AdS/CFT and Integrability:
Cognate structure at weak and strong coupling for three-point functions

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Based on the series of works

on the 3-point functions in the prototypical duality between

$\mathcal{N} = 4$ SYM in 4D and string in $AdS_5 \times S^5$

- **arXiv:1512.????? (KKN)** **to appear soon**
  weak and strong coupling
- arXiv:1205.6060 (KK) strong coupling
- arXiv:1312.3727(KK) strong coupling
- arXiv:1410.8533(KKN) weak coupling
- arXiv:1506.03203(KKN) weak coupling

KK = Yoichi Kazama and Shota Komatsu (Perimeter)

KKN = Yoichi Kazama, Shota Komatsu and Takuya Nishimura (U. of Tokyo)

Proper explanation requires much more time than is available today.

So **I will just focus on the essence of the story**.

1 Introduction and motivation

- **Discovery of Higgs** $\Rightarrow$ **establishment of the standard model** (at least as an accurate effective theory)

- One of the **most profound remaining problems in high energy physics in years to come** is undoubtedly that of **physics at the Planck scale**, which necessarily includes

$\boxed{\textbf{Quantum gravity}}$ $\supset$

- **true nature of quantum black holes**
- **problem of observer dependence**
- **role of Planck scale**
- **necessarily a string theory ?**

etc. etc.

- Unfortunately **not much reliable hints exist so far**.

♦ **AdS/CFT** e.g. $\boxed{\begin{array}{c} \mathcal{N}=4 \text{ SYM theory in 4D} \\ \Updownarrow \\ \text{string theory in } AdS_5 \times S^5 \end{array}}$

$=$ **a concrete possibility that may reveal the crucial nature of quantum gravity through the study of** **gauge theories** which are well-defined.

♦ **Enormous investigations have been already performed but not revealing what we want:**

- Only the computations and the comparisons of results on the two sides (impressive as they may be)
- Various "applications" (in condensed matter physics and high energy QCD, etc) often based on rather unwarrantable foundations.

♦ What we crave for is **the understanding of the dynamical mechanism** of this **weak/strong duality** so that we may get new firmer hints for quatum gravity problems.

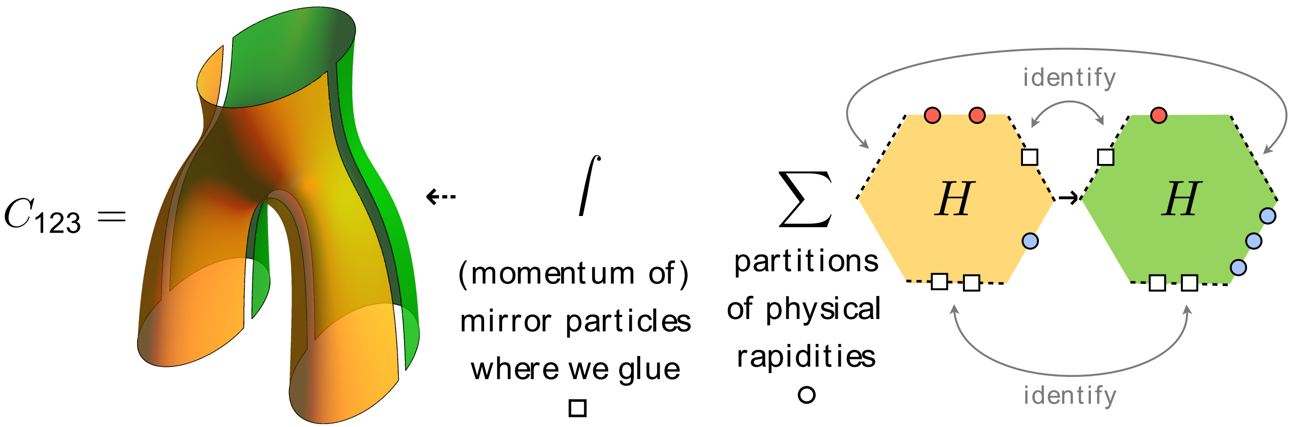
With this motivation deep in mind, I have been studying in the past few years

the simplest basic dynamical objects

$=$ $\boxed{\text{the non-BPS three-point functions}}$

Recent important development:

♦ **Hexagon form factor approach** for computation of certain three-point functions at **all finite coupling** (B. Basso, S. Komatsu and P. Vieira, 1505.06745)



(©Basso,Komatsu,Vieira)

Characteristic features of this work:

> ⋆ **Strong assumption: integrability holds to all orders**, **which is an unproven miracle.**
>
> ⋆ **The three-point structure constant C_{123} is computed from several integrability axioms, where neither SYM nor string theory is visible.**

Our series of works can be said to be $\boxed{\textbf{complementary}}$ **to such an approach**:

- ♦ Our results (so far) are at the tree (or partly 1-loop) level for SYM at weak coupling and at the semi-classical level for string theory at strong coupling.
- ♦ **However**, except for small assumptions, **the results are based on the integrability properties which have been firmly established.**

♦ Both SYM and string theory are clearly visible and our emphasis will be on the observation of the

structure common to	♦	**SYM (weak coupling)**
		and
	♦	**string theories (strong coupling)**

This should **help reveal the mechanism of strong/weak AdS/CFTduality**.

♦ In particular, we compare the **semiclassical behaviors**, **which are physically most intresting and suggestive.**

We will concentrate on the most well-studied prototypical duality

$$\boxed{\mathcal{N} = 4 \ \textbf{SYM} \Leftrightarrow \textbf{String theory on } AdS_5 \times S^5}$$

primarily for the "SU(2) sector "

Plan of the talk

1. Introduction and motivation
2. A "lightening" review of the weak coupling computation of the 3-point functions in the SU(2) sector
3. A "short" review of the essential ingredients for strong coupling computation using string theory in AdS space
4. New weak coupling technology for more general 3-point functions in SYM: Wick-contraction as singlet projection
5. Monodromy relation: A cognate integrability structure at weak and strong coupling
6. New computation of semiclassical three-point functions at weak coupling by strong coupling technology with the use of the Landau-Lifshitz formulation
7. Summary and future problems

2 A lightening review of the weak coupling computation of the 3-point functions in the SU(2) sector

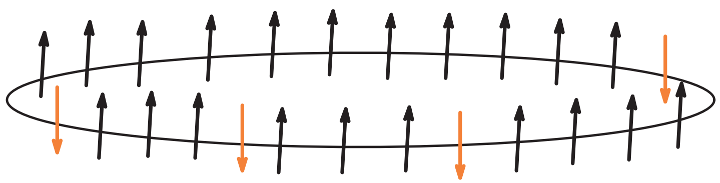
♦ Minahan and Zarembo made the following remarkable discovery in $\mathcal{N} = 4$ SYM theory (2002):

1-loop dilatation operator or **renormalization mixing matrix** for the gauge-invariant composite operators in the so-called **"SU(2)" sector**[1] takes the form of the **Hamiltonian of the $\mathrm{XXX}_{1/2}$ Heisenberg spin chain** with the identification

$$\uparrow \quad Z = \Phi_1 + i\Phi_2 \quad \text{(pseudo) vacuum component}$$

$$\downarrow \quad X = \Phi_3 + i\Phi_4 \quad \text{excitation} = \text{magnon}$$

Example: "4-magnon" configuration:



$$\sim \quad \mathrm{Tr}\,(ZZZ\cdots XZZ\cdots XZZ\cdots XZ\cdots X) + \cdots$$

mixed under renormalization with other similar configurations

[1] Actually, Minahan and Zarembo discussed the case of SO(6) sector $\supset$ SU(2) sector.

Mixing operator $\sim$ Hamiltonian of the periodic Heisenberg spin $1/2$ chain of length L

$$H = -J \sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1}\,, \qquad \vec{S} = \frac{1}{2}\vec{\sigma}$$

□ **Diagonalization of H via algebraic Bethe ansatz method**:

Minimal ingredients and results:

$$L_n(u) = \begin{pmatrix} u + iS_n^3 & iS_n^- \\ iS_n^+ & u - iS_n^3 \end{pmatrix} = \textbf{Lax matrix at site } n$$

u = **magnon "rapidity" (momentum) or the spectral parameter**

$$= \frac{1}{2}\cot\frac{p}{2}\,, \quad \text{or} \quad e^{ip} = \frac{u + \frac{i}{2}}{u - \frac{i}{2}}$$

$$\boxed{\begin{aligned}\Omega(u) = L_1(u) L_2(u) \cdots L_L(u) &= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \\ &= \textbf{monodromy matrix}\end{aligned}}$$

$$\boxed{B(u) \simeq \text{“creation operator”}\ , \quad C(u) \simeq \text{“annihilation operator”}}$$

$$\hat{T}(u) = \mathrm{Tr}\,\Omega(u) = \textbf{transfer matrix} = \sum_{i=0}^{L} \hat{T}_n u^n$$

$\{\hat{T}_n\}$ = **mutually commuting and hence are conserved charges**

since H itself can be constructed out of them

M-magnon eigenstates of $\hat{T}(u)$ (and hence of H)

$$\hat{T}(u) \prod_{i=1}^{M} B(u_i)|0\rangle = T(u) \prod_{i=1}^{M} B(u_i)|0\rangle$$

provided the following **Bethe equations** for the rapidities are satisfied

Bethe equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{i=1, i\neq k}^{M} \frac{u_k - u_i + i}{u_k - u_i - i}, \qquad k = 1, 2, \ldots, M$$

$$\Leftrightarrow \quad e^{ip_k L} = \prod_{i=1, i\neq k}^{M} S(u_k, u_i)\,, \qquad S(u_k, u_i) = \text{two-particle S-matrix}$$

Expresses consistency of the phase for the magnon with momentum p_k as it goes around the spin-chain once, hitting other magnons.

Hereafter, **all the composite operators (spin chains) are taken to be eigenstates of H (*i.e.* with definite conformal dimension).**

The corresponding states $\prod_{i=1}^{M} B(u_i)|0\rangle$ are called $\boxed{\textbf{on-shell}}$ **Bethe states**.

