Entanglement renormalization and integral geometry

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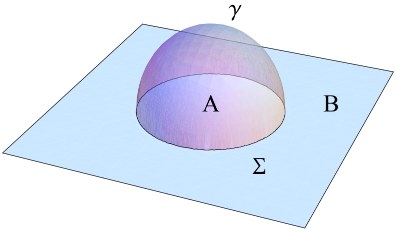
Nov 20th @ NTHU

Emergent geometry in AdS/CFT

Our framework

- $\mathrm{AdS}_{d+1} \leftrightarrow \mathrm{CFT}_d$
- large N, strong coupling $\Rightarrow$ classical gravity in the bulk
- entanglement = geometry

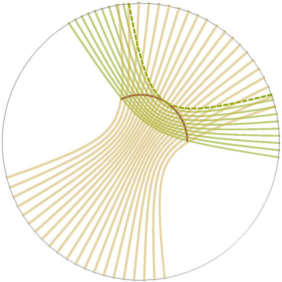
Ryu-Takayanagi formula

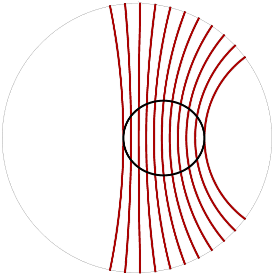


$$\boxed{S_{\mathrm{EE}} = \frac{A(\gamma)}{4G}}$$

γ: minimal surface ending on ∂A

Crofton's formula

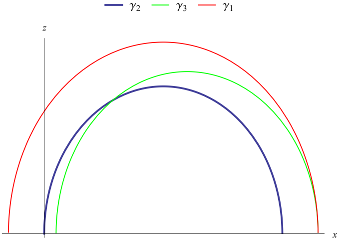


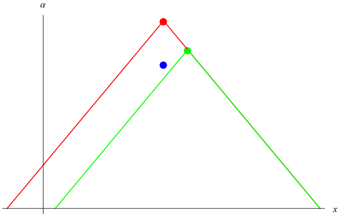


$$\boxed{\frac{\sigma(\gamma)}{4G} = \frac{1}{4}\int_{\gamma\cap\Gamma\neq\varnothing} N(\gamma\cap\Gamma)\ \epsilon_{\mathcal{K}}}$$

figures taken from Czech etal. arXiv:1505.05515

Kinematic space





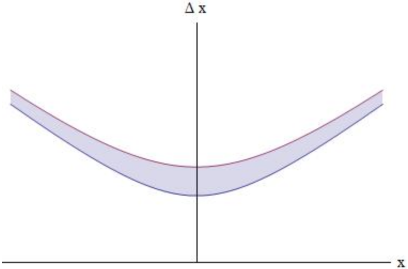
$$\mathrm{d}s^2 = \frac{1}{z^2}\left(\mathrm{d}z^2 + \mathrm{d}x^2\right)$$

$$\mathrm{d}s^2 = \frac{1}{\alpha^2}\left(-\mathrm{d}\alpha^2 + \mathrm{d}x^2\right)$$

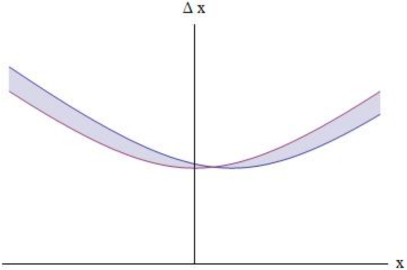
Measure in the kinematic is given by the second derivative of entanglement entropy

$$\boxed{\epsilon_{\mathcal{K}}(u, v) = \frac{\partial^2 S(u,v)}{\partial u \partial v}\mathrm{d}u \wedge \mathrm{d}v}$$

Point curve and distance



A (blue): (x,z), B (red): (x,z+δ)



A (blue): (x,z), B (red): (x-δ,z)

The distance is given by integration over the region between the point curves

$$\frac{\mathrm{d}(A,B)}{4G} = \frac{1}{4}\int_{(\tilde{p}_A \cup \tilde{p}_B)-(\tilde{p}_A \cap \tilde{p}_B)} \epsilon_{\mathcal{K}}$$

Kinematic space of geodesics in general dimensions

Crofton's formula in Higher dimensions:

$$\int_{M^q \cap L_r \neq \varnothing} \sigma_{q+r-d}(M^q \cap L_r)\, \epsilon_{\mathcal{K}} = \frac{O_d \ldots O_{d-r} O_{q+r-d}}{O_r \ldots O_1 O_0 O_q} \sigma_q(M^q)$$

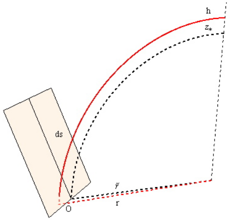
$$\epsilon_{\mathcal{K}} = \frac{1}{4G} \det\left[\frac{\partial^2 S(\vec{x}_1, \vec{x}_2)}{\partial \vec{x}_1 \partial \vec{x}_2}\right] \prod_{i=1}^{d-1} \mathrm{d}x_2^i \wedge \mathrm{d}x_1^i$$

- The measure follows from second derivative of S
- S is no longer related to entropy even though it can be computed from field theory

Line element

We work in a space with translation symmetry

$$\mathrm{d}s^2 = -\tilde{f}(z)\mathrm{d}t^2 + \sum_{i=1}^{d-1} g(z)(\mathrm{d}x^i)^2 + g(z)f(z)\mathrm{d}z^2$$

