Witten Index for M-theory
Old and New

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Korea Institute for Advanced Study

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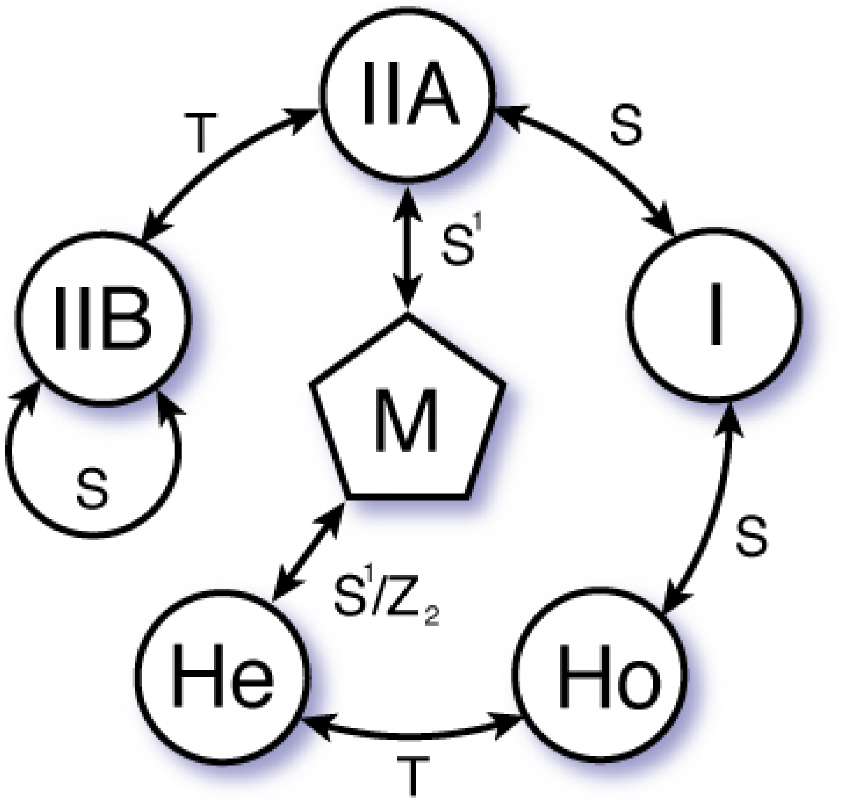
M / IIA / IA dualities and D-Particle Quantum Mechanics



Witten 1995
Horava-Witten 1995

M theory on a circle

M on $\mathbf{S}^1 \times \mathcal{M}_{9+1}$ = IIA on $\mathcal{M}_{9+1}$

IIA theory must remember
this M-theory origin

by forming an infinite tower of
multi D-particle bound states
moving freely on $\mathcal{M}_{9+1}$

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Witten 1995

P.Y. 1997
Sethi+Stern 1997
Gutperle+Green 1997
Moore+Nekrasov+Shatashvili 1998

⋮

S.J.Lee+P.Y. 2016

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=16} = 1$$

i.e., there has to be a unique threshold bound state
in maximally supersymmetric *SU(N)* quantum mechanics

A_0

$X_{i=1,2,\ldots,9}$

$\Psi_{\alpha=1,2,\ldots,16}$

$\in \quad \mathrm{adj}(U(N))$

center
of mass
motion

$$U(N) = U(1) \times SU(N)$$

mutual
interaction

$$\mathcal{I}^{SU(N)}_{N=16} = 1$$

P.Y. 1997
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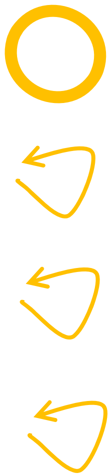
⋮

S.J.Lee+P.Y. 2016

which is a particular case of $\mathcal{N} = 4$ quiver quantum mechanics



A_0 $\quad X_{i=1,2,3}$

$\Phi_1 = X_4 + iX_5$

$\Phi_2 = X_6 + iX_7$

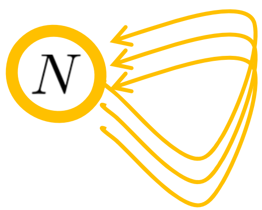
$\Phi_3 = X_8 + iX_9$

center of mass motion

$$U(N) = U(1) \times SU(N)$$

mutual interaction

$$\mathcal{I}_{\mathcal{N}=4}^{SU(N);\Phi_{1,2,3}} = 1$$



M theory on a circle

M on $\mathbf{S}^1 \times \mathcal{M}_{9+1}/\Gamma$ = IIA on $\mathcal{M}_{9+1}/\Gamma$

IIA theory must remember
this M-theory origin

by forming an infinite tower of
multi D-particle bound states
along fixed points of the orbifold

M theory on a circle

M on $\mathbf{S}^1 \times \mathcal{R}^{8+1} \times \mathcal{R}^1/Z_2$ = IIA on $\mathcal{R}^{8+1} \times \mathcal{R}^1/Z_2$

IIA theory must remember
this M-theory origin

by forming an infinite tower of
multi D-particle bound states
along the O8 orientifold
with eight D8-branes

Horava+Witten 1995
Kachru+Silverstein 1997

$\vdots$

Hwang+Kim+Kim+Park 2014

$\vdots$

M theory on a circle

M on $\mathbf{S}^1 \times \mathcal{R}^{4+1} \times \mathcal{R}^5/Z_2$ = IIA on $\mathcal{R}^{4+1} \times \mathcal{R}^5/Z_2$