□ 3-point functions in the SU(2) sector:

Actually to construct 3-point functions in the "SU(2)" sector, we need to include all the single trace operators made out of the 4 scalars Φ_i (among the 6 scalars present in $\mathcal{N} = 4$ SYM)

$$Z \equiv \Phi_1 + i\Phi_2\,, \qquad \bar{Z} = \Phi_1 - i\Phi_2\,,$$
$$X \equiv \Phi_3 + i\Phi_4\,, \qquad \bar{X} \equiv \Phi_3 - i\Phi_4\,.$$

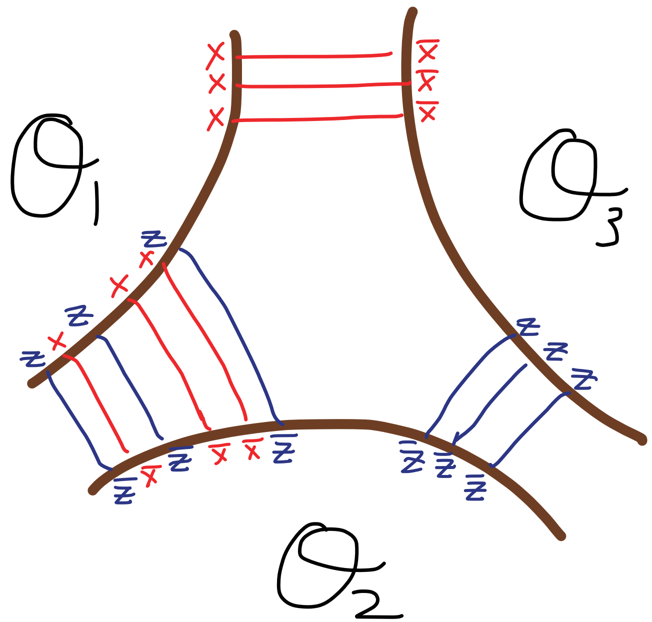
There is an **SO(4)=SU(2)$_{\rm L}\times$SU(2)$_{\rm R}$ symmetry** rotating the 4 scalars, **becoming important later**

The choice of operators to form a **proper 3-point function** is essentially unique[2]

$$\boxed{\mathcal{O}_1 :\ (Z, X)\,, \qquad \mathcal{O}_2 :\ (\bar{Z}, \bar{X})\,, \qquad \mathcal{O}_3 :\ (Z, \bar{X})}$$

[2]This was generalized in our work arXiv:1410.8533.

- Lines connecting the fields forming the operators represent the **Wick contraction** in $N = 4$ SYM at the tree level, like $\boldsymbol{X}(\boldsymbol{x}_1)\ \bar{\boldsymbol{X}}(\boldsymbol{x}_2) \propto \frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^2}$, ***etc.***
- The lines representing the operators are actually closed chains
- Each operator $\mathcal{O}_i$ actually consists of superpositions of many terms due to mixing.



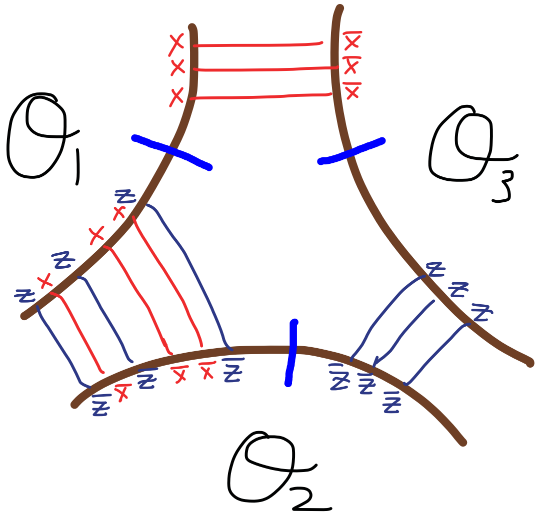
☐ Systematic "tailoring" procedure for constructing 3-point functions

Basic idea due to [Escobedo, Gromov, Sever, Vieira (2010)][3]:

3-point function can be constructed by **cutting, flipping and sewing**.

[3]Earlier works include: Okuyama-Tsent, Roiban-Volovich, Alday-David-Gava-Narain,etc

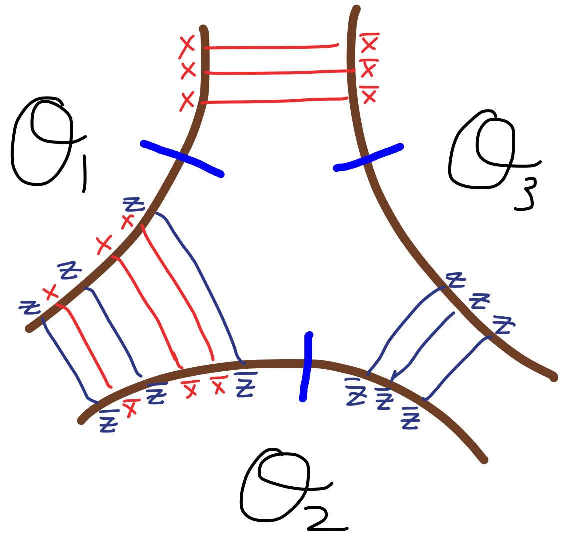
♦ Cutting



Many ways of cutting $\Rightarrow$ sum

♦ Flipping

- We should **contract the left part of $\mathcal{O}_1$ with the right part of $\mathcal{O}_2$** and so on.
- One wishes to **express the Wick-contraction** as an operation in the **spin-chain Hibert space.**
- So one should **flip the order** of, say, the **right part of the operator and make it into a "bra" state.**



♦ After some analysis, the effect of **flipping operation** $\mathcal{F}\circ$ turns out to be remarkably simple:

$$\mathcal{F} \circ \left(\prod_{i=1}^{M} B(u_i) |0\rangle \right) = \langle 0| \prod_{i=1}^{M} C(u_i)$$

♦ **Sewing**

Finally, we can contract (**sew**) the appropriate left and the right parts and express them in terms of spin-chain "inner products"

$$\langle v|u\rangle \equiv \langle \uparrow^L | \prod_{i=1}^{M} C(v_i) \prod_{j=1}^{M} B(u_j) | \uparrow^L \rangle$$

When the set $\{v_i\}$ or $\{u_i\}$ satisfies the Bethe equations ($i.e.$ on-shell), this is known to be expressible in terms of the $M \times M$ **Slavnov determinant**.

For example, if $\{u_i\}$ are on-shell,

$$
\begin{aligned}
\langle v|u\rangle &= \Delta^{-1}(v)\Delta^{-1}(u)\prod_{i=1}^{M} Q^{+}(v_i)Q^{-}(u_i) \\
&\quad \times \det\left(\frac{1}{v_i-u_i}\left(\prod_{k\neq j}^{M}(v_i-u_k-i)-\prod_{k\neq j}^{M}(v_i-u_k+i)\prod_{l=1}^{L}\frac{v_i-\frac{i}{2}}{v_i+\frac{i}{2}}\right)\right) \\
\Delta(x) &= \text{Vandermonde determinant for } M \text{ variables } \{x\}=(x_1,\ldots,x_M) \\
Q^{\pm}(u) &\equiv (u\pm\frac{i}{2})^{L}
\end{aligned}
$$

Explicit and exact but appears complicated for large L and M.

Result of tailoring:

With a clever trick, the result can expressed in terms of the ⟨off-shell|on-shell⟩ products: ($z = \{z_1, z_2, \ldots, z_L\}$ with $z_i = i/2$)

$$\boxed{C_{123}^{(0)} = \sqrt{L_1 L_2 L_3}\,\frac{\langle v \cup z|u\rangle_{L_1} \langle z|w\rangle_{L_3}}{\sqrt{\langle u|u\rangle \langle v|v\rangle \langle w|w\rangle}}}$$

This can be expressed in terms of the Slavnov determinant.

Semiclassical expression

Computing the semiclassical limit ($L, M \rightarrow \infty$ with M/L =fixed) of $C_{123}^{(0)}$ is quite non-trivial . Nevertheless, with a lot of effort, people[4] have succeeded in obtaining the following **remarkably compact expression** for the log of the structure constant in this limit

[4] Gromov, Sever and Vieira 1111; More general non-BPS case: Kostov 1203

$$
\ln C_{123}^{(0)} \simeq \oint_{\mathcal{A}_u \cup \mathcal{A}_v} \frac{du}{2\pi} \mathrm{Li}_2(e^{ip_u + ip_v + iL_3/(2u)}) + \oint_{\mathcal{A}_w} \frac{du}{2\pi} \mathrm{Li}_2(e^{ip_w + i(L_3 - L_1)/(2u)}) - \frac{1}{2} \oint_{\mathcal{A}_u} \frac{du}{2\pi} \mathrm{Li}_2(e^{2ip_u}) - \frac{1}{2} \oint_{\mathcal{A}_v} \frac{du}{2\pi} \mathrm{Li}_2(e^{2ip_v}) - \frac{1}{2} \oint_{\mathcal{A}_w} \frac{du}{2\pi} \mathrm{Li}_2(e^{2ip_w})
$$

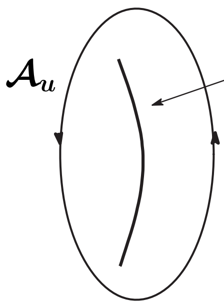
where $p_u(u)$ and $p_v(u)$ are called **"quasimomenta"** defined here by

$$
p_u(u) = \sum_{i=1}^{M} \frac{1}{u - u_i} - \frac{L}{2u}, \quad p_v(u) = \sum_{i=1}^{M} \frac{1}{u - v_i} - \frac{L}{2u}
$$

$\mathrm{Li}_2(z)$ is the dilog function

$$
\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}
$$

The contours $\mathcal{A}_u, \mathcal{A}_v, \mathcal{A}_w$ encircle the "cut" formed by the on-shell Bethe roots corresponding to u, v, w counter-clockwise.



3 A short review of the semiclassical strong coupling computation using string theory in AdS space

3.1 Structure of semiclassical 3-point functions at strong coupling

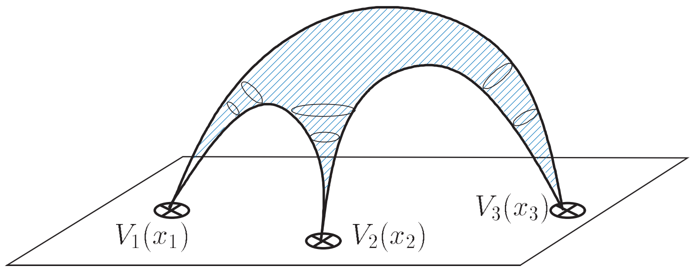
Basic structure in the saddle-point approximation

$$G(x_1, x_2, x_3) = e^{-S[X_*]} \prod_{i=1}^{3} V_i[X_*; z_i, x_i, Q_i]$$

$$= \text{Action part} \times \text{Vertex operator part}$$

$S \sim \log V_i[Q_i] \sim \mathcal{O}(\sqrt{\lambda})$, large action, large charge

$\lambda =$ large 't Hooft coupling



X_* = solution to the saddle-point equation

$$\frac{\delta}{\delta X}\left(-S[X] + \sum_i \log V_i[X]\right)\Bigg|_{X_*} = 0$$

x_i = Points on the boundary of AdS

- $V_i(x_i) = (1, 1)$ conformal primary vertex operator.

Serious obstacles

- No systematic method to construct proper $V_i(x_i)$ of interest in curved spacetime.
- No three-pronged saddle solutions in curved spacetime are known.

So we know almost nothing !

Nontheless, we have been able to overcome these difficulties by exploiting

the classical integrability of the string in $AdS_\star \times S^*$ and certain analyticity properties.

