(b)\hat{Z}_{n-m}(b)}{(n-1)(1+nb/2)}. \tag{14}$$

Thus, the problem amounts to solving the recursive equation (14) with the initial condition (12).

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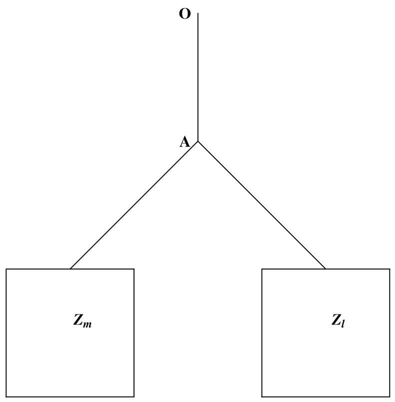

Figure 2. The graphics for calculation of Z_n recursively fracturing $n = l + m$. The groups with l and m endpoints meet at the hierarchic level v at the point A.

Equation (10) defines the so-called multifractal scaling. We can rewrite it in the form

$$\langle Z(l,b)^n \rangle \equiv \mathrm{e}^{(n+bn(n-1)/2)\ln(l/L)} \hat{Z}_n. \tag{15}$$

Let us now define a characteristic function

$$u(x) = \sum_{n=1} \hat{Z}_n x^n \tag{16}$$

for those values of β where the sum converges. We rewrite equation (14) as

$$(n-1)(1+nb/2)\hat{Z}_n(b) = \sum_{1\le m<n} \hat{Z}_m(b)\hat{Z}_{n-m}(b). \tag{17}$$

Using the equations

$$\sum_n x^n \sum_{1\le m<n} \hat{Z}_m(b)\hat{Z}_{n-m}(b) = \left(\sum_n x^n \hat{Z}_n(b)\right)^2, \qquad \sum_n n\hat{Z}_n(b)x^n = x\frac{\mathrm{d}}{\mathrm{d}x}\sum_n \hat{Z}_n(b)x^n,$$
$$\sum_n n^l \hat{Z}_n(b)x^n = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^l \sum_n \hat{Z}_n(b)x^n, \tag{18}$$

we derive the following ODE for the characteristic function:

$$\frac{bx^2}{2}\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + x\frac{\mathrm{d}u}{\mathrm{d}x} = u(x) + u^2(x), \qquad u(0) = 1, u'(0) = 1. \tag{19}$$

Unfortunately we could not solve equation (19) to derive explicitly analytical expressions for Z_n in the case of general (complex) values of n, which precludes us from following the procedure of [19, 20]. Nevertheless, equation (19) allows us to extract a few first moments. We find

$$\hat{Z}_1 = 1, \qquad \hat{Z}_2 = \frac{1}{1+b}, \qquad \hat{Z}_3 = \frac{2}{(1+b)(2+3b)},$$
$$\hat{Z}_4 = \frac{4}{(1+b)(2+3b)(3+6b)} + \frac{1}{(1+b)^2(3+6b)}. \tag{20}$$

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The above relation allows us also to calculate the asymptotic expression of $\hat{Z}_n$ at large n. In doing this we assume that our function $u(x)$ has a singularity at some $\bar{\rho}$ of the form [34]

$$u(x) = \frac{c}{(1-(z/\bar{\rho}))^{\alpha}}. \tag{21}$$

Putting the latter expression into equation (19), we derive

$$\alpha = 2, \qquad c = \frac{3b^2}{\bar{\rho}^2}. \tag{22}$$

Equation (22) gives the asymptotic expression

$$\hat{Z}_n = \frac{3(n+1)b^2}{\bar{\rho}^{2+n}}. \tag{23}$$

We can define the value of $\bar{\rho}$ only numerically, looking at Z_n for large values of the parameter n. A similar problem has been considered in [35]. Slightly modifying their results we arrive at the following algorithm to calculate $\bar{\rho}$. For a given integer n_0 we are looking for the minimum over j and obtain

$$\bar{\rho} = \min \left[\frac{6(n-1+n(n-1)b^2/2)}{n(n+2)p_j}\right]^{1/(2+n)}\Bigg|_{0\le j\le n_0}. \tag{24}$$

It is possible to get accurate values of $\bar{\rho}$ by increasing the value of n_0.

In [15, 17] the following reaction–diffusion equation has been derived for the same model:

$$\frac{\partial G(x,v)}{\partial v} = \frac{\partial^2 G(x,v)}{2\partial x^2} + G(x,v)\ln G(x,v), \tag{25}$$

where $0 \le v \le V$ played the role of time parameter in the reaction–diffusion equation. We should solve that equation with the initial value of G

$$G(x,0) = \exp[-\mathrm{e}^{-\beta x}]. \tag{26}$$

Therefore, there must be a relation between ODE (19) and PDE (25): solving equation (25) one can find the solutions of equation (19). Equation (25) allows one to be investigated via the traveling wave approach [17] and $\beta^2 = 2$ is its critical point. The critical point of equation (19) should therefore correspond to $b = -2$, and is associated with the anticipated freezing transition from the high-temperature phase to the spin-glass-like phase.