IIA theory must remember
this M-theory origin

by forming an infinite tower of
multi D-particle bound states
along the O4 orientifold
with some D4-branes

Aharony+Berkooz+Kachru
+Seiberg+Silverstein 1998

⋮

Hwang+Kim+Kim 2016

⋮

M theory on a circle

M on $\mathbf{S}^1 \times \mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$ = IIA on $\mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$

IIA theory must remember
this M-theory origin

by forming an infinite tower of
multi D-particle bound states
localized at the orientifold point

Dasgupta+Mukhi 1995
Kol+Hanany+Rajaraman 1999
Kac+Smilga 1999

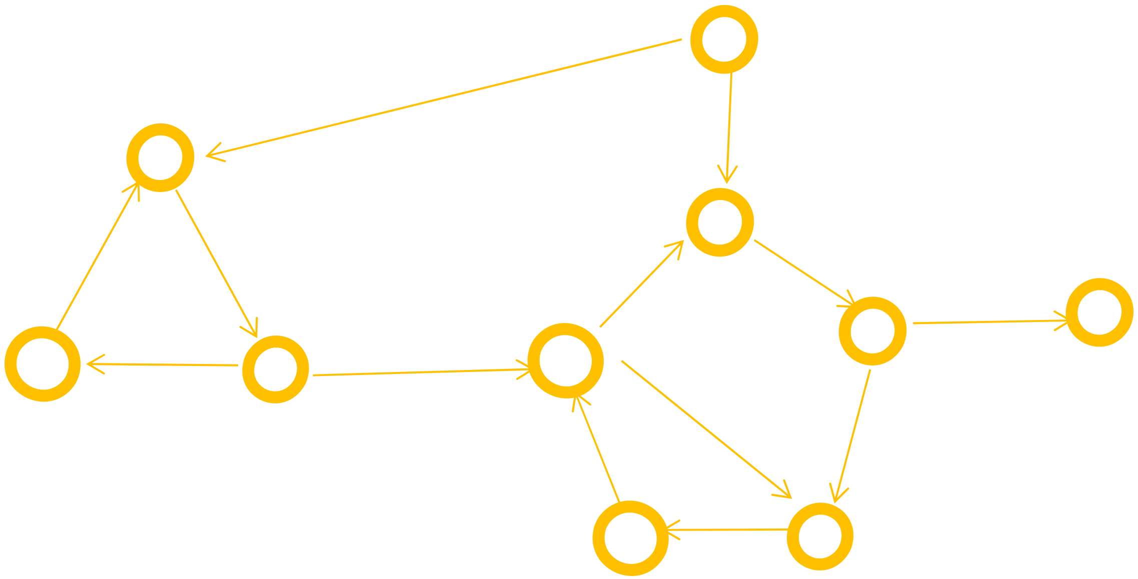
⋮

S.J.Lee + P.Y. 2016 & to appear

numerous D-brane counting problems, including these,
call for general Witten index program for
supersymmetric gauged theories,

which saw a reasonable completion only last few years.

such as for quiver gauge theories, relevant for
calabi-yau compactification, wall-crossing,
and black hole microstate counting



Witten index as a twisted partition function

(Witten 1982)

prototype : supersymmetric harmonic oscillators

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$$

$$\{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1 \qquad\qquad b^2 = 0 = (b^\dagger)^2$$

$$H = \hbar w \left[\left(a^\dagger a + aa^\dagger \right)/2 + \left(b^\dagger b - bb^\dagger \right)/2 \right] = \hbar w \left(a^\dagger a + 1/2 \right) + \hbar w \left(b^\dagger b - 1/2 \right)$$

$$= H_B + H_F$$

partition function vs.

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$$

$$\{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1$$

$$H = \hbar w \left[\left(a^\dagger a + a a^\dagger \right) /2 + \left(b^\dagger b - b b^\dagger \right) /2 \right] = \hbar w \left(a^\dagger a + 1/2 \right) + \hbar w \left(b^\dagger b - 1/2 \right)$$

$$= H_B + H_F$$

$$Z = \mathrm{tr} \left[e^{-\beta H} \right] = \mathrm{tr}_B e^{-\beta H_B} \times \mathrm{tr}_F e^{-\beta H_F}$$

$$= (1/2 sinh(\beta \hbar w/2)) \times 2 \cosh(\beta \hbar w/2)$$

$$= 1/\tanh(\beta \hbar w/2)$$

twisted partition function

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$$

$$\{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1$$

$$H = \hbar w \left[\left(a^\dagger a + aa^\dagger\right)/2 + \left(b^\dagger b - bb^\dagger\right)/2 \right] = \hbar w \left(a^\dagger a + 1/2\right) + \hbar w \left(b^\dagger b - 1/2\right)$$

$$= H_B + H_F$$

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{tr} e^{-\beta H_B} \times \mathrm{tr}(-1)^F e^{-\beta H_F}$$

$$= (1/2sinh(\beta\hbar w/2)) \times 2\sinh(\beta\hbar w/2)$$

$$= 1$$

why an integer ?