3.2 String in EAdS$_3\times$ S^3 and its classical integrability

3.2.1 Spacetime carrying the same symmetries as those of the SU(2) sector of SYM

We want to consider the string background having the same symmetry structure as the SU(2) sector of SYM: This is given by

$$\boxed{EAdS_3 \times S^3} \; (\subset AdS_5 \times S^5)$$

Embedding string coordinates with constraints, expressing the relevant backgrounds

$$AdS_5: \quad X^M X_M \equiv X^M \eta_{MN} X^N = -1\,, \quad M,N = -1,0,1,2,3,4\,,$$

$$\eta_{MN} = \mathrm{diag}\,(-1,1,1,1,1,1)\,,$$

$$EAdS_3: \quad X^\mu X_\mu \equiv X^\mu \eta_{\mu\nu} X^\nu = -1\,, \quad \mu,\nu = -1,1,2,4\,,$$

$$\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)\,,$$

$$S^3: \quad Y^I Y_I \equiv Y^I \delta_{IJ} Y^J = 1\,, \quad I,J = 1,2,3,4\,.$$

Action of the string in $EAdS_3 \times S^3$

$$S = \frac{\sqrt{\lambda}}{\pi} \int d^2z \left(\partial X^\mu \bar{\partial} X_\mu + \Lambda (X^\mu X_\mu + 1) + \partial Y^I \bar{\partial} Y_I + \tilde{\Lambda} (Y^I Y_I - 1) \right)$$

$$\text{string tension} = \frac{\sqrt{\lambda}}{2\pi} = \frac{1}{2\pi\alpha'}$$

't Hooft coupling $= \lambda = g_{YM}^2 N = \dfrac{1}{(\alpha')^2}\,,$ in the unit $R_{AdS_5} = R_{S^5} = 1$

Eq. of motion are non-linear after eliminating the Lagrange multipliers

$$\partial\bar{\partial}X^{\mu} - (\partial X^{\nu}\bar{\partial}X_{\nu})X^{\mu} = 0\,,$$
$$\partial\bar{\partial}Y^{I} + (\partial Y^{J}\bar{\partial}Y_{J})Y^{I} = 0\,.$$

Two sectors are connected solely by the Virasoro constraints for the total system

$$T_{AdS}(z) + T_{S}(z) = 0\,, \qquad \bar{T}_{AdS}(\bar{z}) + \bar{T}_{S}(\bar{z}) = 0\,,$$
$$T_{AdS}(z) = \partial X^{\mu}\partial X_{\mu}\,, \qquad T_{S}(z) = \partial Y^{I}\partial Y_{I}\,,$$
$$\bar{T}_{AdS}(\bar{z}) = \bar{\partial}X^{\mu}\bar{\partial}X_{\mu}\,, \qquad \bar{T}_{S}(\bar{z}) = \bar{\partial}Y^{I}\bar{\partial}Y_{I}\,.$$

3.2.2 Formulations to deal with the string equations using the the classical integrability of the system

For simplicity, we will focus on the **string on the S^3 part.**
($EAdS$ part is quite similar.)

There are **2 formulations**[5] **to deal with a string in such a background using classical integrability.**

- **Sigma model formulation**: Uses variables **covariant** under the global symmetry group G
- **Pohlmeyer reduction**: Uses variables **invariant** under G

- Although we actually need both formulations, we will only sketch the Pohlmeyer reduction, relevant for the computation of the action part.

[5] They are actually related by a (field-dependent) gauge transformation. (Y. Kazama and S. Komatsu, arXiv:1312.3727)

□ Method of Pohlmeyer reduction for a string in S^3:

The method of **Pohlomeyer reduction** uses the **"moving frame"** made out of the string coordinate $Y\ (\equiv \vec{Y} = (Y^1, Y^2, Y^3, Y^4))$ and then constructs the **connections** (gauge fields) invariant under G.

Moving frame q_i

Basic frame of 4-component fields q_i are defined as

$$q_1 \equiv Y\,, \quad q_2 \equiv a\partial Y + b\bar{\partial} Y\,, \quad q_3 \equiv c\partial Y + d\bar{\partial} Y\,, \qquad q_4 \equiv N$$

N is made out of $Y, \partial Y, \bar{\partial} Y$ such that it is orthogonal to q_1, q_2, q_3:

$$N \cdot Y = N \cdot \partial Y = N \cdot \bar{\partial} Y = 0$$

From these definitions it is immediate that

$$q_1^2 = 1\,, \quad q_1 \cdot q_2 = q_1 \cdot q_3 = 0$$

Further one can **impose the simple conditions**

$$q_2 \cdot q_3 = -2\,, \qquad q_2^2 = q_3^2 = 0$$

They provide conditions for the coefficients $a \sim d$.

Further it is convenient to define $SO(4)$-invariant quantity γ through an expression of the **kinetic term**:

$$\boxed{\partial Y \cdot \bar{\partial} Y = \sqrt{T\bar{T}} \cos 2\gamma}$$

Then one can solve a, b, c, d in terms of $T, \bar{T}$ and γ and obtain q_2 and q_3 in terms of them.

Closure under differentiation

It is easy to check that the **derivatives of q_i's are expressed in terms of q_i again**. This gives the compact closed equations

Assemble q_i's in the form

$$\boxed{W = \frac{1}{2}\begin{pmatrix} q_1 + iq_4 & q_2 \\ q_3 & q_1 - iq_4 \end{pmatrix}}$$

Then we get the $\boxed{W \text{ equations}}$

$$\boxed{\partial W + B_z^L W + W(B_z^R)^t = 0\,, \quad \bar{\partial} W + B_{\bar{z}}^L W + W(B_{\bar{z}}^R)^t = 0}$$

B_z^L, B_z^R and $B_{\bar{z}}^L, B_{\bar{z}}^R$ are called **connections** and are given by

$$B_z^L \equiv \begin{pmatrix} -\frac{i\partial\gamma}{2} & \frac{\rho e^{i\gamma}}{\sqrt{T}\sin 2\gamma} - \frac{\sqrt{T}}{2}e^{-i\gamma} \\ \frac{\rho e^{-i\gamma}}{\sqrt{T}\sin 2\gamma} - \frac{\sqrt{T}}{2}e^{i\gamma} & \frac{i\partial\gamma}{2} \end{pmatrix}$$

$$B_z^R \equiv \begin{pmatrix} \frac{i\partial\gamma}{2} & -\frac{\rho e^{-i\gamma}}{\sqrt{T}\sin 2\gamma} - \frac{\sqrt{T}}{2}e^{i\gamma} \\ -\frac{\rho e^{i\gamma}}{\sqrt{T}\sin 2\gamma} - \frac{\sqrt{T}}{2}e^{-i\gamma} & -\frac{i\partial\gamma}{2} \end{pmatrix}$$

$$B^L_{\bar z} \equiv \begin{pmatrix} \frac{i\bar\partial\gamma}{2} & \frac{\tilde\rho e^{-i\gamma}}{\sqrt{T}\sin 2\gamma} + \frac{\sqrt{T}}{2}e^{i\gamma} \\ \frac{\tilde\rho e^{i\gamma}}{\sqrt{T}\sin 2\gamma} + \frac{\sqrt{T}}{2}e^{-i\gamma} & -\frac{i\bar\partial\gamma}{2} \end{pmatrix}$$

$$B^R_{\bar z} \equiv \begin{pmatrix} -\frac{i\bar\partial\gamma}{2} & -\frac{\tilde\rho e^{i\gamma}}{\sqrt{T}\sin 2\gamma} + \frac{\sqrt{T}}{2}e^{-i\gamma} \\ -\frac{\tilde\rho e^{-i\gamma}}{\sqrt{T}\sin 2\gamma} + \frac{\sqrt{T}}{2}e^{i\gamma} & \frac{i\bar\partial\gamma}{2} \end{pmatrix}$$

where we have introduced some $SO(4)$ invariant quantities ρ and $\tilde\rho$.

♦ Now for the **connections** B^L **and** B^R, the W **equations** imply the **flatness (zero-curvature) conditions** $([\partial_\mu + A_\mu, \partial_\nu + A_\nu] = F_{\mu\nu} = 0)$

$\therefore$ **Original equations of motion can be written as**

$$\boxed{[\partial + B^L_z, \bar\partial + B^L_{\bar z}] = 0\,, \qquad [\partial + B^R_z, \bar\partial + B^R_{\bar z}] = 0}$$

♦ **The flatness conditions still hold** with an introduction of a complex **spectral parameter** ζ and leads to the **Lax equation:**

$$\boxed{\begin{aligned}&[\partial + B_z(\zeta), \bar\partial + B_{\bar z}(\zeta)] = 0\\ &B_z(\zeta) \equiv \frac{\Phi_z}{\zeta} + A_z\,, \quad B_{\bar z}(\zeta) \equiv \zeta\Phi_{\bar z} + A_{\bar z}\end{aligned}}$$

with

$$\Phi_z \equiv \begin{pmatrix} 0 & -\frac{\sqrt{T}}{2}e^{-i\gamma} \\ -\frac{\sqrt{T}}{2}e^{i\gamma} & 0 \end{pmatrix}, \quad \Phi_{\bar z} \equiv \begin{pmatrix} 0 & \frac{\sqrt{\bar T}}{2}e^{i\gamma} \\ \frac{\sqrt{\bar T}}{2}e^{-i\gamma} & 0 \end{pmatrix}$$

$$A_z \equiv \begin{pmatrix} -\frac{i\partial\gamma}{2} & \frac{\rho e^{i\gamma}}{\sqrt{T}\sin 2\gamma} \\ \frac{\rho e^{-i\gamma}}{\sqrt{T}\sin 2\gamma} & \frac{i\partial\gamma}{2} \end{pmatrix}, \quad A_{\bar z} \equiv \begin{pmatrix} \frac{i\bar\partial\gamma}{2} & \frac{\tilde\rho e^{-i\gamma}}{\sqrt{\bar T}\sin 2\gamma} \\ \frac{\tilde\rho e^{i\gamma}}{\sqrt{\bar T}\sin 2\gamma} & -\frac{i\bar\partial\gamma}{2} \end{pmatrix}$$

♦ Lax equation, as it is an extension of the eq. of motion, is **non-linear**.

To analyze it, one introduces the **auxiliary linear problem (ALP)** in the following way.

$$\boxed{(\partial + B_z(\zeta))\,\hat\psi = 0\,, \quad (\bar\partial + B_{\bar z}(\zeta))\,\hat\psi = 0}$$

- Existence of such $\hat{\psi}$ $\Leftrightarrow$ **Lax equation**.
- $\hat{\psi}$ $\Rightarrow$ **Reconstuction** of **string solutions**

Remark: **Often, instead of ζ, one uses x as the spectral parameter**

$$\boxed{\zeta = \frac{1-x}{1+x}, \qquad \text{or} \quad x = \frac{1-\zeta}{1+\zeta}}$$

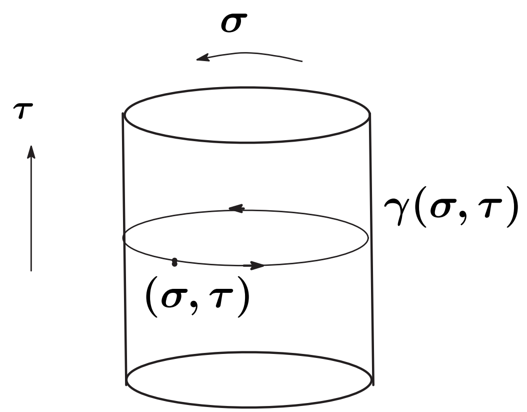
Monodromy matrix

Zero-curvature condition is extremely important for integrability.

Let $J(x) = J_\tau(x)d\tau + J_\sigma(x)d\sigma$ be a connection 1-form.

