

ODE/IM correspondence and its application to Supersymmetric Gauge Theories

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Introduction

- The ODE/IM correspondence is a relation between spectral analysis of **ordinary differential equation** (ODE), and the “functional relations” approach to 2d quantum **integrable model** (IM). [Dorey-Tateo 1998]
- This is an example of non-trivial correspondence between **classical** and **quantum** integrable models

- Dorey-Tateo (1998) studied the spectral determinant of the **second order differential equation**

$$\left(-\frac{d^2}{dx^2} + x^{2M} - E\right) \psi(x, E) = 0$$

and its relation to the A_{2M-1} -type **Thermodynamic Bethe-ansatz (TBA) equations**.

- Lukyanov-Zamolodchikov (2010) studied the relation for the **linear problem** associated with the **sinh-Gordon equation**

$$\varphi_{tt} - \varphi_{xx} + \sinh \varphi = 0$$

and the **quantum XXZ model**. In the conformal limit, this relation reduces to the above ODE/IM correspondence.

- This suggests existence of Lie algebraic structure behind this mysterious correspondence.

Motivation

- Understand mathematical structure and physical meaning of the ODE/IM correspondence
Making (complete) dictionaries
use this relation to understand the relation between integrable models
- Application to the AdS/CFT correspondence
Gluon scattering amplitudes [Alday-Maldacena](#), Form factors,
Correlation functions at strong coupling
- [Application to \$\mathcal{N} = 2\$ SUSY gauge theories](#)
Bethe-Gauge correspondence [\[Nekrasov-Shatashvili\]](#)

We will

- introduce some basic concepts of the ODE/IM correspondence
- and discuss its application to the Argyres-Douglas theory

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ODE/IM correspondence

[Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov]

- ODE

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^{2M} - E \right] y(x, E, \ell) = 0$$

$x \in \mathbb{C}$, E complex, ℓ : real, $M > 0$

- **large**, real positive x : we have two (divergent and convergent) solutions

$$y(x, E, \ell) \sim \frac{x^{-\frac{M}{2}}}{\sqrt{2i}} \exp\left(-\frac{x^{M+1}}{M+1}\right) \quad (x \rightarrow \infty)$$

subdominant (small) solution in the sector $|\arg x| < \frac{\pi}{2M+2}$

- **small** x asymptotics:

$$y(x, E, \ell) \sim x^{\ell+1}, x^{-\ell}$$

- The ODE is invariant under the rotation $x \rightarrow ax$, $E \rightarrow a^{2M}E$:

$$\left[\frac{1}{a^2} \left(-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} \right) + a^{2M}(x^{2M} - E) \right] y(ax, a^{2M}E, \ell) = 0$$

for $a = \omega = \exp(\frac{2\pi i}{2M+2})$.

- Symanzik rotation

$$y_k(x, E, \ell) = \omega^{\frac{k}{2}} y(\omega^{-k}x, \omega^{2k}E, \ell)$$

is also the solution of the ODE. ($k \in \mathbf{Z}$)

- y_k is subdominant in the sector $\mathcal{S}_k = \left\{ x \mid \left| \arg x - \frac{2k\pi}{2M+2} \right| < \frac{\pi}{2M+2} \right\}$

- $\{y_k, y_{k+1}\}$ forms a basis of the solutions.
- The Wronskian $W[f, g] := fg' - f'g$
 - ▶ If f, g are the solutions, then $W[f, g]$ is a constant, independent of x .
 - ▶ $W[f, g] = -W[g, f]$
 - ▶ $W_{k_1, k_2}(E, \ell) := W[y_{k_1}, y_{k_2}]$
- $W_{0,1} = 1$ (evaluated by asymptotic behaviour of y_0 and y_1)
- Periodicity (Symanzik rotation)

$$W_{k_1+1, k_2+1}(E, \ell) = W_{k_1, k_2}(\omega^2 E, \ell)$$

•

$$W_{k, k+1}(E, \ell) = 1$$

- $\{y_0, y_1\}$ are chosen as a fixed basis

$$y_k = -\frac{W_{1,k}}{W_{0,1}}y_0 + \frac{W_{0,k}}{W_{0,1}}y_1 = -T_{k-2}^{[k+1]}y_0 + T_{k-1}^{[k]}y_1$$