$$H = \hbar w Q^2 \qquad (-1)^F Q + Q(-1)^F = 0$$

$(-1)^F = 1$ bosonic

fermionic $(-1)^F = -1$

$$(a^\dagger)^{n+1} |0\rangle_B \otimes |0\rangle_F$$

$$|0\rangle_B \otimes |0\rangle_F$$

$Q = a^\dagger b + b^\dagger a$ →

← $Q = a^\dagger b + b^\dagger a$

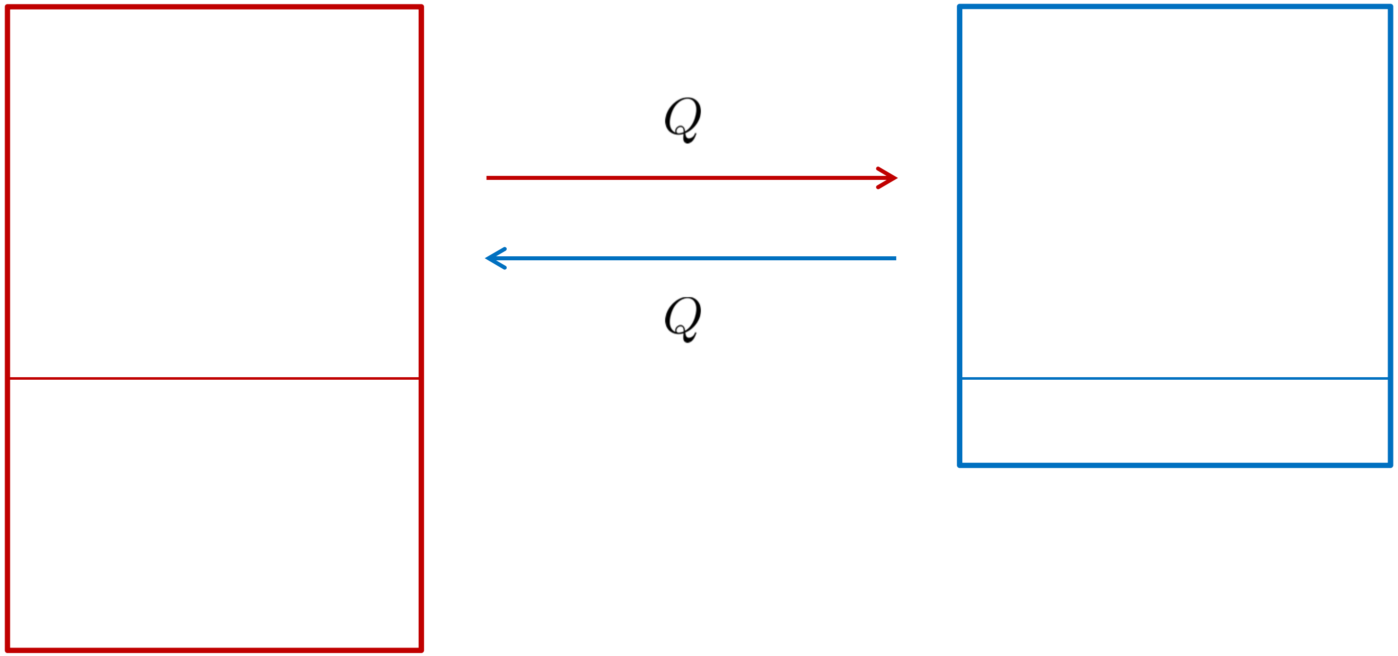
$$(a^\dagger)^{n} |0\rangle_B \otimes b^\dagger |0\rangle_F$$