Monodromy matrix is defined by

$$\Omega(x) = \mathrm{P}\exp\left(-\oint_\gamma J(x)\right)$$
$$= \mathrm{P}\exp\left(-\int_0^{2\pi} d\sigma' J_\sigma(x;\sigma',\tau)\right)$$



Zero-curvature condition $\Rightarrow$ **monodromy matrix** $\Omega(x)$ is **independent of the contour** γ and hence actually **τ-independent**:

♦ Expanding $\Omega(x)$ in powers of $x \Rightarrow$ $\boxed{\infty \text{ number of conserved charges}}$.

quasimomentum

By diagonalizing $\Omega(x)$, one defines **quasimomentum** $p(x)$

$$u(x)\Omega(x)u^{-1}(x) = \begin{pmatrix} e^{ip(x)} & 0 \\ 0 & e^{-ip(x)} \end{pmatrix}$$

spectral curve$\simeq$ Riemann surface in x (with singularities)

x **can be thought of as parametrizing a spectral curve** Γ defined by the characteristic equation

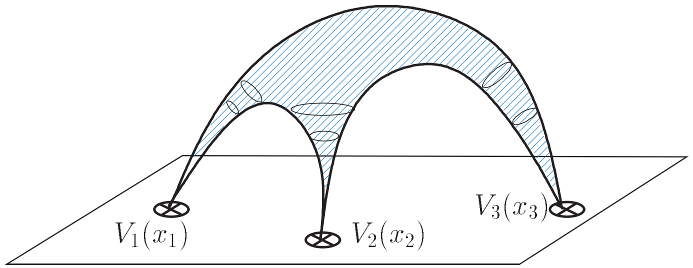
$$\det\left(y\mathbf{1} - \Omega(x)\right) = 1$$

In the present case, $\Omega(x)$ is 2×2, and hence Γ is a **hyperelliptic curve** $y^2 = f(x)$

3.3 Computation of the action part of the 3-point function

Recall the structure of the 3-point function we want to compute

$$\langle V_1(x_1)V_2(x_2)V_3(x_3)\rangle$$
$$= e^{-S[X_*]}\prod_{i=1}^{3} V_i[X_*; z_i, x_i, Q_i]$$
$$= \text{Action part} \times \text{Vertex operator part}$$



3.3.1 The structure of the action part of S^3

Write the structure of the 3-point function as

$$\langle V_1V_2V_3\rangle = \exp\left(F_{S^3} + F_{EAdS_3}\right)$$

We will first focus on the action part of F_{S^3}.

$$F_{S^3} = \boxed{\mathcal{F}_{\text{action}}} + \mathcal{F}_{\text{vertex}}\,,$$

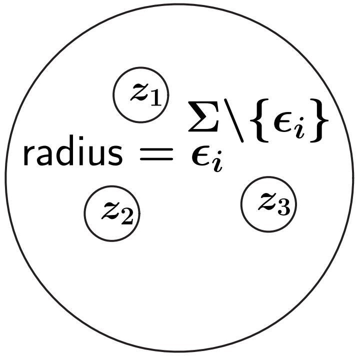
Action part is **invariant under the global symmetry** transformation.

$\Rightarrow$ Natural to make use of the **Pohlmeyer reduction**. We can then write the action as

$$S_{S^3} = \frac{\sqrt{\lambda}}{\pi} \int_{\Sigma \backslash \{\epsilon_i\}} d^2z\, \partial Y_I \bar{\partial} Y_I = \frac{\sqrt{\lambda}}{\pi} \int_{\Sigma \backslash \{\epsilon_i\}} d^2z \sqrt{T\bar{T}} \cos 2\gamma$$

where $T(z) \equiv T_{AdS}(z) = -T_S(z)$

$\Sigma \backslash \{\epsilon_i\}$ = a two-sphere with a small disk of radius ϵ_i cut out at each vertex operator insertion point z_i (for regularization purpose).



To simplify the integral, introduce the following **closed 1-forms**:

$$\varpi \equiv \sqrt{T(z)}dz$$

$$\eta \equiv -\sqrt{\bar{T}(\bar{z})}\cos 2\gamma d\bar{z} + \underbrace{\frac{2}{\sqrt{T}}\left(-(\partial\gamma)^2 + \frac{\rho^2}{T}\right)dz}_{\text{added to make } d\eta = 0}$$

$$d\varpi = d\eta = 0$$

Then using the closedness of η, we can write

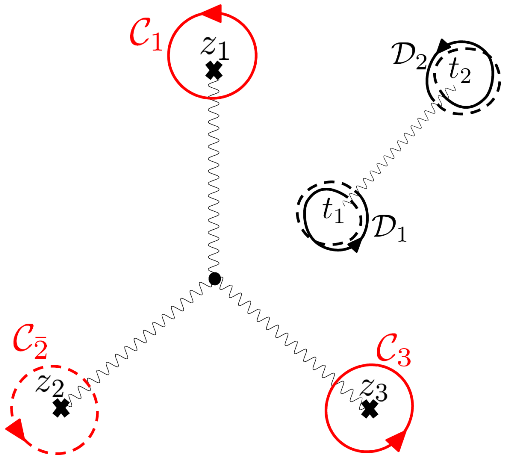
$$S_{S^3} = \frac{i\sqrt{\lambda}}{2\pi}\int_{\Sigma\backslash\{\epsilon_i\}} \varpi \wedge \eta = \frac{i\sqrt{\lambda}}{2\pi}\int_{\Sigma\backslash\{\epsilon_i\}} d(\Pi\eta)$$

where Π is the integral of ϖ:

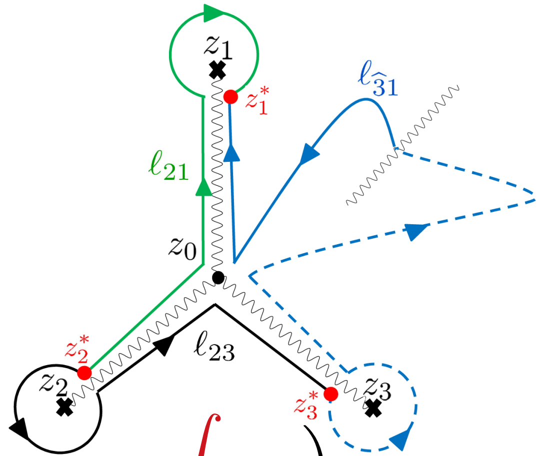
$$\Pi(z) = \int_{z_0}^{z} \varpi(z')dz'$$

♦ Now we can use **Stokes theorem** to rewrite the action as a **contour integral along a boundary $\partial\tilde{\Sigma}$** of a certain region $\tilde{\Sigma}$.

It turns out that there is a $\sqrt{}$ **branch cut** (as well as log branch cuts) $\Rightarrow$ $\tilde{\Sigma}$ **=two-sheeted Riemann surface**



$$\int_{\tilde{\Sigma}} \varpi \wedge \eta = \textbf{Local} + \textbf{Double} + \textbf{Global} + \textbf{Extra}\,,$$



$$\text{Local} = \sum_i \oint_{\mathcal{C}_i} \varpi \oint_{\mathcal{C}_i} \eta + \sum_{i<j} \Big(\oint_{\mathcal{C}_i} \varpi \oint_{\mathcal{C}_j} \eta - (\varpi \leftrightarrow \eta) \Big)$$

$$\text{Double} = -2 \sum_i \oint_{\mathcal{C}_i} \eta \int_{z_i^*}^{z} \varpi\,.$$

$$\text{Global} = \Big(\oint_{\mathcal{C}_1 + \mathcal{C}_{\bar 2} - \mathcal{C}_3} \varpi \int_{\ell_{21}} \eta + \oint_{\mathcal{C}_{\bar 2} + \mathcal{C}_3 - \mathcal{C}_1} \varpi \int_{\ell_{23}} \eta + \oint_{\mathcal{C}_3 + \mathcal{C}_1 - \mathcal{C}_{\bar 2}} \varpi \int_{\ell_{\bar 3 1}} \eta \Big) - (\varpi \leftrightarrow \eta)\,.$$

$$\text{Extra} = \sum_k \oint_{\mathcal{D}_k} \Pi \eta$$ (integrals around the zeros of $\sqrt{T}$)

♦ Almost all the integrals can be easily evaluated or only give phases, which we ignore.

♦ But the crucial line integrals $\boxed{\int \eta_{\ell_{ij}}}$ in **Global** **involve the contours intertwining the three punctures and hence give the essential information of the 3-point function**. **But they are impossible to compute directly !**

3.3.2 A sketch of the logic of how $\int \eta_{\ell_{ij}}$ can be computed

- ♦ Study the solutions of the **ALP(auxiliary linear problem) for the Pohlmeyer reduction** formalism.

$$\left(\partial + \frac{1}{\zeta}\Phi_z + A_z\right)\hat{\psi} = 0\,, \quad \left(\bar{\partial} + \zeta\Phi_{\bar{z}} + A_{\bar{z}}\right)\hat{\psi} = 0$$

and make “WKB” expansion around $\zeta = 0$ **at each** $z_i =$ **vertex insertion point**

$\Rightarrow$ The leading behaviors near $\zeta = 0$

$$\hat{\psi}_1^d \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp\left[\frac{1}{2\zeta}\int_{z_0}^{z} \varpi\right], \quad \hat{\psi}_2^d \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left[\frac{-1}{2\zeta}\int_{z_0}^{z} \varpi\right]$$

Denote the solutions of ALP around z_i which are eigenfunctions of Ω_i with eigenvalues ± 1 to be $\boxed{i_\pm}$

One of the $i_\pm$ is exponentially small and the other is big.

Only the small solution is unambiguous. (big' $=$ big $+\, a \times$ small)

- Now we define the skew-symmetric product called **"Wronskian"** between two-component vectors ϕ and χ as

$$\boxed{\langle \phi, \chi \rangle \equiv \phi_\alpha \epsilon^{\alpha\beta} \chi_\beta}$$

This will play a central role in what follows.

♦ After a tedious computations, we find that the Wronskians between small solutions can be expressed as

For $\mathbf{Re}\,\zeta > 0$, ($\langle i_+, j_+\rangle$ is small)

$$\langle 2_+\,, 1_+\rangle = \exp\left(-S_{2\to 1}\right)\,,\quad \langle 2_+\,, 3_+\rangle = \exp\left(-S_{2\to 3}\right)$$

$$\langle 3_+\,, 1_+\rangle = \exp\left(-S_{\hat{3}\to 1}\right)$$

For $\mathbf{Re}\,\zeta < 0$, ($\langle i_-, j_-\rangle$ is small)

$$\langle 2_-\,, 1_-\rangle = \exp\left(S_{2\to 1}\right)\,,\quad \langle 2_-\,, 3_-\rangle = \exp\left(S_{2\to 3}\right)$$

$$\langle 1_-\,, 3_-\rangle = \exp\left(S_{\hat{3}\to 1}\right)$$

where

$$\boxed{S_{i\to j} = \frac{1}{2\zeta}\int_{\ell_{ij}} \varpi + \int_{\ell_{ij}} \alpha + \frac{\zeta}{2}\int_{\ell_{ij}} \eta + \mathcal{O}(\zeta^2)}$$

The quantity $\int_{\ell_{ij}} \eta$ we needed for computing the action part appeared in the expressions of these Wronskians of small solutions !