- T-function

$$T_s(E, \ell) := \left(\frac{W_{0,s+1}(E, \ell)}{W_{0,1}} \right)^{[-(s+1)]}, \quad s \in \mathbf{Z}$$

$$f(E, \ell)^{[m]} := f(\omega^m E, \ell)$$

- The Plücker relation

$$W[y_{k_1}, y_{k_2}]W[y_{k_3}, y_{k_4}] = W[y_{k_1}, y_{k_4}]W[y_{k_3}, y_{k_2}] + W[y_{k_3}, y_{k_1}]W[y_{k_4}, y_{k_2}]$$

for $(k_1, k_2, k_3, k_4) = (1, s+2, 0, s+1)$ leads to the functional relation called the **T-system**

$$T_s^{[+1]}T_s^{[-1]} = T_{s-1}T_{s+1} + 1 \quad s \in \mathbf{Z}$$

- The **Y-functions**: $Y_s = T_{s-1}T_{s+1}$ define the **Y-system**:

$$Y_s^{[+1]}Y_s^{[-1]} = (1 + Y_{s+1})(1 + Y_{s-1})$$

The boundary conditions for the T-system

- generic M and ℓ ($T_s \propto W_{0,s+1}$)

$$T_{-1} = 0, \quad T_0 = 1$$

T_s ($s \geq 0$) are non-zero.

- $\ell = 0$, $2M + 2 = n \geq 4$: integer
no singularity (monodromy) at $x = 0$

$$y_n(x) \propto y_0(e^{-2\pi i}x) = y_0(x)$$

$T_{n-1} = 0 \implies A_{n-2}$ -type T-system

- $\ell \neq 0$, $2M + 2 = n \geq 4$: integer
There is a **monodromy** around $z = 0$. $\implies D_{n-2}$ -type T-system

Baxter's T-Q relation

The basis of the ODE around $x = 0$

$$\psi_+(x, E, \ell) := x^{\ell+1} + \dots, \quad \psi_-(x, E, \ell) := x^{-\ell} + \dots$$

- Monodromy around the origin: $\psi_+(e^{2\pi i}x) = e^{2\pi i(\ell+1)}\psi_+(x)$
- Q-function: $Q_{\pm}(E, \ell) = W[y_0, \psi_{\pm}](E, \ell)$

$$W[y_k, \psi_{\pm}](E, \ell) = \omega^{\pm(\ell+\frac{1}{2})} W[y_0, \psi_{\pm}](\omega^{2k}E, \ell)$$

Baxter's T-Q relation: $-T_{-3}y_0 = y_{-1} + y_1$

$$(-T_{-3})Q_{\pm}(E, \ell) = \omega^{\mp(\ell+\frac{1}{2})}Q_{\pm}(\omega^{-2}E, \ell) + \omega^{\pm(\ell+\frac{1}{2})}Q_{\pm}(\omega^2E, \ell)$$

- $E = E_n$ such that $T(E_n, \ell) = 0$ (y_0, y_1 become linearly dependent)

$$\frac{Q_{\pm}(\omega^{-2}E_n, \ell)}{Q_{\pm}(\omega^2E_n, \ell)} = -\omega^{\pm(2\ell+1)}$$

Bethe ansatz equation of the **twisted six-vertex model**

Y-system and TBA equation

- [Zamolodchikov] The Y-system can be transformed into the non-linear integrable equations called the **Thermodynamic Bethe-ansatz (TBA) equation**. $\epsilon_a(\theta) = \log Y_a(\theta)$: pseudo-energy ($E = \exp(2M\theta/(M+1))$)

$$\epsilon_a(\theta) = m_a L e^\theta - \sum_b \int_{-\infty}^{+\infty} \phi_{ab}(\theta - \theta') \log(1 + e^{-\epsilon_b(\theta')}) d\theta'$$

- free energy

$$F(L) = -\frac{1}{4\pi} \sum_a \int_{-\infty}^{+\infty} m_a e^\theta \log(1 + e^{-\epsilon_a(\theta)}) d\theta = -\frac{\pi c_{eff}}{6L}$$

- The UV effective central charge becomes

$$c_{eff}^{UV} = 1 - \frac{6 \left(\ell + \frac{1}{2}\right)^2}{M+1}$$

which agrees with the one of the **twisted six-vertex model**.