a twisted partition function is enumerative because

$$H = \hbar w Q^2 \qquad (-1)^F Q + Q(-1)^F = 0$$

$(-1)^F = 1$ bosonic

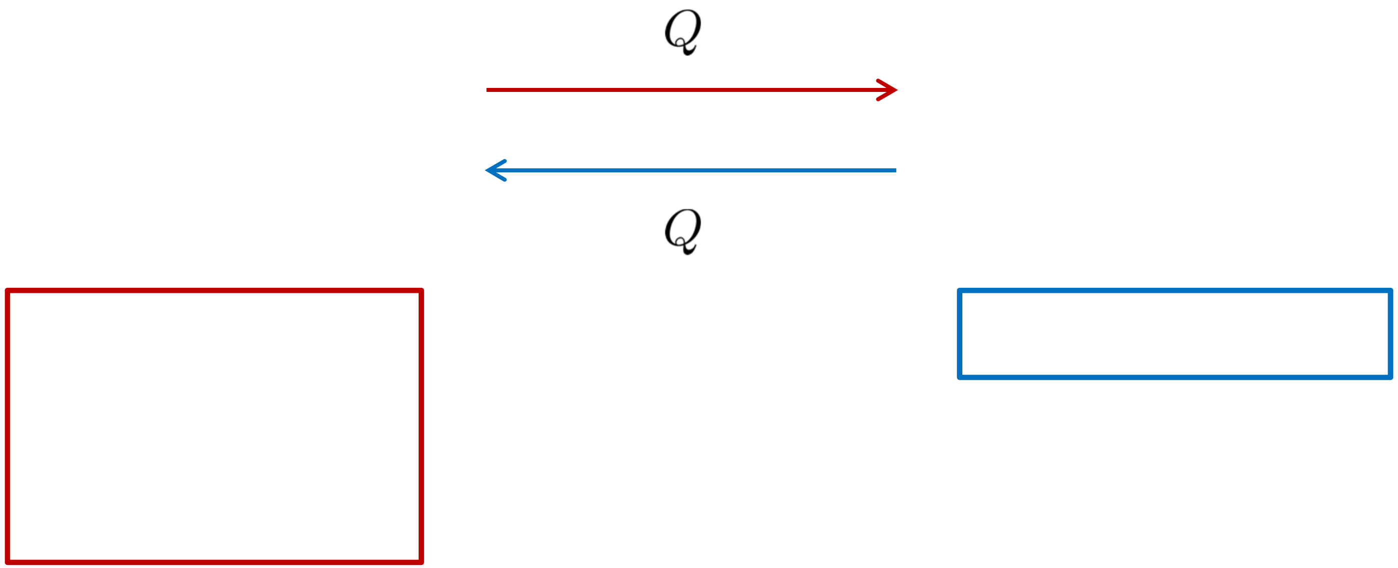
fermionic $(-1)^F = -1$



and defines the Witten index

$$Z_{twisted} = \mathrm{tr}_{\mathrm{bosonic}} 1 - \mathrm{tr}_{\mathrm{fermionic}} 1 \quad = \quad \mathrm{Index}(Q)$$

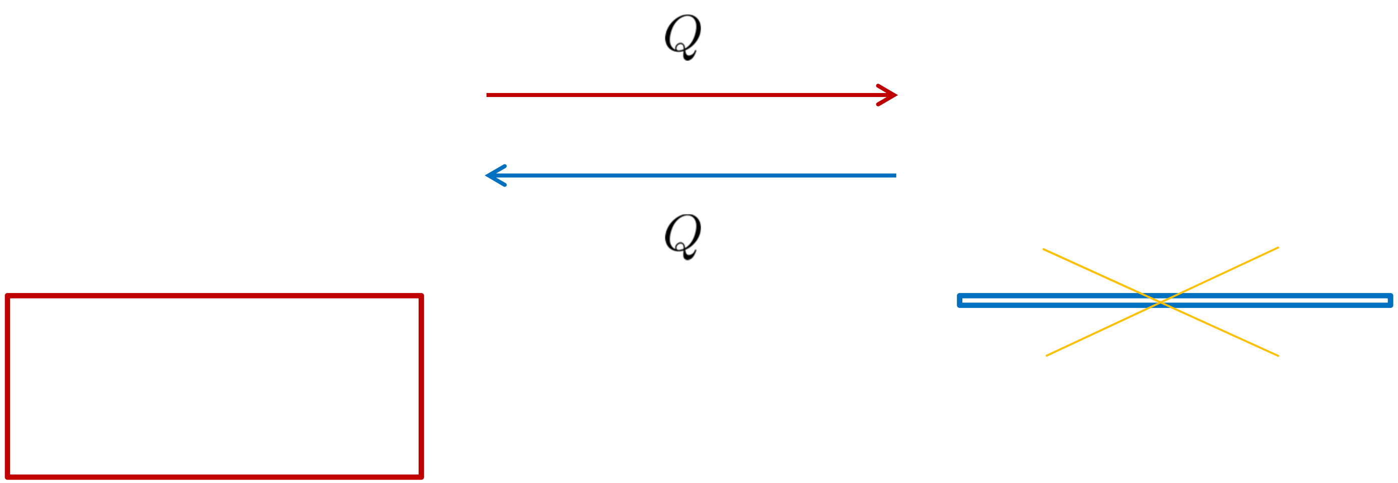
$(-1)^F = 1$ $\qquad\qquad$ $(-1)^F = -1$



which sometimes counts the entire ground state sector

$$Z_{twisted} = \mathrm{tr}_{\text{bosonic}} 1 - \cancel{\mathrm{tr}_{\text{fermionic}} 1} \quad = \quad \mathrm{Index}(Q)$$

$(-1)^F = 1$ $\qquad\qquad\qquad\qquad$ $(-1)^F = -1$



twisted partition function as a path integral

(Alvarez-Gaume 1983)

can one compute such things as a path integral?

$$H_B \;\leftarrow\; \quad L_B = \frac{1}{2}\left(\dot{x}^2 - w^2 x^2\right)$$

$$H_F \;\leftarrow\; \quad L_F = i\psi^\dagger\dot{\psi} - w\psi^\dagger\psi$$

can one compute such things as a path integral?

$$H_B \ \leftarrow \quad L_B = \tfrac{1}{2}\left(\dot{x}^2 - w^2 x^2\right)$$

$$\mathrm{tr}\left[e^{-\beta H_B}\right] = \int [dx]_{\mathrm{periodic\,BC}}\, e^{-\int_0^\beta L_B^{Euclidean} d\tau}$$

$$= 1/\sqrt{Det\left(-\partial_\tau^2 + w^2\right)}$$

$$= 1/\left(\prod_{n\in\mathbf{Z}}\left((2\pi n/\beta)^2 + w^2\right)\right)^{1/2}$$

$$= 1/w \times 1/\prod_{n>0}\left((2\pi n/\beta)^2 + w^2\right)$$

can one compute such things as a path integral?

$$H_F \;\leftarrow\quad L_F = i\psi^\dagger \dot{\psi} - w\psi^\dagger \psi$$

$$\int [d\psi^\dagger d\psi]_{\text{which BC ?}} \, e^{-\int_0^\beta L_F^{Euclidean} d\tau}$$

$$= Det\,(\partial_\tau + w)_{\text{which BC ?}}$$

$$= \begin{cases} \omega \times \prod_{n>0} \left((2\pi n/\beta)^2 + w^2\right) & \text{periodic BC} \\ \prod_{n\geq 0} \left((2\pi(n+1/2)/\beta)^2 + w^2\right) & \text{antiperiodic BC} \end{cases}$$

can one compute such things as a path integral?