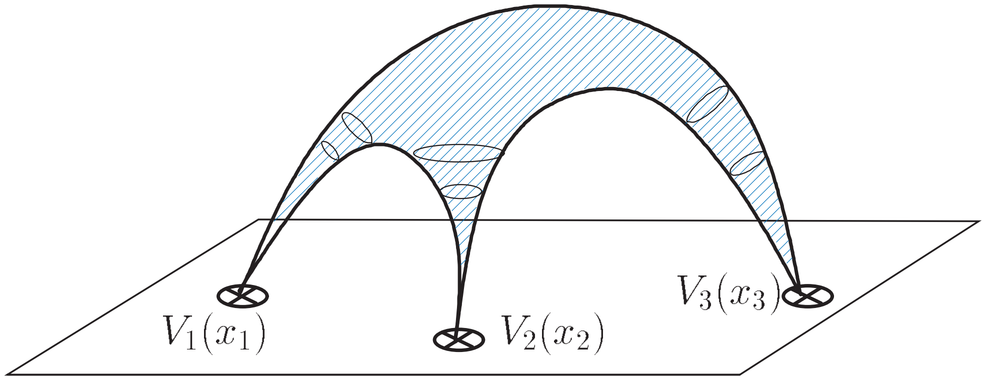
Thus, the crucial task for the evaluation of the action part will be to compute the Wronskians $\langle i_\pm, j_\pm \rangle$

3.4 Contribution from the vertex operators

3.4.1 Problems

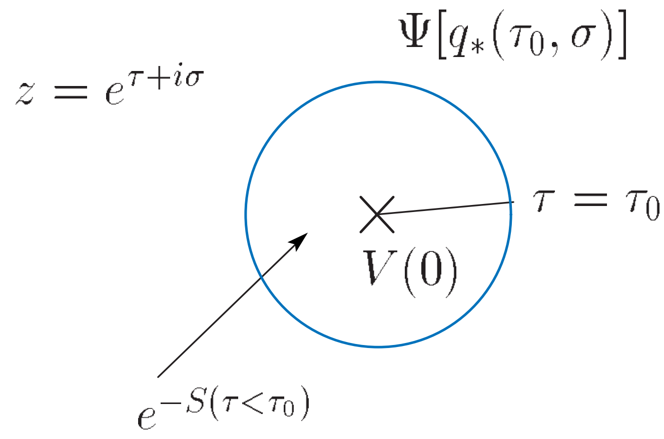
Recall the **serious problems**

- **No proper $V_i(x_i)$'s are known**
- **No saddle point solution with three prongs are known**



3.4.2 The way out: State-operator correspondence

In the saddle point approximation



$$V[q_*(z=0)]e^{-S_{q*}(\tau<\tau_0)} = \Psi[q_*(\tau_0,\sigma)]$$

$q_*(\tau,\sigma)$ = saddle point configuration in some canonical variable $q(\tau,\sigma)$.

If we can employ the **action-angle variables** (S_n, ϕ_n), the **wave function** can be expressed simply as (with Virasoro constraints used)

$$\Psi[\{\phi\}] = \exp\left(i\sum_n S_n\phi_n\right)$$

♠ **It is extremely hard to construct action-angle variables for non-linear systems.**

⋆ But for integrable systems, **Sklyanin's method** allows one to construct action-angle variables

3.5 Construction of the action-angle variables by Sklyanin's "magic recipe"

Consider the **normalized eigenvector** $h(x;\tau)$ of $\Omega(x;\tau,\sigma=0)$

$$(\star) \quad \Omega(x;\tau,\sigma=0)h(x;\tau) = e^{i\hat{p}(x)}h(x;\tau)$$

$$\boxed{n \cdot h = 1}, \qquad n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \text{normalization vector}, \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Important theorem (on a Riemann surface Γ of genus g)

$h(x;\tau)$ has $g+1$ poles, as a function of x.
Their positions on $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_g, \gamma_\infty)(\tau)$
$\gamma_i(\tau)$ depends on n

♦ **Sklyanin[6] explicitly constructed canonical variables associated to each pole γ_i of $h(x;\tau)$**

Canonical pairs "(q,p)"$\sim (z(\gamma_i), \hat{p}(\gamma_i))$

$$\left\{ \frac{\sqrt{\lambda}}{4\pi i} \hat{p}(\gamma_j) \, , z(\gamma_i) \right\}_P = \delta_{ij}$$

$$\{z(\gamma_i) \, , z(\gamma_j)\}_P = \{\hat{p}(\gamma_i) \, , \hat{p}(\gamma_j)\}_P = 0$$

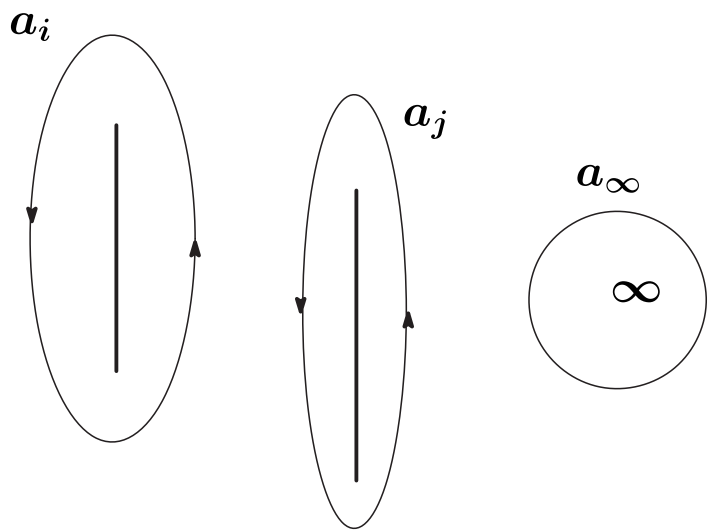
$\hat{p}$ = quasimomentum

$z = x + \frac{1}{x}$

= Zhukovski variable

[6] Applied to string in $R \times S^3$ by Dorey and Vicedo. Applicable to Euclidean AdS_3 case as well.

Then, the **conserved** **action variables S_i** can be defined in the usual way as $\sim \oint pdq$



$$S_i \equiv \frac{i\sqrt{\lambda}}{8\pi^2} \int_{a_i} \hat{p}(x) dz$$

$$= \text{“filling fraction”}$$

$$(i = 1, 2, \ldots, g, \infty)$$

Angle variables ϕ_i conjugate to S_i:

Generating function $F(S_i \,, z(\gamma_i))$ for the canonical transformation

$$(*) \quad \frac{\partial F}{\partial z(\gamma_i)} = -\frac{\sqrt{\lambda}}{4\pi i} \hat{p}(\gamma_i) \,, \qquad (**) \quad \frac{\partial F}{\partial S_i} = \phi_i$$

They can be solved and we get the **angle variable ϕ_i conjugate to S_i** in the nice form

$$\phi_i(\tau) = \frac{\partial F}{\partial S_i} = 2\pi \sum_k \int_{x_0}^{\gamma_k(\tau)} \omega_i = \textbf{Abel map}$$

$$\omega_i = \text{normalized holomorphic differential}\,, \oint_{a_j} \omega_i = \delta_{ij}$$

3.6 Construction of the wave functions corresponding to the vertex operators

Now that we have the action and angle variables, we can construct the **wave function** essentially as $\boxed{\sim \exp(iS_i\phi_i)}$.

♦ We need to take into account several points, which require rather non-trivial considerations. (We skip all the details.)

♦ **Normalization with respect to 2-point function:** $\Rightarrow$ **the difference of angle variables is important**

$$\Delta\phi = \phi_{\text{(3-point)}} - \phi_{\text{(2-point)}}$$

♦ **The angle variables that actually contribute to the wave functions are $\Delta\phi_R$ and $\Delta\phi_L$, conjugate to the global SU(2) charges R and L.**

♦ **Theorem**: The vertex operator corresponding to the SYM on-shell Bethe state is a **highest weight state of the global $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$.**

$\Rightarrow$ SU(2) property of the vertex operators can be characterized by 2-component **"polarization spinors" n (for R) and $\tilde{n}$ (for L)** which conincide with **normalization vectors**.

After a considerable amount of analysis, we obtain

$$\Psi_R^{S^3} = \exp\left(-i\sum_{i=1}^{3}(-R_i)\,\Delta\phi_{R,i}\right) = \prod_{\{i,j,k\}}\left(\frac{\langle n_i, n_j\rangle}{\langle i_-, j_-\rangle\big|_\infty}\right)^{R_i+R_j-R_k}$$

$$\Psi_L^{S^3} = \exp\left(-i\sum_{i=1}^{3}L_i\Delta\phi_{L,i}\right) = \prod_{\{i,j,k\}}\left(\frac{\langle \tilde{n}_i, \tilde{n}_j\rangle}{\langle i_+, j_+\rangle\big|_0}\right)^{L_i+L_j-L_k}$$

Again Wronskians are the building blocks!

3.7 Evaluation of the Wronskians $\langle i_\pm, j_\pm \rangle$

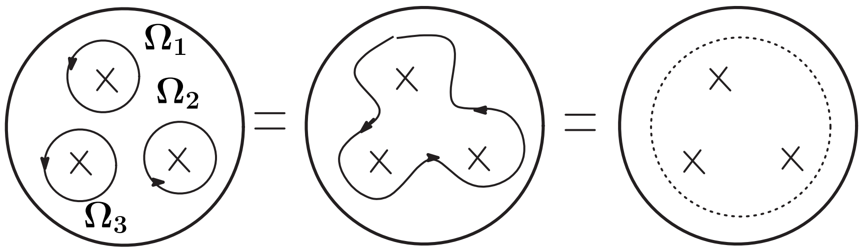
3.7.1 Relations following from the global monodromy condition:

$\Omega_i =$ local monodromy for ALP equations around z_i

Crucial ingredient:

Monodromy relation $\boxed{\Omega_1 \Omega_2 \Omega_3 = 1}$ $\Rightarrow$ **Global information**

$\Leftrightarrow$ **Monodromy around all the singularities at the positions z_i of the vertex operators on the worldsheet must be trivial**[7]**.**



[7] This was first utilzed for $\boldsymbol{AdS_2}$ case by Janik and Weresczenski.

Consequence of the monodromy relation

Take Ω_1-diagonal basis: $\Omega_1 = \begin{pmatrix} e^{ip_1} & 0 \\ 0 & e^{-ip_1} \end{pmatrix}$

$\Omega_{2,3}$, which are not diagonal in this basis, can be expressed in terms of the Wronskians $\langle i_\pm, j_\pm \rangle$ and $p_2(u)$ and $p_3(u)$.