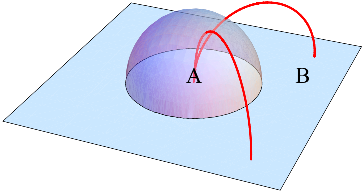


The area of the strip can be reproduced from the flux of geodesic

$$\begin{aligned} \mathrm{vol}_{S_{d-2}\times ds} &= \sqrt{f(z)g(z)}\mathrm{d}z \times g(z)^{\frac{d-2}{2}}\delta_{d-2}(\delta) \\ \Rightarrow \mathrm{d}s &= \sqrt{f(z)g(z)}\mathrm{d}z \end{aligned}$$

We can read off the line element from an infinitesimal strip

Reproducing RT-formula



- Count all the pairs with only one point in A
- The result is divergent

$$\frac{\sigma(\gamma_D)}{4G} = \frac{a_{d-2}}{\delta^{d-2}} \cdots + \frac{a_{d-2-2k}}{\delta^{d-2-2k}} \cdots + \begin{cases} a_0 \log \delta & \text{, d even} \\ a_0 & \text{, d odd} \end{cases} , \quad k \in \text{integers}$$

$$a_0 = \frac{L^{d-1}}{4G} \times \begin{cases} \frac{(-1)^{\frac{d}{2}} O_{d-2} (d-3)!!}{(d-2)!!} & \text{, d even} \\ \pi^{\frac{d}{2}-1} \Gamma\left(1 - \frac{d}{2}\right) & \text{, d odd} \end{cases}$$

Entanglement contour

Entanglement contour $s_A(i)$ (Chen&Vidal 2015)

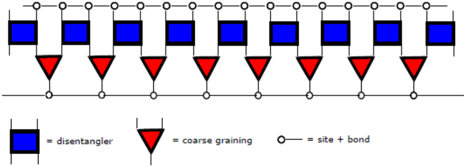
$$\sum_{i \subset A} s_A(i) = S_A$$

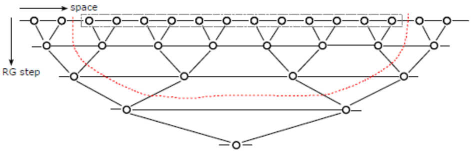
The entanglement entropy of a point can be expressed in the form above

$$S_{EE}(\mathrm{d}x;\tilde{z}) := \mathrm{d}x \left(\int_{-\infty}^{\tilde{x}} \frac{\partial^2 S(u,v;\tilde{z})}{\partial u \partial v}\Big|_{v=\tilde{x}} \mathrm{d}u + \int_{\tilde{x}}^{\infty} \frac{\partial^2 S(u,v;\tilde{z})}{\partial u \partial v}\Big|_{u=\tilde{x}} \mathrm{d}v \right)$$

and hence $\partial^2 S(u,v)$ can be identified as entanglement contour

Multi-scale entanglement renormalization ansatz





LUs (disentangler, isometry) modify the entanglement structure

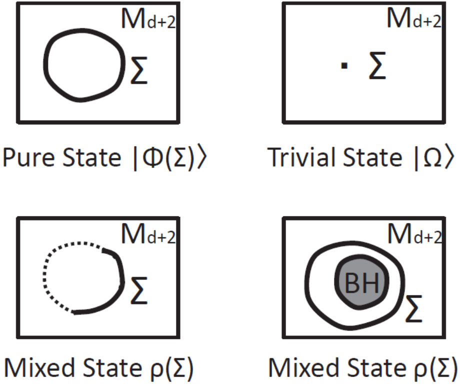
MERA version of RT-formula

$$S_A \leq \#(\mathrm{LUs\ cut})$$

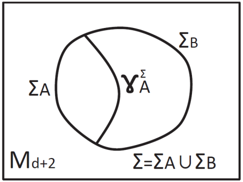
MERA = Discretized AdS (Swingle 2009)

figures taken from Swingle arXiv:1209.3304

Surface/state correspondence

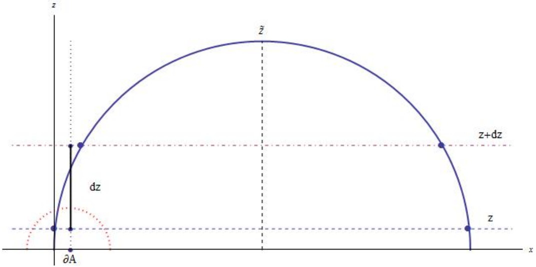


- AdS/CFT applies to $\Phi(\Sigma)$, an excited state
- RT remains valid for $\Phi(\Sigma)$



figures taken from Miyaji and Takayanagi arXiv:1503.03542

Renormalization of entanglement contour



The short-distance entanglement (red) is removed while the long-distance (blue) entanglement is invariant under RG flow but it is reshuffled to shorter scale.

Summary

- The punch line is that we have a duality between the kinematic space $\leftrightarrow$ real space. We can reconstruct the kinematic space from field theory.
- The number density of geodesics can be understood as a measure of two-point entanglement (entanglement contour).
- Entanglement contour can be defined on arbitrary surface and hence we can study the RG behavior

Future Problems

- ER=EPR (eternal AdS BH) and quench
- Co-dimension two surface higher dimensions
- Construction of bulk operator using Radon transform (Lin etal. 2014)
- DS/CFT and dynamics in the kinematic space (de Boer etal. 2015)