Affine Toda Field Equations and ODE/IM correspondence

the Lie algebraic structure behind the ODE/IM correspondence

Lie algebra notations:

\mathfrak{g} : a simple Lie algebra of rank r :

- $\{H^i, E_\alpha\}$: basis of \mathfrak{g} , $i = 1, \dots, r$, α : roots ($\alpha^\vee = \frac{2\alpha}{\alpha^2}$: coroots)
- $\alpha_1, \dots, \alpha_r$: the simple roots of \mathfrak{g}
- $\alpha_0 = -\theta$ (θ : the highest root) $\theta = \sum_{i=1}^r n_i \alpha_i$
 n_i : Coxeter labels, $n_0 \equiv 1$
- the Coxeter number: $h = \sum_{i=0}^r n_i$
- the fundamental weights: ω_i ($i = 1, \dots, r$), $\omega_i \cdot \alpha_j^\vee = \delta_{ij}$
- the Weyl vector $\rho = \sum_{i=1}^r \omega_i$, $\rho \cdot \alpha_i^\vee = 1$

affine Lie algebra

affine Lie algebra: $\hat{\mathfrak{g}} = X_r^{(s)}: \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ simple roots

- $X_r^{(1)}$: untwisted ($X = ABCDEFG$)
- $A_{2r-1}^{(2)}, A_{2r}^{(2)}, D_{r+1}^{(2)}, E_6^{(2)}, D_4^{(3)}$: twisted

- **Langlands (GNO) dual:**

$\hat{\mathfrak{g}}$: simple roots $\alpha_i \iff \hat{\mathfrak{g}}^\vee: \alpha_i^\vee$ simple coroots

- ▶ $\hat{\mathfrak{g}}^\vee = \hat{\mathfrak{g}}$ for simply-laced $\hat{\mathfrak{g}} = A_r^{(1)}, D_r^{(1)}, E_r^{(1)}$
- ▶ $\hat{\mathfrak{g}}^\vee = X_r^{(s)}$ for non-simply laced $\hat{\mathfrak{g}}$ (twisted affine Lie algebra)
 - $(B_r^{(1)})^\vee = A_{2r-1}^{(2)}$
 - $(C_r^{(1)})^\vee = D_{r+1}^{(2)}$
 - $(F_4^{(1)})^\vee = E_6^{(2)}$
 - $(G_2^{(1)})^\vee = D_4^{(3)}$
 - $(A_{2r}^{(2)})^\vee = A_{2r}^{(2)}$

affine Toda field equations

two-dimensional affine Toda field Theory based on $\hat{\mathfrak{g}}$:

r -component scalar fields: $\phi(z, \bar{z}) = (\phi^1, \dots, \phi^r)$

complex coordinates: $z = \frac{1}{2}(x^0 + ix^1)$, $\bar{z} = \frac{1}{2}(x^0 - ix^1)$ ($z = \rho e^{i\theta}$)

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \cdot \partial_\mu \phi - \left(\frac{m}{\beta} \right)^2 \sum_{i=0}^r n_i [\exp(\beta \alpha_i \cdot \phi) - 1],$$

affine Toda field equation:

$$\partial_z \partial_{\bar{z}} \phi + \left(\frac{m^2}{\beta} \right) \sum_{i=0}^r n_i \alpha_i \exp(\beta \alpha_i \phi) = 0.$$

- without the potential term $e^{\beta \alpha_0 \phi}$, the theory is conformally invariant (e.g. Liouville theory)
- with the potential term $e^{\beta \alpha_0 \phi}$, it becomes massive theory. The equation of motion changes under the conformal transformation.

modified affine Toda field equation

conformal transformation (ρ^\vee : co-Weyl vector)

$$z \rightarrow \tilde{z} = f(z), \quad \phi \rightarrow \tilde{\phi} = \phi - \frac{1}{\beta} \rho^\vee \log(\partial f \bar{\partial} \bar{f}),$$

modified affine Toda equations:

$$\partial \bar{\partial} \phi + \left(\frac{m^2}{\beta} \right) \left[\sum_{i=1}^r n_i \alpha_i \exp(\beta \alpha_i \phi) + p(z) \bar{p}(\bar{z}) n_0 \alpha_0 \exp(\beta \alpha_0 \phi) \right] = 0,$$