$\because$ **Set of eignevectors $j_\pm$ at z_j form a complete basis**

$\Rightarrow$ One can expand $i_\pm$ in terms of $j_\pm$: $i_\pm = \langle i_\pm, j_- \rangle j_+ - \langle i_\pm, j_+ \rangle j_-$

$\Rightarrow$ Expression of Ω_2 in the Ω_1-diagonal basis

$$\Omega_2 = M_{12} \begin{pmatrix} e^{ip_2} & 0 \\ 0 & e^{-ip_2} \end{pmatrix} M_{21}\,, \quad M_{ij} = \begin{pmatrix} -\langle i_-\,, j_+ \rangle & -\langle i_-\,, j_- \rangle \\ \langle i_+\,, j_+ \rangle & \langle i_+\,, j_- \rangle \end{pmatrix}$$

Substituting such forms into the relation $\Omega_1\Omega_2\Omega_3 = 1$, we get **the crucial equations expressing the product of Wronskians in terms of the quasi-momenta**

$$
\begin{aligned}
\langle 1_+\,,2_+\rangle\langle 1_-\,,2_-\rangle &= \frac{\sin\frac{p_1+p_2+p_3}{2}\,\sin\frac{p_1+p_2-p_3}{2}}{\sin p_1\,\sin p_2}\,,\\
\langle 2_+\,,3_+\rangle\langle 2_-\,,3_-\rangle &= \frac{\sin\frac{p_1+p_2+p_3}{2}\,\sin\frac{-p_1+p_2+p_3}{2}}{\sin p_2\,\sin p_3}\,,\\
\langle 3_+\,,1_+\rangle\langle 3_-\,,1_-\rangle &= \frac{\sin\frac{p_1+p_2+p_3}{2}\,\sin\frac{p_1-p_2+p_3}{2}}{\sin p_3\,\sin p_1}\,,\\
\langle 1_+\,,2_-\rangle\langle 1_-\,,2_+\rangle &= \frac{\sin\frac{p_1-p_2+p_3}{2}\,\sin\frac{p_1-p_2-p_3}{2}}{\sin p_1\,\sin p_2}\,,\\
\langle 2_+\,,3_-\rangle\langle 2_-\,,3_+\rangle &= \frac{\sin\frac{p_1+p_2-p_3}{2}\,\sin\frac{-p_1+p_2-p_3}{2}}{\sin p_2\,\sin p_3}\,,\\
\langle 3_+\,,1_-\rangle\langle 3_-\,,1_+\rangle &= \frac{\sin\frac{-p_1+p_2+p_3}{2}\,\sin\frac{-p_1-p_2+p_3}{2}}{\sin p_3\,\sin p_1}\,.
\end{aligned}
$$

Major task = Extract individual Wronskian from the product

3.7.2 Extraction of individual Wronskian from analytic properties

Consider one of the relations

$$\langle 1_+ , 2_+ \rangle \langle 1_- , 2_- \rangle = \frac{\sin \frac{p_1+p_2+p_3}{2} \sin \frac{p_1+p_2-p_3}{2}}{\sin p_1 \sin p_2}$$

analyticity structure

$\boxed{\text{RHS}}$

- **poles** when $\sin p_1 = 0$ or $\sin p_2 = 0$
- **zeros** when $\sin \frac{p_1+p_2+p_3}{2} = 0$ or $\sin \frac{p_1+p_2-p_3}{2} = 0$

$\boxed{\text{LHS}}$: **Study which factor on the LHS has a pole or a zero under what condition**. (This requires a fairly delicate analysis.)

- We need to make some **natural global analyticity assumption** to proceed:

We assume: No singularity for the string coordinate on the world-sheet except at the positions of the vertex operators.

$$\Downarrow$$

♦ **Rules to tell when which of the Wronskian factor has a pole or a zero.**

Wiener-Hopf decomposition to project out individual Wronskian

$F(x)$ = a function which decreases sufficiently fast at ∞.

Suppose $F(x)$ can be decomposed as

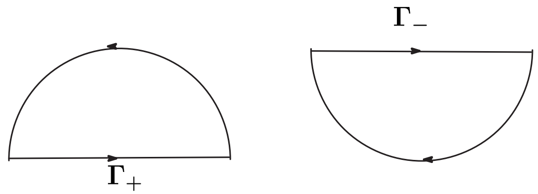
$$\boxed{F(x) = F_+(x) + F_-(x)\,, \quad F_\pm(x) = \text{analytic in } H_\pm}$$

Wiener-Hopf decomposition formla gives the integral representation of $F_\pm(x)$ in terms of the function $F(x)$. It reads

$$F_+(x) = \int_{\Gamma_+} \frac{dx'}{2\pi i} K(x';x) F(x') \qquad (\mathrm{Im}\, x > 0)$$

$$F_-(x) = -\int_{\Gamma_-} \frac{dx'}{2\pi i} K(x';x) F(x') \qquad (\mathrm{Im}\, x < 0)$$

kernel $\quad K(x';x) = \dfrac{1}{x' - x}$

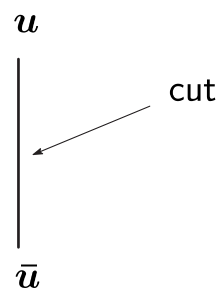


□ Some remarks on the Wiener-Hopf integral:

♦ For the product $\boldsymbol{F(x) = F_+(x) F_-(x)}$,
take the log $\boldsymbol{\ln F(x) = \ln F_+(x) + \ln F_-(x)}$ and use the decomposition.

♦ **A caution:** $p_i(x)$ **is defined on a two-sheeted Riemann surface.** Thus **we must modify the simple kernel $1/(x' - x)$, such that the pole occurs only when x' and x coincide on the same sheet.** This is achieved by the kernel

$$\widehat{\mathcal{K}}_i(x'; x) \equiv \frac{1}{2(x' - x)} \left(\sqrt{\frac{(x - u_i)(x - \bar{u}_i)}{(x' - u_i)(x' - \bar{u}_i)}} + 1 \right)$$



♦ **Integration contour** must be chosen so that it divides the regions with different analyticity.

3.8 Remark: Contribution from the $EAdS_3$ part

Since the logic is very much the same as for the S^3 case, we just list the different features.

♦ The right and the left polarization spinors $\hat{n}, \hat{\tilde{n}}$ specifying the vertex operators are related in this case to their positions (x_1, x_2) on the boundary of $EAdS_3$

$$x \equiv x^1 + ix^2\,, \quad \bar{x} \equiv x^1 - ix^2$$

$$\hat{n} = \begin{pmatrix} 1 \\ x \end{pmatrix}, \qquad \hat{\tilde{n}} = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}$$

$\Rightarrow$ The **wave function part**, containing the Wronskians $\langle \hat{n}_i, \hat{n}_j \rangle$ etc., produces precisely the **correct coordinate dependence of the 3-point function with non-zero spins**

$$\prod_{\{i,j,k\}} (x_i - x_j)^{-(R_i+R_j-R_k)} (\bar{x}_i - \bar{x}_j)^{-(L_i+L_j-L_k)}$$

♦ Various quantities of $EAdS_3$, such as **the energy-momentum tensor, have opposite signs compared to the S^3 case**. $\Rightarrow$ **Contributions to the structure constant $\ln C_{123}$ come with opposite signs.**

3.9 Final answer for the 3-point function at strong coupling

$$\begin{aligned}\langle V_1 V_2 V_3\rangle =& \frac{1}{N}\frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}\\ &\times \langle n_1\,, n_2\rangle^{R_1+R_2-R_3}\langle n_2\,, n_3\rangle^{R_2+R_3-R_1}\langle n_3\,, n_1\rangle^{R_3+R_1-R_2}\\ &\times \langle \tilde{n}_1\,, \tilde{n}_2\rangle^{L_1+L_2-L_3}\langle \tilde{n}_2\,, \tilde{n}_3\rangle^{L_2+L_3-L_1}\langle \tilde{n}_3\,, \tilde{n}_1\rangle^{L_3+L_1-L_2}\end{aligned}$$

where the log of the structure constant C_{123} is given by

$$\ln C_{123} = \ln C_{123}^{S^3} + \ln C_{123}^{EAdS_3} + \text{Contact}$$

$$
\begin{aligned}
&\ln C_{123}^{S^3} \\
&= \int_{\mathcal{M}^{uuu}_{---}} \frac{z(x)\,(dp_1 + dp_2 + dp_3)}{2\pi i} \ln\sin\left(\frac{p_1 + p_2 + p_3}{2}\right) \\
&+ \int_{\mathcal{M}^{uuu}_{--+}} \frac{z(x)\,(dp_1 + dp_2 - dp_3)}{2\pi i} \ln\sin\left(\frac{p_1 + p_2 - p_3}{2}\right) \\
&+ \int_{\mathcal{M}^{uuu}_{-+-}} \frac{z(x)\,(dp_1 - dp_2 + dp_3)}{2\pi i} \ln\sin\left(\frac{p_1 - p_2 + p_3}{2}\right) \\
&+ \int_{\mathcal{M}^{uuu}_{+--}} \frac{z(x)\,(-dp_1 + dp_2 + dp_3)}{2\pi i} \ln\sin\left(\frac{-p_1 + p_2 + p_3}{2}\right) \\
&- 2\sum_{j=1}^{3} \int_{\Gamma^{u}_{j_-}} \frac{z(x)\,dp_j}{2\pi i} \ln\sin p_j
\end{aligned}
$$

$$\ln C_{123}^{EAdS_3}$$

$$= -\int_{\hat{\mathcal{M}}^{uuu}_{---}} \frac{z(x)\,(d\hat{p}_1 + d\hat{p}_2 + d\hat{p}_3)}{2\pi i} \ln\sin\left(\frac{\hat{p}_1 + \hat{p}_2 + \hat{p}_3}{2}\right)$$

$$- \int_{\hat{\mathcal{M}}^{uuu}_{--+}} \frac{z(x)\,(d\hat{p}_1 + d\hat{p}_2 - d\hat{p}_3)}{2\pi i} \ln\sin\left(\frac{\hat{p}_1 + \hat{p}_2 - \hat{p}_3}{2}\right)$$

$$- \int_{\hat{\mathcal{M}}^{uuu}_{-+-}} \frac{z(x)\,(d\hat{p}_1 - d\hat{p}_2 + d\hat{p}_3)}{2\pi i} \ln\sin\left(\frac{\hat{p}_1 - \hat{p}_2 + \hat{p}_3}{2}\right)$$

$$- \int_{\hat{\mathcal{M}}^{uuu}_{+--}} \frac{z(x)\,(-d\hat{p}_1 + d\hat{p}_2 + d\hat{p}_3)}{2\pi i} \ln\sin\left(\frac{-\hat{p}_1 + \hat{p}_2 + \hat{p}_3}{2}\right)$$

$$+ 2\sum_{j=1}^{3} \int_{\hat{\Gamma}^{u}_{j_-}} \frac{z(x)\,d\hat{p}_j}{2\pi i} \ln\sin\hat{p}_j + \text{Contact}$$

Remark:

- The structure that we obtained above for the strong coupling is **remarkably similar to the weak coupling counterpart**. **(They need not be.)** In fact it is not difficult to show that the integral over the $\mathbf{Li_2}$ function that appeared at weak coupling is related to the characteristic integral for the strong coupling as

$$\oint \frac{dz}{2\pi i} \mathrm{Li}_2(e^{2ip}) = 2i \oint \frac{dz}{2\pi i} z \frac{dp}{dz} \ln \sin p + \oint \frac{dz}{2\pi i} \left(p^2(z) - 2i \ln(-2i) p(z) \right)$$

4 New weak coupling technology for more general 3-point functions: Wick-contraction as singlet projection

Note:

♦ At weak coupling, the 3-point function obtained by the tailoring was of quite special type.