$$H_F \;\leftarrow\quad L_F = i\psi^\dagger\dot{\psi} - w\psi^\dagger\psi$$

$$\int [d\psi^\dagger d\psi]_{\text{which BC ?}}\, e^{-\int_0^\beta L_F^{Euclidean} d\tau}$$

$$= Det\,(\partial_\tau + w)_{\text{which BC ?}}$$

$$= \left\{ \begin{array}{lr} 2\sinh(\beta w/2) & \text{periodic BC} \\ 2\cosh(\beta w/2) & \text{antiperiodic BC} \end{array} \right\} = \left\{ \begin{array}{l} \mathrm{tr}(-1)^F e^{-\beta H_F} \\ \mathrm{tr} e^{-\beta H_F} \end{array} \right\}$$

therefore,

$$Z_{twisted} = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{tr} e^{-\beta H_B} \times \mathrm{tr}(-1)^F e^{-\beta H_F}$$

$$= \int [dx\, d\psi^\dagger d\psi]_{\text{periodic BC for all !}}\, e^{-\int_0^\beta L^{Euclidean} d\tau}$$

$$= \frac{1}{\sqrt{Det\,(-\partial_\tau^2 + w^2)}} \times Det(\partial_\tau + \omega)\Big|_{\text{periodic}}$$

$$= 1 \quad = \quad \mathrm{Tr}(-1)^F e^{-\beta H}$$

this trivial example contains many basic structures
that one needs to attack arbitrary interacting gauge theories

this program for Witten index, now revived
under the new flag of localization,
produced exact formulae for diverse susy gauge theories

supersymmetric localization is a way to reduce path-integral
to that of harmonic oscillators,
relying on the robust nature of the twisted partition function

$$Z_t(\mathcal{O}) = \int e^{-S-tQ\Psi}\mathcal{O}$$

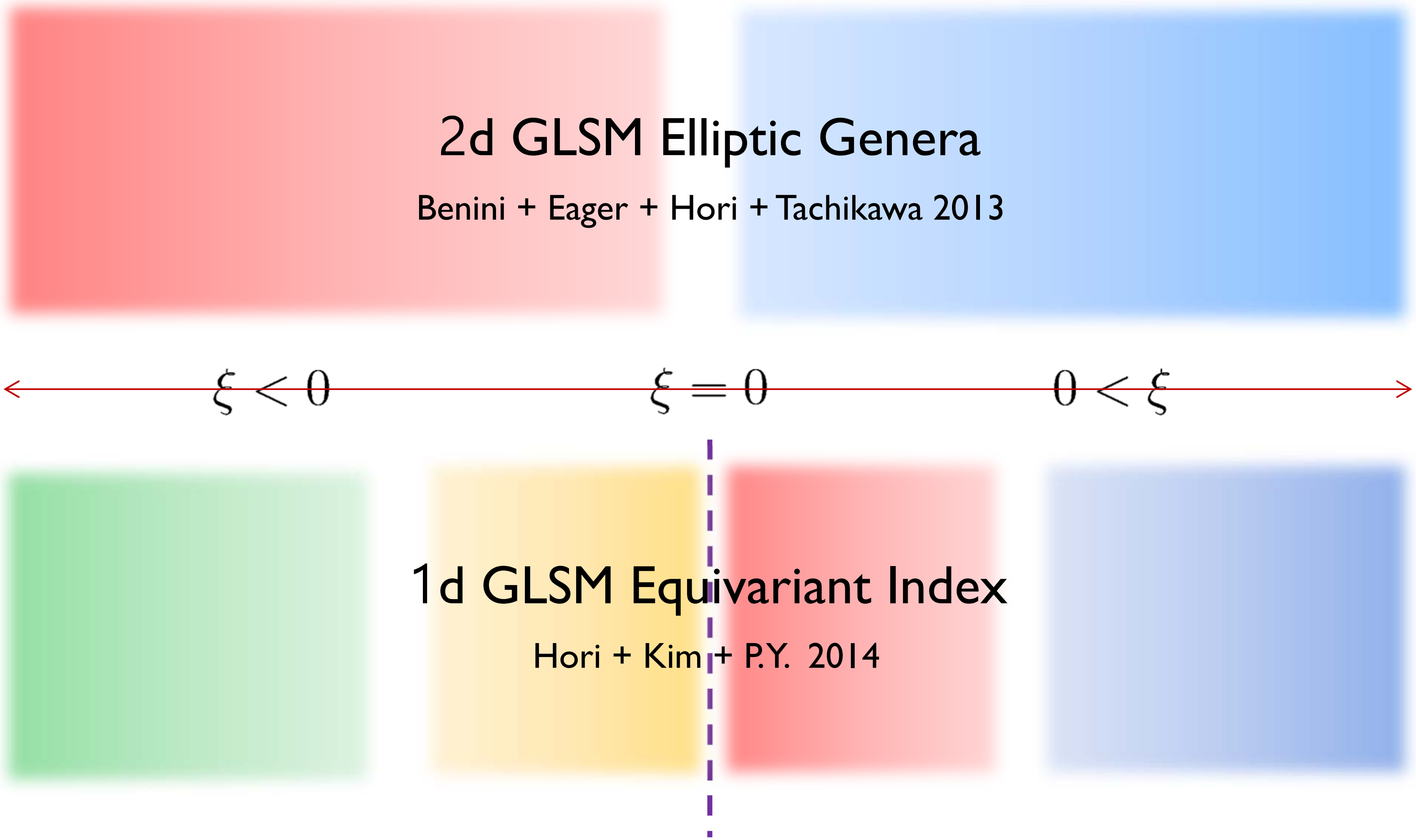
$$\partial_t Z_t(\mathcal{O}) = 0$$

$$[Q, \mathcal{O}] = 0$$

$$Q^2 = G$$

$$[G, \mathcal{O}] = 0$$

in particular, powerful and comprehensive formalism have emerged for low-dimensional gauged linear sigma models