$$p(z) = (\partial f)^h, \quad \bar{p}(\bar{z}) = (\bar{\partial} \bar{f})^h.$$

Lax formalism

- The modified affine Toda equation can be expressed as the **zero-curvature condition**: $[\partial + A, \bar{\partial} + \bar{A}] = 0$

$$A = \frac{\beta}{2} \partial \phi \cdot H + m e^{\lambda} \left\{ \sum_{i=1}^r \sqrt{n_i^{\vee}} E_{\alpha_i} e^{\frac{\beta}{2} \alpha_i \phi} + p(z) \sqrt{n_0^{\vee}} E_{\alpha_0} e^{\frac{\beta}{2} \alpha_0 \phi} \right\},$$
$$\bar{A} = -\frac{\beta}{2} \bar{\partial} \phi \cdot H - m e^{-\lambda} \left\{ \sum_{i=1}^r \sqrt{n_i^{\vee}} E_{-\alpha_i} e^{\frac{\beta}{2} \alpha_i \phi} + \bar{p}(\bar{z}) \sqrt{n_0^{\vee}} E_{-\alpha_0} e^{\frac{\beta}{2} \alpha_0 \phi} \right\}$$

λ : **spectral parameter**

- **linear problem**: $(\partial + A)\Psi = 0$ and $(\bar{\partial} + \bar{A})\Psi = 0$.
- **gauge transformation**: $\Psi \rightarrow U\Psi$, $A \rightarrow UAU^{-1} + U\partial U^{-1}$

Example: modified sinh-Gordon equation: $A_1^{(1)}$

modified Sinh-Gordon equation [Lukyanov-Zamolodchikov 1003.5333]

$$\partial_z \partial_{\bar{z}} \phi - e^{2\phi} + p(z) \bar{p}(\bar{z}) e^{-2\phi} = 0, \quad p(z) = z^{2M} - s^{2M}$$

- zero curvature condition $[\partial + A, \bar{\partial} + \bar{A}] = 0$

$$A = \begin{pmatrix} \frac{1}{2} \partial \phi & -e^\lambda e^\phi \\ p(z) e^\lambda e^\phi & -\frac{1}{2} \partial \phi \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} -\frac{1}{2} \bar{\partial} \phi & -e^{-\lambda} e^\phi \\ \bar{p}(\bar{z}) e^{-\lambda} e^\phi & \frac{1}{2} \bar{\partial} \phi \end{pmatrix}$$

- asymptotic behavior of $\phi(z, \bar{z})$ at $\rho \rightarrow 0, \infty$ ($z = \rho e^{i\theta}$)

- ▶ $\phi(\rho, \theta) \rightarrow M \log \rho$ ($\rho \rightarrow \infty$)
- ▶ $\phi(\rho, \theta) \rightarrow \ell \log \rho$ ($\rho \rightarrow 0$)

We can introduce a new parameter ℓ for the boundary condition at $\rho = 0$.

linear problem and its asymptotic solutions

- linear problem $(\partial + A)\Psi = (\bar{\partial} + \bar{A})\Psi = 0$
- linear problem is invariant under
Symanzik rotation Ω : $\theta \rightarrow \theta + \frac{\pi}{M}$, $\lambda \rightarrow \lambda - \frac{i\pi}{M}$
- At $\rho \rightarrow \infty$, the subdominant solution is

$$\Psi \sim \begin{pmatrix} e^{\frac{iM\theta}{2}} \\ e^{-\frac{iM\theta}{2}} \end{pmatrix} \exp\left(-\frac{2\rho^{M+1}}{M+1} \cosh(\lambda + i(M+1)\theta)\right)$$

- $\rho \rightarrow 0$ basis $\Psi_+(\rho, \theta|\lambda) \rightarrow \begin{pmatrix} 0 \\ e^{(i\theta+\lambda)\ell} \end{pmatrix}$, $\Psi_-(\rho, \theta|\lambda) \rightarrow \begin{pmatrix} e^{(i\theta+\lambda)\ell} \\ 0 \end{pmatrix}$
-

$$\Psi = Q_-(\lambda)\Psi_+ + Q_+(\lambda)\Psi_-$$

$Q_{\pm}(\lambda)$ defines the **Q-function** satisfying the Bethe ansatz equation.

- T-functions and the Y-functions are also defined. They satisfy the D -type Y-system.