♦ At strong coupling, we could compute 3-point functions for fairly **general vertex operators characterized by the polarization spinors n and $\tilde{n}$** transforming under $SU(2)_L \times SU(2)_R$.

⋆ **We wish to treat the more general 3-point functions at weak coupling in a parallel manner.**

There are two observations which make it possible and will be extremely useful.

4.1 Double-spin-chain formalism and Wick contraction

To make the weak coupling treatment more parallel to the strong coupling case, it is convenient to recall

★ **Symmetry of the "SU(2)" sector is actually SO(4) $=$ SU(2)$_L \otimes$ SU(2)$_R$.** $\Rightarrow$ These fundamental fields can be naturally mapped to the

♦ $\boxed{\textbf{double-spin-chain}\text{ states}}$

Usual spin chain $=$ tensor product of two spin chains.

For the fundamental fields,

$$Z \mapsto |\uparrow\rangle_L \otimes |\uparrow\rangle_R\,, \quad X \mapsto |\uparrow\rangle_L \otimes |\downarrow\rangle_R\,,$$
$$\bar{Z} \mapsto |\downarrow\rangle_L \otimes |\downarrow\rangle_R\,, \; -\bar{X} \mapsto |\downarrow\rangle_L \otimes |\uparrow\rangle_R\,,$$

$\Rightarrow$ $\boxed{\textbf{All the quantities get factorized into L and R parts}}$

♦ In this point of view, **More general spin-chain states built on rotated vacuum** can be defined and characterized by two polarization spinors

$\mathfrak{n} = (\mathfrak{n}^1, \mathfrak{n}^2)^t$ **and** $\tilde{\mathfrak{n}} = (\tilde{\mathfrak{n}}^1, \tilde{\mathfrak{n}}^2)^t$, just as in the strong coupling case.

$$|\psi\rangle = |\mathfrak{n}\rangle_L \otimes |\mathfrak{n}\rangle_R$$

$$|\mathfrak{n}\rangle_L \equiv \mathfrak{n}^1|\uparrow\rangle_L + \mathfrak{n}^2|\downarrow\rangle_L = n^a|a\rangle_L\,, \quad |1\rangle \equiv |\uparrow\rangle\,, |2\rangle \equiv |\downarrow\rangle$$

$$|\tilde{\mathfrak{n}}\rangle_R \equiv \tilde{\mathfrak{n}}^1|\uparrow\rangle_R + \tilde{\mathfrak{n}}^2|\downarrow\rangle_R = \tilde{\mathfrak{n}}^{\tilde{a}}|\tilde{a}\rangle_R$$

♦ **Wick contraction can be interpreted as group singlet projection**

The basic contractions are (omitting coordinate dependence)

$$\underbrace{Z\,Z}=0\,,\quad \underbrace{Z\,X}=0\,,\quad \underbrace{Z\,\bar{X}}=0\,,\quad \underbrace{Z\,\bar{Z}}=1\,,\quad \text{etc}\,.$$

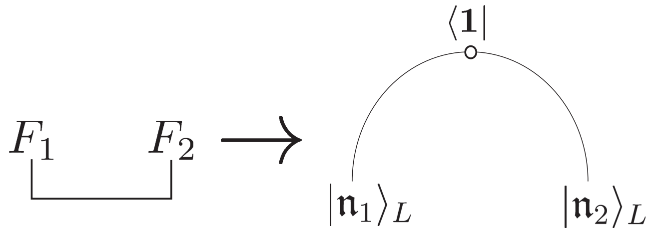
For general operators $F_i \mapsto |\mathfrak{n}_i\rangle_L \otimes |\tilde{\mathfrak{n}}_i\rangle_R$, the contraction rules above lead to

$$(\star)\quad \underbrace{F_1\,F_2} = (\mathfrak{n}_1^a \epsilon_{ab} \mathfrak{n}_2^b)(\tilde{\mathfrak{n}}_1^{\tilde{c}} \epsilon_{\tilde{c}\tilde{d}} \tilde{\mathfrak{n}}_2^{\tilde{d}}) = \langle 1|\,(|\mathfrak{n}_1\rangle_L \otimes |\mathfrak{n}_2\rangle_L)\,\langle 1|\,(|\tilde{\mathfrak{n}}_1\rangle_R \otimes |\tilde{\mathfrak{n}}_2\rangle_R)$$

where $\langle 1|$ is the singlet projector[8] $\boxed{\langle 1| \equiv \epsilon_{ab}\langle a| \otimes \langle b|}$

[8]Singlet projector for the entire psu(2,2|4) sector can also be constructed (Jiang, Kostov, Petrovskii and Servan; KKN)

The contraction ($\star$) can be dipicted by



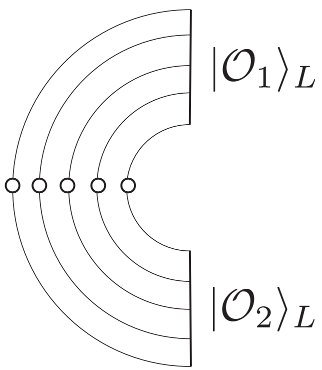
This can be easily generalized for the contraction of two and three composite operators

We will denote the **Wick-contraction** as

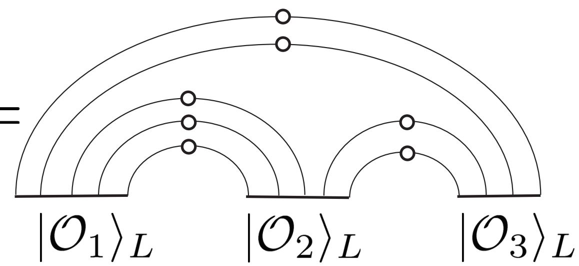
$$\langle |\mathcal{O}_1\rangle_L, |\mathcal{O}_2\rangle_L \rangle \equiv \langle \mathbf{1} | (|\mathcal{O}_1\rangle_L \otimes |\mathcal{O}_2\rangle_L)$$

The contractions of two and three spin chain states can be dipicted as

$$\langle |\mathcal{O}_1\rangle_L \, , |\mathcal{O}_2\rangle_L \rangle \;=$$



$$\langle |\mathcal{O}_1\rangle_L \, , |\mathcal{O}_2\rangle_L \, , |\mathcal{O}_3\rangle_L \rangle =$$



5 Monodromy relation for weak coupling: Expression of "integrability" for correlation functions

Recall that for the strong coupling computation, the **trivality of the product of local monodromies**

$$\boxed{\Omega_1 \Omega_2 \Omega_3 = 1}$$

played a crucial role, giving the precious **global information** of the 3-point function.

♦ **This should represents an important part of the "integrability" of the 3-point function.**

What is the counterpart in the weak coupling SYM theory ?

5.1 Monodromy relation for 2-point function

First consider the case of the **2-point function**.

Key is the two basic properties of the Lax matrix

$$L(u) = \begin{pmatrix} u + iS^3 & iS^- \\ iS^+ & u - iS^3 \end{pmatrix}$$

(1) Unitarity (or inversion) relation

$$\mathrm{L}(-u + i/2)\mathrm{L}(u - i/2) = f(u) \cdot 1 \quad \text{(unit matrix)}$$
$$f(u) = -(u^2 + 1)$$

(2) Crossing relation

$$\langle 1 | (L(u)|\psi_1\rangle \otimes |\psi_2\rangle) = \langle 1 | (|\psi_1\rangle \otimes \mathcal{C} \circ \mathrm{L}(u)|\psi_2\rangle)$$

where $\quad \mathcal{C} \circ \mathrm{L}(u) = -\mathrm{L}(-u)$

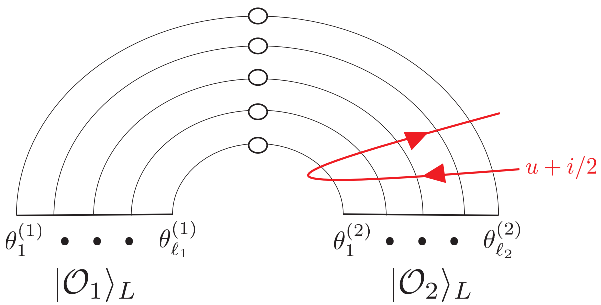
Apply **unitarity relation** to the monodromy matrix: $\quad \Omega(u) = L_1(u)L_2(u) \cdots L_\ell(u)$

$\Rightarrow$ **"Unitarity" relation for the monodromy matrices**

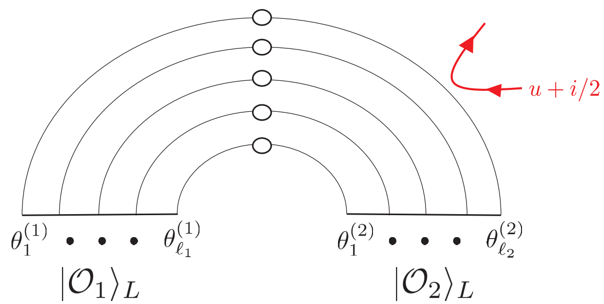
$\Rightarrow$ **Relate 2-point functions with and without monodromy matrix insertions**

$$(\star) \qquad \left\langle |\mathcal{O}_1\rangle_L \,, \left(\overleftarrow{\Omega}_2(-u+i/2)\right)_{ij} \left(\Omega_2(u+i/2)\right)_{jk} |\mathcal{O}_2\rangle_L \right\rangle$$
$$= \delta_{ik}(-1)^{\ell} f_{12}(u) \left\langle |\mathcal{O}_1\rangle_L \,, |\mathcal{O}_2\rangle_L \right\rangle$$

This is illustrated in the following figures



$=$



♦ Make use of the **crossing relation** for the Lax operator constituting $\overleftarrow{\Omega}_2$ repeatedly, the LHS of $(\star)$ can be rewritten as

$$(-1)^{\ell}\left\langle \Big(\Omega_1^-(u)\Big)_{ij}|\mathcal{O}_1\rangle_L \,, \Big(\Omega_2^+(u)\Big)_{jk}|\mathcal{O}_2\rangle_L \right\rangle .$$

Combining, we get the monodromy relation for the 2-point function

$$\boxed{\left\langle \Big(\Omega_1^-(u)\Big)_{ij}|\mathcal{O}_1\rangle_L \,, \Big(\Omega_2^+(u)\Big)_{jk}|\mathcal{O}_2\rangle_L \right\rangle = \delta_{ik}\, f_{12}(u)\left\langle |\mathcal{O}_1\rangle_L \,, |\mathcal{O}_2\rangle_L \right\rangle}$$

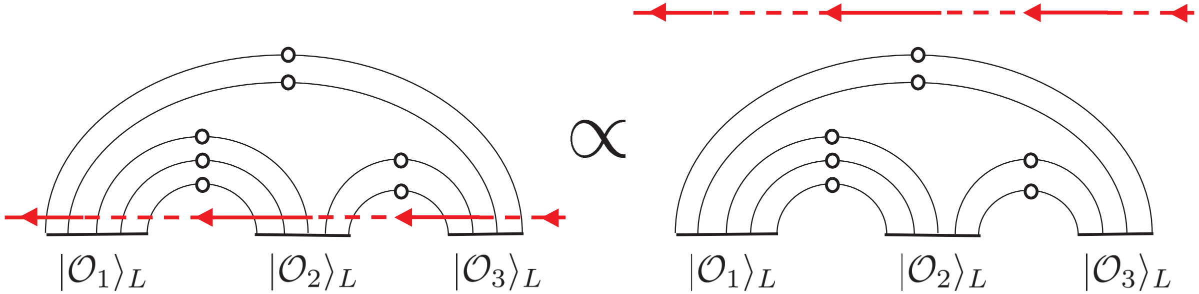
where $\boxed{\Omega^{\pm}(u) \equiv \Omega(u \pm \tfrac{i}{2})}$.