2.2. The case of general distribution

Instead of equation (4) we now consider

$$\rho\left(\frac{V_0}{K}, x\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}h\, \exp\left[\frac{V_0}{K}\phi(\mathrm{i}h) - \mathrm{i}hx\right]. \tag{27}$$

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Then for the sum y_i of K random variables x, we can compose the distribution $\rho(V_0, \epsilon)$ simply by multiplying ϕ in the exponent of equation (27) by K:

$$\rho(V_0, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}h \, \exp[V_0 \phi(\mathrm{i}h) - \mathrm{i}h\epsilon]. \tag{28}$$

We define

$$Z = \sum_i \mathrm{e}^{-\beta y_i - V_0 \phi(\beta)}. \tag{29}$$

We calculate the correlations in $\langle Z^n \rangle$ using the following trick: the part v of the trajectory of w_i which belongs only to w_i (no intersection with trajectories of other points) gives a factor $\mathrm{e}^{\phi(\beta)v}$, while the part v of the trajectory common to n trajectories gives the factor $\mathrm{e}^{\phi(n\beta)v}$.

Repeating the calculations of section 2.1 we obtain

$$\langle Z^2 \rangle = \int_0^V \mathrm{d}v \, \mathrm{e}^{(V-v)\phi(2\beta)} \mathrm{e}^{2v\phi(\beta) - 2V_0\phi(\beta)} \mathrm{e}^{2v} \mathrm{e}^{V-v}. \tag{30}$$

The factor $\mathrm{e}^{(V-v)\phi(2\beta)}$ corresponds to the part of trajectories $[A, O]$, see figure 1(a), while any of the lines $[I, A]$ and $[J, A]$ gives the factor $\mathrm{e}^{2v\phi(\beta)}$, with e^{2v} being the number of possible locations of I, J at the hierarchic distance v, whereas e^{V-v} is the number of possible positions of the point A.

Performing the integration over all $0 \le v \le V$, we arrive at

$$Z_2(\mathrm{e}^V, \beta) = \hat{Z}_2(\beta) \mathrm{e}^{(V-V_0)(2\phi(\beta) - \phi(2\beta) + 2)} \mathrm{e}^{2V_0}, \qquad \hat{Z}_2(\beta) = \frac{1}{1 - \phi(2\beta) + 2\phi(\beta)}. \tag{31}$$

Let us derive recursive equations to calculate $Z_n, n > 1$, defined as

$$Z_n(\beta, V) \equiv \mathrm{e}^{(V-V_0)(n\phi(\beta) - \phi(n\beta) + n)} \mathrm{e}^{nV_0} \hat{Z}_n(\beta). \tag{32}$$

We need to consider all possible splittings $n = m + (n - m)$, with $1 \le m \le n$. Let us assume that two groups with m and $n - m$ endpoints of our hierarchic tree are separated by the hierarchic distance v, and their trajectories meet at some point A. Calculations similar to those used to derive equation (14) give

$$\hat{Z}_n(\beta, V) = \sum_{1 \le m \le n} \hat{Z}_m(\beta, v) \hat{Z}_{n-m}(\beta, v) \mathrm{e}^{(V-v)(1 + \phi(n\beta) - n\phi(\beta))}. \tag{33}$$

Then the scaling in equation (32) yields

$$\hat{Z}_n(\beta) = \frac{\sum_{1 \le m \le n} \hat{Z}_m(\beta) \hat{Z}_{n-m}(\beta)}{n - 1 - \phi(n\beta) + n\phi(\beta)}. \tag{34}$$

One can use equations (31) and (34) to calculate any positive integer moment of the partition function under the condition

$$n\phi(\beta) + n < 1 + \phi(n\beta). \tag{35}$$

Equation (32) implies multifractal behavior with

$$\xi(n) = n\phi(\beta) - \phi(n\beta) + 1. \tag{36}$$

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For the corresponding generating function we now obtain an equation

$$(1+\phi(\beta))x\frac{\mathrm{d}u(x)}{\mathrm{d}x} - \phi\left(\beta x\frac{\mathrm{d}}{\mathrm{d}x}\right)u(x) = u(x) + u^2(x), \qquad u(0)=1, u'(0)=1. \tag{37}$$

For the same model in [17], another equation has been derived:

$$\frac{\partial G(x,v)}{\partial v} = \phi(\partial x)G(x,v) + G(x,v)\ln G(x,v). \tag{38}$$

Thus the two equations must be related.

2.3. Comparison of the logarithmic 1d REM and the hierarchic model with normal distribution

In [20] the following partition function has been considered:

$$Z = \epsilon^{\hat{\beta}^2}\int_0^1 \mathrm{d}y\, \mathrm{e}^{\hat{\beta}y(x)}, \qquad \langle y(x)y(x')\rangle = 2\ln\frac{|x-x'|}{\epsilon}, \tag{39}$$

with the parameter ϵ being an ultraviolet cutoff needed to regularize the model.