$$Q = H$$

$$\{Q, (-1)^F\} = 0$$

$$[Q, G_F] = 0$$

refined Witten index of d=1 N≥2 GLSM

$$\mathcal{I}(x) \equiv \lim_{\beta\to\infty} \mathrm{Tr}\left[(-1)^F x^{G_F} e^{-\beta H}\right]$$

K.Hori + H.Kim + P.Y. 2014

$$Q = H$$

$$\{Q, (-1)^{2J_3}\} = 0$$

$$[Q, G_F] = 0$$

$$[Q, R + J_3] = 0$$

refined Witten index of d=1 N≥4 GLSM

$$\mathcal{I}(\mathbf{y}; x) \equiv \lim_{\beta\to\infty} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2(R+J_3)}x^{G_F}e^{-\beta H}\right]$$

K.Hori + H.Kim + P.Y. 2014

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2(R+J_3)}x^{G_F}e^{-\beta H}\right]$$

N≥4 compact and geometric $\rightarrow$

$$= \sum_{p,q}(-1)^{p+q-d}\mathbf{y}^{2p-d}\dim H^{(p,q)}(\mathcal{M})$$

$$= (-\mathbf{y})^{-d}\mathcal{I}_{\text{Hirzebruch}}(z=-\mathbf{y}^2)$$

x-independent

$$\mathcal{I}_{\text{Hirzebruch}}(z) = \sum_p z^p \sum_q (-1)^q h^{p,q}(\mathcal{M})$$

$$\begin{array}{ccccccc}
 & & & h^{d,d} & & & \\
 & & h^{d,d-1} & & h^{d-1,d} & & \\
 & \cdots & & \cdots & & \cdots & \\
h^{d,0} & & \cdots & & \cdots & & h^{0,d} \\
 & \cdots & & \cdots & & \cdots & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

the localization = reduction to a Gaussian path integral via continuous and small parameter shift that will not change the path integral

$$\mathcal{L}_{\text{vector}} = \frac{1}{e^2}\,\mathrm{Re}\left(\int d\theta^2\ \mathrm{tr}\, W_\alpha W^\alpha\right)$$

$$\mathcal{L}_{\text{chiral}} = \frac{1}{g^2}\int d\theta^2 d\bar{\theta}^2\ \mathrm{tr}\,\bar{\Phi} e^V \Phi$$

$$\mathcal{L}_{\text{usperpotential}} = \int d\theta^2 W(\Phi) + c.c.$$

$$\mathcal{L}_{\text{FI}} = \xi\int d\theta^2 d\bar{\theta}^2 \mathrm{tr}\, V$$

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

the limit we take is $e^2 \to 0$ which reduces
the path-integral to that of many harmonic oscillators

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right] \qquad [Q, J_3+R]=0$$

$$= \lim_{e^2\to 0}\int_{\text{periodic}}[dX\cdots d\phi\cdots]\, e^{-\int_0^\beta d\tau \mathcal{L}_E}\Big|_{\partial_\tau\to\partial_\tau+(2J_3+2R)\log(\mathbf{y})/\beta+\cdots}$$

cf) $$\mathcal{I} \equiv \lim_{\beta\to\infty} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

the limit we take is $e^2 \to 0$ which reduces
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$$\Omega \equiv \lim_{e^2 \to 0} \mathrm{Tr}\left[(-1)^{2J_3} \mathbf{y}^{2J_3+2R} x^{G_F} e^{-\beta Q^2}\right] \qquad u = A_3 + iA_\tau \Big|_{\text{zeromode}}^{\text{Cartan}}$$

$$= \int_{M_u} du\, d\bar{u} \int_{\mathbf{R}+i\delta} dD \left[h(u,\bar{u};D) \cdot g(u,\bar{u};D) \cdot e^{-\frac{D^2}{e^2} + i\xi D}\right]$$

zero mode
of gauge multiplets

from integral over
gaugino zero mode

one-loop determinants
of everything else

$$g(u,\bar{u};D) \sim \prod_Q \prod_n \frac{(2\pi n i + Qu - (R-2)\log(\mathbf{y} + \cdots)) \cdot (-2\pi n i + Qu - R\log(\mathbf{y}) + \cdots)}{|2\pi n i + Qu - R\log(\mathbf{y}) + \cdots|^2 - iQD}$$

scale up FI to send $e\xi$ to infinite,

then, after a long, long, long song and dance, ……

path integral reduces to a boundary integral on

$$M_u = (C^*)^{\text{rank}} \setminus \cup H_*^Q$$

singular hyperplanes

of a meromorphic function

$$g(u, \bar{u}; D = 0) \quad = \quad \left(\frac{1}{\mathbf{y} - \mathbf{y}^{-1}} \right)^{\text{rank}} \prod_\alpha \frac{t^{-\alpha/2} - t^{\alpha/2}}{t^{\alpha/2}\mathbf{y}^{-1} - t^{-\alpha/2}\mathbf{y}}$$

$$u = \log(t) \qquad \times \prod_i \frac{t^{-Q_i/2} x^{-F_i/2} \mathbf{y}^{-(R_i/2-1)} - t^{Q_i/2} x^{F_i/2} \mathbf{y}^{R_i/2-1}}{t^{Q_i/2} x^{F_i/2} \mathbf{y}^{R_i/2} - t^{-Q_i/2} x^{-F_i/2} \mathbf{y}^{-R_i/2}}$$