From MShG to ODE

- Take the **light-cone limit** $\bar{z} \rightarrow 0$. The linear system reduced to a holomorphic differential equation. $(\partial + A_z)\Psi = 0$.
- Under the gauge transformation by $U = \text{diag}(e^\phi, e^{-\phi})$, it becomes

$$(\partial_z + \tilde{A}_z) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad \tilde{A}_z = \begin{pmatrix} \partial\phi & e^\lambda \\ p(z)e^\lambda & -\partial\phi \end{pmatrix}$$

- linear system \implies ODE (**Miura transformation**)

$$\left[(\partial_z - \partial_z \phi)(\partial_z + \partial_z \phi) - e^{2\lambda} p(z) \right] \psi_1(z) = 0$$

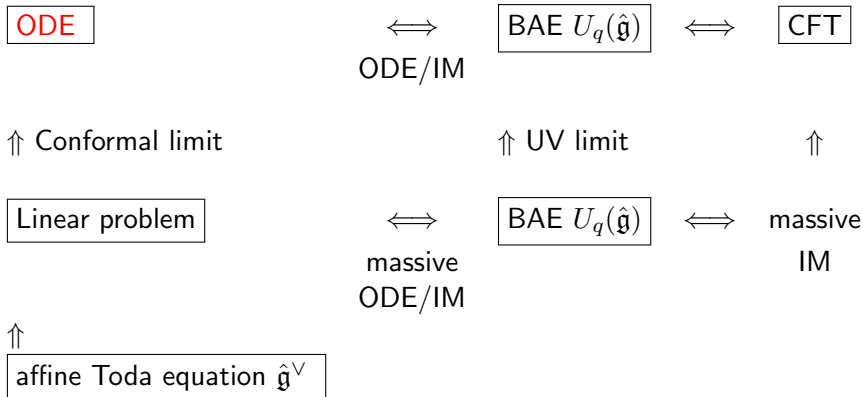
- **conformal limit:**

$$z \rightarrow 0, \lambda \rightarrow \infty \text{ with fixed } x = ze^{\frac{\lambda}{M+1}}, E = s^{2M} e^{\frac{2\lambda M}{1+M}}, \phi \sim \ell \log x$$

$$\left[-(\partial_x - \frac{\ell}{x})(\partial_x + \frac{\ell}{x}) + x^{2M} \right] \psi = \left[-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + x^{2M} \right] \psi = E\psi$$

This is the ODE of **[Dorey-Tateo, BLZ]**

ODE/IM correspondence for $\hat{\mathfrak{g}}^\vee$ modified affine Toda field equations



Langlands Duality: [Masoero-Raimondo-Valeri, Kl-Locke] The modified affine Toda equation for the Langlands dual $\hat{\mathfrak{g}}^\vee$ corresponds to the \mathfrak{g} -type Bethe ansatz equation [Reshetikhin-Wiegmann, Kuniba-Suzuki].

ODE for affine Lie algebras

[KI-Locke,1312.6759] (ABCD type:
[Dorey-Dunning-Masoero-Suzuki-Tateo 2006])

$A_r^{(1)}$	$D(\mathbf{h})\psi = (-me^\lambda)^h p(z)\psi$
$D_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial^{-1}D(\mathbf{h})\psi = 2^{r-1}(me^\lambda)^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$B_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^r(me^\lambda)^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$A_{2r-1}^{(2)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = -2^{r-1}(me^\lambda)^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$C_r^{(1)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = (me^\lambda)^h p(z)\psi$
$D_{r+1}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r+1}(me^\lambda)^{2h} p(z)\partial^{-1}p(z)\psi$
$A_{2r}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = -2^r \sqrt{2}(me^\lambda)^h p(z)\psi$
$G_2^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 8(me^\lambda)^h \sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$D_4^{(3)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi + (\omega+1)2\sqrt{3}(me^\lambda)^4 D(\mathbf{h}^\dagger)p(z) \\ -(\omega+1)2\sqrt{3}(me^\lambda)^4 pD(\mathbf{h}) - 8\sqrt{3}\omega(me^\lambda)^3 D(-h_1)\sqrt{p}\partial\sqrt{p}D(h_1) \\ +(\omega-1)^3 12(me^\lambda)^8 p\partial^{-1}p\}\psi = 0$

$D(\mathbf{h}) = D(h_r) \cdots D(h_1)$, $D(\mathbf{h}^\dagger) = D(-h_1) \cdots D(-h_r)$ for $\mathbf{h} = (h_r, \dots, h_1)$ set of weight vectors for the fundamental representation.