The above model shares the multifractal properties with our model for normal distribution of random energies, with the mapping

$$\epsilon = 1/\mathrm{e}^{V_0}, \qquad 2\hat{\beta}^2 = \beta^2. \tag{40}$$

Let us compare the moments. In [20] the following expression is given for the moments:

$$\hat{Z}_n = \prod_{j=1}^{j=n}\frac{\Gamma(1-(j-1)\gamma)^2\Gamma(1-\gamma j)}{\Gamma(2-(n+j-2)\gamma)\Gamma(1-\gamma)}, \tag{41}$$

where $\gamma = \hat{\beta}^2$.

We have, naturally, $\hat{Z}_1 = 1$, and further

$$\begin{aligned}
\hat{Z}_2 &= \frac{\Gamma(1-2\gamma)}{\Gamma(2-\gamma)\Gamma(2-2\gamma)} = \frac{1}{(1-\gamma)(1-2\gamma)},\\
\hat{Z}_3 &= \frac{\Gamma(1-2\gamma)^3\Gamma(1-3\gamma)}{\Gamma(2-2\gamma)\Gamma(2-3\gamma)\Gamma(2-2\gamma)},\\
\hat{Z}_4 &= \frac{\Gamma(1-2\gamma)^3\Gamma(1-3\gamma)^3\Gamma(1-4\gamma)}{\Gamma(1-\gamma)\Gamma(2-3\gamma)\Gamma(2-4\gamma)\Gamma(2-5\gamma)}.
\end{aligned} \tag{42}$$

Comparing with the results of section 2.1, we see that the second moment $\hat{Z}_2$ has a different expansion for small β^2,

$$\hat{Z}_2 \approx 1 + 3\gamma. \tag{43}$$

More important information contained in the moments is, however, the *smallest* real pole γ_n as a function of γ (respectively, b). Indeed, precisely these poles define the critical temperatures $T_n = \gamma_n^{-1/2}$ below which the given moment of the partition function starts to diverge. It is easy to see from equation (41) that for the one-dimensional model $\gamma = 1/n$, $n = 2, 3, \ldots$, while equation (42) gives the smallest poles at $b/2 = 1/2$ for Z_2, $b/2 = 1/3$ for Z_3, $b/2 = 1/4$ for Z_4, etc. The results for a few lower moments indicate that the two models share the same sequence of 'transition temperatures'. Although we are unable to prove this statement in full generality, the factor $(1-bn/2)$ appearing in the denominator of the right-hand side of the recursion equation (14) makes the statement very plausible, if no special cancelations occur. Assuming that the correspondence is correct, the latter property would imply that the probability densities for the partition function in both models share the same power law tail: $\mathcal{P}(Z) \propto Z^{-1-(1/\beta^2)}$ valid everywhere in the high-temperature phase $\beta < \beta_c = 1$. Such a tail was indeed proposed as one of the universal features characterizing the class of models with logarithmic correlations [20].

We obtained analytical expressions for the hierarchic model's Z_n at positive b, equation (14). Having large series of Z_n for positive b, one can try to construct an approximate expression for the probability distribution and fractional moments.

3. Conclusion

The investigation of the statistical mechanics of REM-like models in finite dimension and their multifractal properties remains an active field of research, especially since the advances achieved in [27] and [20]. In this paper we confirmed that the multifractal properties are shared by the third class of REM-like models: directed polymers on a hierarchic tree. We calculated the corresponding moments important for the applications. In particular, we used those moments to conjecture the power law tail of the distribution of the partition function Z in the high-temperature phase. Unfortunately it is impossible to derive the exact probability distribution, as has been achieved in [20] for the 1d logarithmic REM. Nevertheless, our formulas allow the calculation of infinite series of moments for some values of parameters. We hope that such information will be able be applied to recover the probability distribution, a well-known problem in probability theory [32].

Our results are interesting for the mathematics of reaction–diffusion equations as well, as we related them to a certain nonlinear ODE. Moreover, we found the connection between the multifractal spectrum and the hierarchic tree model equation (36), while the latter has been mapped to some reaction–diffusion equations [17], see equation (38) in the present article. Thus the reaction–diffusion equations have some multi-scaling structures. One can connect the reaction–diffusion equation and the nonlinear ODE with the interesting versions of the multifractal phenomenon, i.e. [37], and try to investigate them to derive the multifractal scaling from the reaction–diffusion equation (38). Such a work is currently in the progress.

The directed walk model on hierarchic trees describes a rather rich physics, as well as some connections with quantum models in finite dimension. The relation to 2d conformal field models is rather well known [12, 13], and the work [17] mentions the relations to the quantum disorder problem in finite dimensions [36]. It will be interesting to try to apply the methods of the current work or our renormalization group (38) to the finite dimensional quantum disorder problem.

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