Hori + Kim + P.Y. 2014

which often translates to a Jeffrey-Kirwan contour integral

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$$= \sum \text{JK-Res}_{\eta:\{Q_i\}}\, g(u,\bar{u};0)$$

Hori + Kim + P.Y. 2014

Szenes + Vergne 2004
Brion + M.Vergne 1999
Jeffrey + Kirwan 1993

$$M_u = (C^*)^{\mathrm{rank}} \setminus \cup H_*^Q$$

$$\partial M_u = \cup_Q \partial\Delta^Q + \partial M_\infty$$

$$\{Q_i\} = \{Q^{\mathrm{chiral}}\} \cup \{Q^{\mathrm{vector}}\} \cup \{Q_\infty = -\xi\}$$

N=4 CP(N -1)

chirals	$U(1)$	$U(N)_F$
X	1	N

null

$\xi < 0$ | $0 < \xi$

1
0 0
0 1 0
0 0 0 0
0 0 1 0 0
0 0 0 0 0 0
0 0 1 0 0
0 0 0 0
0 1 0
0 0
1

$N = 6$

quintic CY3 hypersurface in CP4

	P	$X_{1,2,3,4,5}$
$U(1)$	-5	1

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

$\xi < 0$

$0 < \xi$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

N=4 rank 2 GLSM Q.M. for CY3 in $WCP_{(11222)}$

	P	$X_{1,2}$	$Y_{1,2,3}$	Z
$U(1)_1$	-4	0	1	1
$U(1)_2$	0	1	0	-2

$$\begin{matrix} &&&0&&& \\ &&0&&0&& \\ &0&&0&&0& \\ 1&&86&&86&&1 \\ &0&&0&&0& \\ &&0&&0&& \\ &&&0&&& \end{matrix}$$

hybrid

$$\begin{matrix} &&&1&&& \\ &&0&&0&& \\ &0&&2&&0& \\ 1&&86&&86&&1 \\ &0&&2&&0& \\ &&0&&0&& \\ &&&1&&& \end{matrix}$$