ODE/IM correspondence and the Argyres-Douglas Theory

Argyres-Douglas Theory

[Argyres-Douglas, Argyres-Plesser-Seiberg-Witten, Eguchi-Hori-KI-Yang]

- strongly coupled $N=2$ SCFT in four dimensions
- mutually non-local monopole and dyons are both massless
- no microscopic Lagrangian description
- The curve of the AD theory is realized by degeneration of the SW curve
 $SU(3)$ SW curve: genus two Riemann surface \Rightarrow small + big torus

$G = ADE$ type SW theory:

- The SW curve is the spectral curve of periodic affine Toda lattice based on $(G^{(1)})^\vee$. [Gorsky et al. , Martinec-Warner]

$$z + \frac{\mu^2}{z} = W_G(x, u_1, \dots, u_r), \quad \lambda_{SW} = x \frac{dz}{z}$$

$$W_{A_r} = x^{r+1} - u_2 x^{r-2} - \dots - u_{r+1}$$

- The AD point of the theory is realized at

$$u_1 = \dots = u_{r-1} = 0, \quad u_r = \pm 2\mu$$

- rescaling the variables (q_i : exponents of G), $\epsilon \rightarrow 0$

$$u_i = \epsilon^{q_i} \rho_i (i = 1, \dots, r-1), \quad u_r = +2\mu + \epsilon^{q_r}$$

The SW curve of the AD theory (also rescaling of x, z)

$$\xi^2 = W(x, \rho_1, \dots, \rho_{r-1}, 1), \quad \lambda_{SW} = \xi dx$$

$$\xi = \sqrt{z} + \frac{\sqrt{\mu}}{\sqrt{z}}$$

Classification of AD theories

- periodic Toda lattice (A_1, G) :
The SW curve : $x^2 = W_G(z, u_i)$
- hypersurface singularity in the type IIB setup.
 - ▶ (G, G') AD theory [Cecotti-Neitzke-Vafa 1006.3435]

$$f_G(x_1, x_2) + f_{G'}(x_3, x_4) = 0$$

$$f_{A_r}(x, y) = x^2 + y^{r+1}$$

- ▶ Duality: $(G, G') \sim (G', G)$, $(A_1, E_6) \sim (A_2, A_3)$, $(A_1, E_8) \sim (A_2, A_4)$
- Hitchin system with irregular singularity $\mathfrak{g}^{(b)}[k]$ [Xie, Wang-Xie]
For $b = h$ (Coxeter number)

$$\mathfrak{g}^{(h)}[k] = (\mathfrak{g}, A_{k-1})$$

Some properties of the AD theories

- [Cecotti-del Zotto-Vafa, 1006.4708] [Cecotti-del Zotto 1403.7613]
From the BPS spectrum analysis of the AD theory, one finds the the ADE type Y-system [Zamoldchikov] appears.
- 2d/4d correspondence
 - ▶ trace of 4d quantum monodormy operator = character of certain module of 2d chiral algebra [Cordova-Shao 1506.00265]
Schur index =vacuum character of 2d chiral algebra
 - ▶ N=2 4d central charge \leftrightarrow central charge of 2d chiral algebra [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees 1312.5344]

$$c_{4d} = -\frac{1}{12}c_{2d}$$

- ▶ [Xie 1204.2270, Wang-Xie 1509.00847, Xie-Yan-Yau, 1604.02155] For the AD theory $\mathfrak{g}^b[k]$ the corresponding 2d theory is

$$\mathcal{A} = \frac{\mathfrak{g}_\ell \times \mathfrak{g}_1}{\mathfrak{g}_{\ell+1}}, \quad \ell = -\frac{kh - b}{k}$$

which is the $W_{\mathfrak{g}}(p', p) = W_{\mathfrak{g}}(b + k, b)$ minimal model.

AD theories in the NS limit of the Ω -background

- The quantum spectral curve is the SW curve for the $\mathcal{N} = 2$ theory in the Nekrasov-Sahashvili limit ($\epsilon_2 \rightarrow 0$) of the Ω -background.