5.2 Monodromy relation for 3-point function

Similar relation can be derived for the 3-point function (only left part is shown)

$$\left\langle \left(\Omega_1^{-}(u)\right)_{ij} |\mathcal{O}_1\rangle_L\,,\, \left(\Omega_2^{+|-}(u)\right)_{jk} |\mathcal{O}_2\rangle_L\,,\, \left(\Omega_3^{+}(u)\right)_{kl} |\mathcal{O}_3\rangle_L \right\rangle$$
$$= \delta_{il} f_{123}(u) \langle |\mathcal{O}_1\rangle_L\,,\, |\mathcal{O}_2\rangle_L\,,\, |\mathcal{O}_3\rangle_L \rangle$$

♦ This shows that **certain combination of 3-point functions with monodromy matrices inserted equals the one without any insertions.** This is pictorially represented as



♦ This should be interpreted as **the collection of Ward identities of all the (higher) conserved charges.** Hence **it represents the "integrability" of the entire 3-point function.**

♦ $\boxed{\textbf{In the semi-classical limit}}$ **of the spin-chain**,

i.e. $u = \ell u', \ell \to \infty$ **with** u' **fixed** ($\ell =$**length of the spin chain**),

the $\boxed{\textbf{shifts } \pm i/2 \textbf{ can be ignored}}$ and

$\boxed{\textbf{the relation reduces precisely to } \Omega_1\Omega_2\Omega_3 = 1}$,

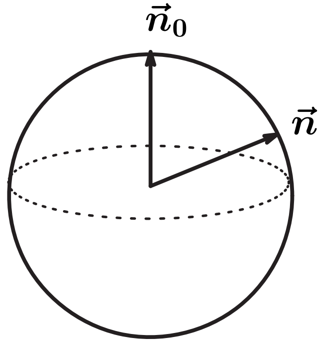
which is identical in form to the one in string theory.

6 New computation of semiclassical three-point functions at weak coupling using the Landau-Lifshitz formulation by strong coupling technology — Monodromy relation as the key structure —

Sketch of the basic logic

1. Take the **semiclassical limit of the quantum integrable $\mathrm{XXX}_{1/2}$ spin chain FIRST**, using the **coherent state representation of SU(2)** parametrized by a vector $\vec{n}$ on a unit sphere S^2

$$|\vec{n}\rangle = \exp\left(i\theta\frac{\vec{n}_0 \times \vec{n}}{|\vec{n}_0 \times \vec{n}|}\cdot \vec{S}\right)|\uparrow\rangle = \cos\frac{\theta}{2}|\uparrow\rangle - e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle$$



$\mathbf{XXX}_{1/2}$ spin chain

$\Rightarrow$ **Classically integrable** $\boxed{\text{Landau Lifshitz model}}$ in the interval $[0, L]$

$$S_{LL} = S_{\text{nearest neighbor}} + S_{\text{Wess-Zumino}}$$

$$S_{\text{nearest neibor}} = -\frac{\lambda}{32\pi^2}\int d\tau \int_0^L d\sigma \partial_\sigma \vec{n} \partial_\sigma \vec{n}$$

$$S_{\text{Wess-Zumino}} = \frac{1}{2}\int d\tau \int_0^L d\sigma \int_0^1 ds \vec{n} \cdot (\partial_\tau \vec{n} \times \partial_s \vec{n})$$

2. 3-point structure constant C_{123} can be obtained by Wick contractions

$$C_{123} = \frac{\sqrt{\ell_1 \ell_2 \ell_3}}{N_c} C_{123}^L \times C_{123}^R$$

$$C_{123}^L = \sum_{a,b,c} \langle |\mathcal{O}_{1a}^r\rangle_L, |\mathcal{O}_{2b}^l\rangle_L\rangle \langle |\mathcal{O}_{2b}^r\rangle_L, |\mathcal{O}_{3c}^l\rangle_L\rangle \langle |\mathcal{O}_{3c}^r\rangle_L, |\mathcal{O}_{1a}^l\rangle_L\rangle$$

similarly for C_{123}^R

It can be expressed as a **path-integral over the coherent state variables**

$$C^L_{123} = \int \mathcal{D}\vec{n}_1 \mathcal{D}\vec{n}_2 \mathcal{D}\vec{n}_3 \psi^L_1[\vec{n}_1]\psi^L_2[\vec{n}_2]\psi^L_3[\vec{n}_3]$$
$$\times \langle|\vec{n}^r_1\rangle, |\vec{n}^l_2\rangle\rangle\langle|\vec{n}^r_2\rangle, |\vec{n}^l_3\rangle\rangle\langle|\vec{n}^r_3\rangle, |\vec{n}^l_1\rangle\rangle$$

where $\quad \mathcal{D}\vec{n}_1 \equiv \mathcal{D}\vec{n}^r_1 \mathcal{D}\vec{n}^l_1 \,,\quad$ etc.

$\psi_i[\vec{n}_i] \equiv \langle \vec{n}_i|\mathcal{O}_i\rangle =$ coherent state wave function of the state $|\mathcal{O}_i\rangle$

♦ **In the semiclassical limit** we can make the **saddle point approximation**.

$$C^L_{123} \simeq \psi^L_1[\vec{n}^*_1]\psi^L_2[\vec{n}^*_2]\psi^L_3[\vec{n}^*_3]$$
$$\times \langle|\vec{n}^{*,r}_1\rangle, |\vec{n}^{*,l}_2\rangle\rangle\langle|\vec{n}^{*,r}_2\rangle, |\vec{n}^{*,l}_3\rangle\rangle\langle|\vec{n}^{*,r}_3\rangle, |\vec{n}^{*,l}_1\rangle\rangle$$

$\vec{n}^*_i =$ saddle point configuration

3. To compute C_{123},

study how it changes when we infinitesimally increase the filling fraction (action variable) $S^{(n)}_i$ for the operator $\mathcal{O}_n$ **by adding a small number of Bethe roots.**

One can show that this consideration gives one of the key relations

$$\boxed{\frac{\partial \ln C_{123}}{\partial S_i^{(n)}} = i\phi_i^{(n)}}$$

where $\phi_i^{(n)}$ **is the angle variable conjugate to $S_i^{(n)}$.**

$\Leftrightarrow$ **$\ln C_{123}$ plays the role of the generating function.**

4. In the semiclassical limit, the quantum monodromy relation for the $\mathrm{XXX}_{1/2}$ system reduces to

$$\boxed{(\star) \quad \Omega_1\Omega_2\Omega_3 = 1}$$

At this point, the situation is almost identical to the one at strong coupling and we can use the same logic as in that case.

5. Express angle variables (relative to the 2-point function case) in terms of the "Wronskians"

6. The monodromy relation $(\star)$ allows us to express the **product** of Wronskians in terms of the quasimomenta.

7. Use the Wiener-Hopf method to obtain individual Wronskian by specifying the analytic structure (*i.e.* poles and zeros) for the spectral parameter x on the spectral curve.

   One difference:

   ♦ **In contrast to the string case, we do not have the worldsheet and hence cannot use the smoothness of the worldsheet to infer the analytic structure**

8. We therefore develop **a new more general method**, **valid for both weak and strong couplings**, to determine the analyticity structure.

The logic consists of

the requirement of continuity and consistency between

- **the exact quantum property**

and

- **the semiclassical property**

under the addition of a small number of Bethe roots at the end of a cut: $|\psi\rangle \longrightarrow |\psi + \delta\psi\rangle$

- **Exact quantum treatment:**

$$(\star\star) \qquad \langle\psi|\psi + \delta\psi\rangle = 0$$

if both ψ and $\psi + \delta\psi$ are eigenstates of H ($i.e.$ on-shell)

- **Semiclassical treatment:** One can show that at the saddle point x_*, the angle variable corresponding to the added Bethe root is given by

$$\left.\frac{\partial \ln\langle\psi|\psi+\delta\psi\rangle}{\partial S_{x_*}}\right|_{\delta\psi\to 0} = i\phi_{x_*}$$

Now we impose $(\star\star)$ also for this semiclassical treatment.

Then this constrains the positions of the angle variables corresponding to the added Bethe roots (*i.e.* additional degrees of freedom) to be on the **first sheet of the Riemann surface**, **giving an important analyticity information.**

9. **These natural requirements determine the ananlyticity properties and fix the appropriate contours of the convolution integration** as we use the Wiener-Hopf decomposition formula to separate out the individual Wronskians from their product.

10. **These procedures directly give the correct compact semiclassical formulas for the 3-point functions (*i.e.* without going through the determinant formula) at weak coupling.**

$$\boxed{\begin{aligned}&\ln C_{123}^{(0)}\\&\quad\simeq \oint_{\mathcal{A}_u\cup\mathcal{A}_v}\frac{du}{2\pi}\,\mathrm{Li}_2(e^{ip_u+ip_v+iL_3/(2u)})+\oint_{\mathcal{A}_w}\frac{du}{2\pi}\,\mathrm{Li}_2(e^{ip_w+i(L_3-L_1)/(2u)})\\&\quad-\frac{1}{2}\oint_{\mathcal{A}_u}\frac{du}{2\pi}\,\mathrm{Li}_2(e^{2ip_u})-\frac{1}{2}\oint_{\mathcal{A}_v}\frac{du}{2\pi}\,\mathrm{Li}_2(e^{2ip_v})-\frac{1}{2}\oint_{\mathcal{A}_w}\frac{du}{2\pi}\,\mathrm{Li}_2(e^{2ip_w})\end{aligned}}$$

7 Summary and future problems

Summary

1. Introduction and motivation
2. A "lightening" review of the weak coupling computation of the 3-point functions in the SU(2) sector
3. A "short" review of the essential ingredients for strong coupling computation using string theory in AdS space
4. New weak coupling technology for more general 3-point functions in SYM: Wick-contraction as singlet projection
5. $\boxed{\text{Monodromy relation:}}$ A cognate integrability structure at weak and strong coupling
6. New computation of semiclassical three-point functions at weak coupling using Landau-Lifshitz formulation by strong coupling technology

Future problems

Among many possible directions for research, let us list a few important ones:

- **Derivation of monodromy relations at higher loop level** in SYM
- **Unify the different "pictures" behind monodromy relations**
  - **Worldsheet picture, at strong coupling**
  - **"Unitarity" and "crossing" for Lax matrix, at weak coupling**
- **Draw more refined consequences of the monodromy relations =Ward identities for higher charges**
- Can we derive the **first quantum corrections to the monodromy relations on the string theory side ?**
- **Relation to the hexagon approach for connecting weak and strong coupling ?**

Thank you for your kind attention !