[Mironov-Morozov, ...]

$$(-\epsilon^2 \partial_x^2 - u + z + \frac{1}{z})\psi(x) = 0, \quad z = e^{ix}, \quad \lambda = x \frac{dz}{z}$$

- It is also obtained as a differential equation arising from the insertion of the degenerate primary field (or the surface operator) in the correlators. (AGT correspondence)

[Alday-Gaiotto-Gukov-Tachikawa-Verlinde]

- quantum spectral curve for AD theories

$$\xi^2 = W_G(x, \rho), \quad \xi \rightarrow \epsilon \partial_x$$

- quantization: the SW differential $\lambda = \xi dx$ defines the symplectic structure

$$d\lambda = d\xi \wedge dx$$

$$\{\xi, x\} = 1 \quad \implies [\hat{\xi}, \hat{x}] = i\epsilon$$

ODE/IM correspondence and AD theory

- Gaiotto [1403.6137] has shown that the TBA system for AD_2 model can be obtained from the oper

$$x^2 = z^2 + 2a, \implies (-\epsilon^2 \partial_z^2 + z^2 + 2a)\psi(z) = 0$$

- The light-cone limit of the linear problem associated with the modified affine Toda field equation for $\hat{\mathfrak{g}} = ADE$ gives the first order linear-differential equation:

$$[\partial_z + A_z]\psi(z) = 0$$

We propose that this can be regarded as the quantum spectral curve of the AD theory of (A_1, \mathfrak{g}) type for $p(z) = z^2 - E$.

For $G = A_r$ -type, we have the ODE

$$(\partial^{r+1} - z^2 + E)\psi_1(z) = 0$$

- From this ODE we can derive the T and Y-system of the A_r -type[CDZV]. One can compute the effective central charge from the TBA equations[CS].
- The BAE leads to the NLIE equations [Dorey-Tateo]. We can compute the central charge.
- Spectral duality $(A_1, A_r) \leftrightarrow (A_r, A_1)$ can be seen from the Fourier transformation.

$$((ik)^{r+1} - \frac{\partial^2}{\partial k^2} + E)\tilde{\psi}_1(k) = 0$$

- Dorey-Dunning-Tateo [0712.2010] argued that the ODE

$$(-\partial^2 + (x^{2M} - E))\psi(x, E, \ell) = 0$$

corresponds to the non-unitary minimal model $M_{2,2M+2}$ with central charge

$$c = 1 - \frac{6M^2}{M+1}$$

perturbed by the operator $\phi_{1,M}$ with conformal dimension

$$\Delta_{1,M} = \frac{1 - 4M^2}{16(M+1)}$$

Note that for general ℓ , the effective central charge becomes

$$c_{eff} = c - 24\Delta = 1 - \frac{6(\ell + \frac{1}{2})^2}{M+1}$$

- The $M_{2,M}$ is realized by the fractional coset CFT:

$$\frac{su(2)_L \times su(2)_1}{su(2)_{L+1}}, \quad L = \frac{1}{M} - 2$$

For the ODE and the BAE, Dorey et al. suggest that the ODE for $\mathfrak{g} = ABCD$ with

$$p(x) = x^{h^\vee M} - E$$

corresponds to the coset model

$$\frac{\hat{\mathfrak{g}}_L \times \hat{\mathfrak{g}}_1}{\hat{\mathfrak{g}}_{L+1}} = W_{\mathfrak{g}}(h^\vee M + h^\vee, h^\vee), \quad L = \frac{1}{M} - h^\vee$$

perturbed by the operator.

The central charges agree with those predicted by the 2d/4d correspondence. [Cordova-Shao, Xie-Yan-Yau 1604.02155]

Conclusions and Outlook

- The ODE/IM correspondence between affine Toda field equations $\hat{\mathfrak{g}}^\vee$ and the $\hat{\mathfrak{g}}$ integrable models. (Langlands duality)
- The ODE/IM correspondence describes the relation between 4d SCFT (quantum spectral curve) and the related 2d conformal field theories (non-unitary W-minimal model).
- The AD theory with flavor symmetry is related to the affine Lie algebra with fractional level.
- D_r and E_r spectral curve and the AD theory **work in progress**
- Baxter's T-Q relation can be regarded as the Bohr-Sommerfeld quantization condition and its is useful to compute the non-perturbative correction to the deformed prepotential at strong coupling from the integrable model point of view