geometric

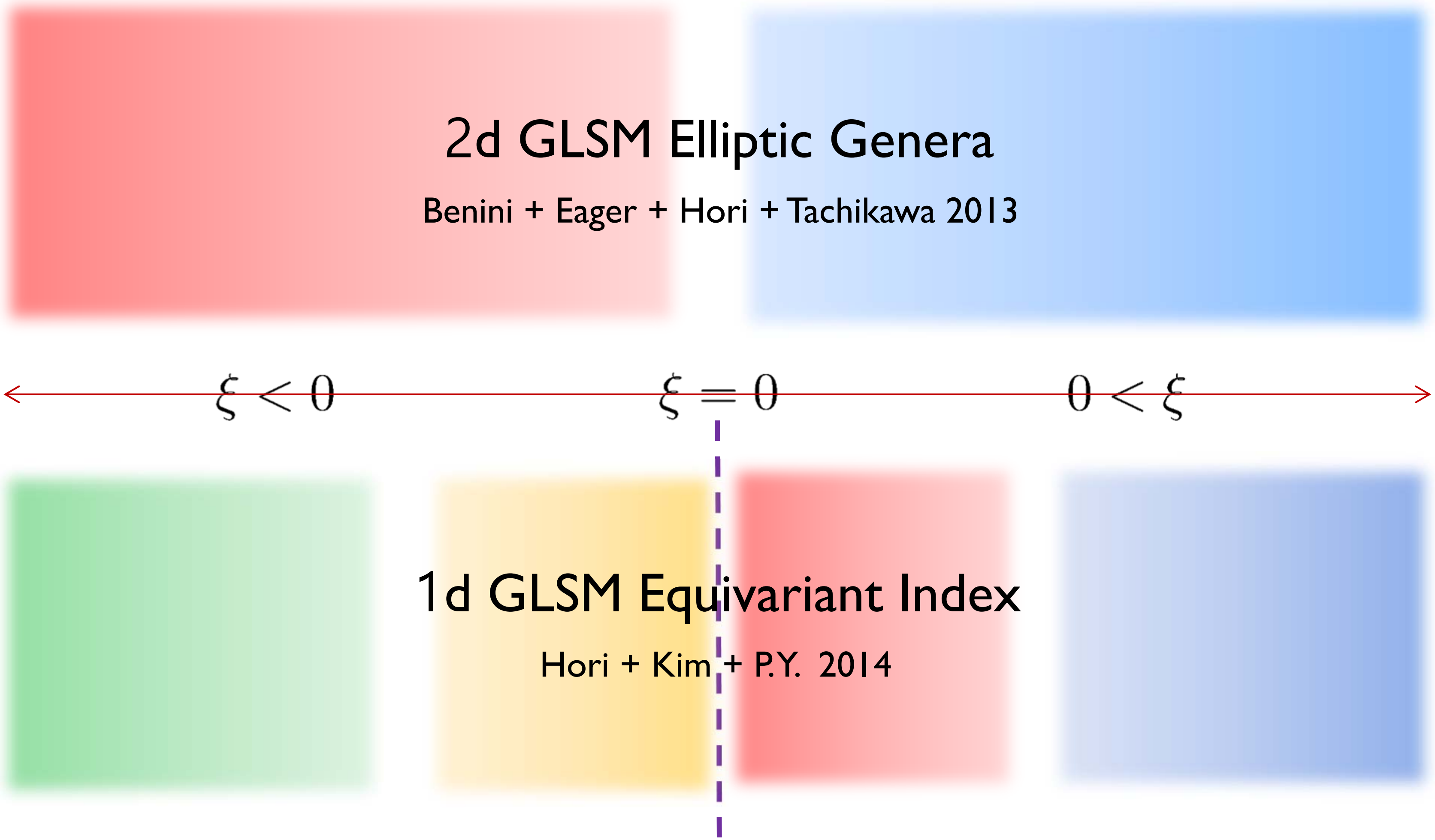
Landau-Ginsburg

$$\begin{matrix} &&&0&&& \\ &&0&&0&& \\ &0&&0&&0& \\ 1&&83&&83&&1 \\ &0&&0&&0& \\ &&0&&0&& \\ &&&0&&& \end{matrix}$$

orbifold

$$\begin{matrix} &&&1&&& \\ &&0&&0&& \\ &0&&1&&0& \\ 1&&83&&83&&1 \\ &0&&1&&0& \\ &&0&&0&& \\ &&&1&&& \end{matrix}$$

2d computations are closely parallel to this,
except that the wall-crossing discontonuity does not occur

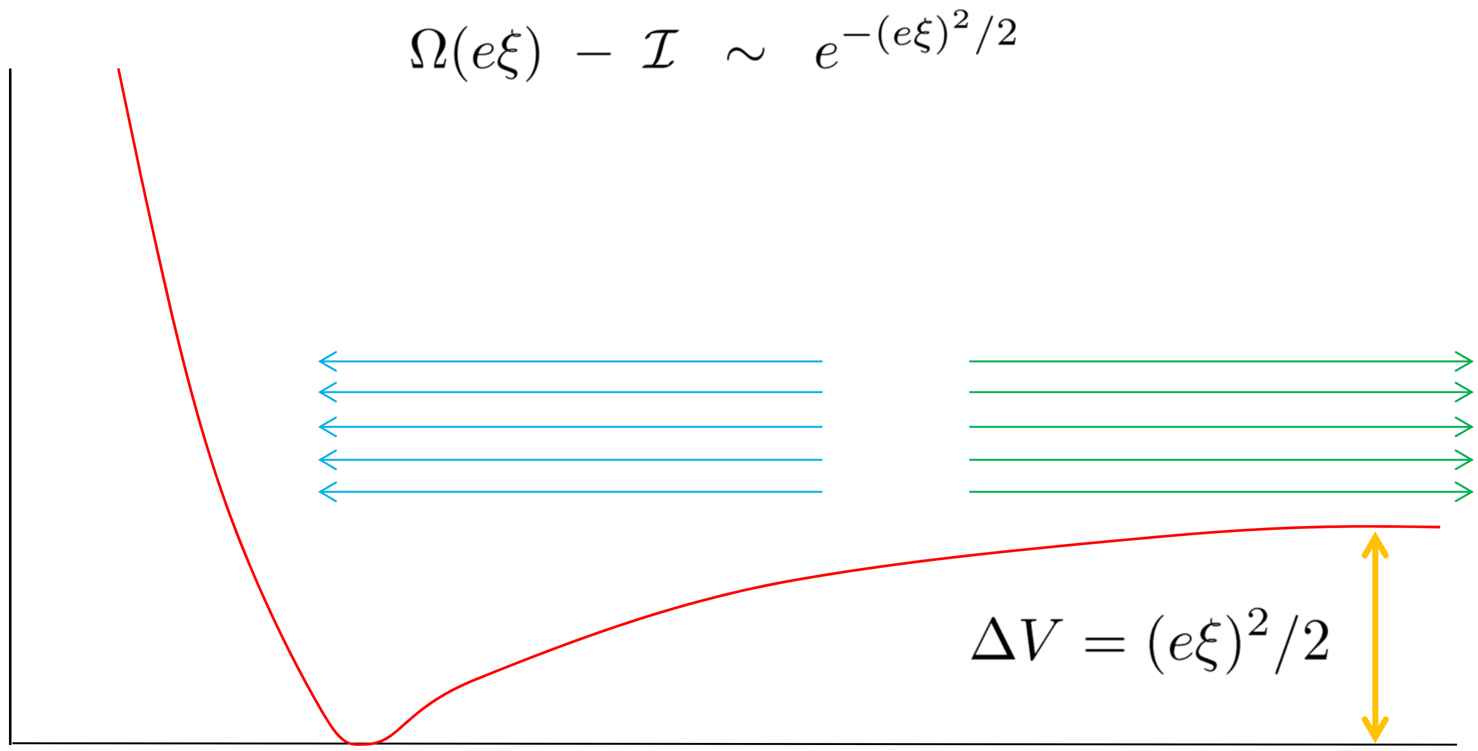


but how was ξ-dependence = wall-crossing possible at all
when the Witten index is supposed to be integral
and thus robust under any continuous parameter shift

such a naïve invariance argument always assumes
“small” deformation of the parameters,
meaning, nothing drastic should happen

however, vanishing FI constants always implies new
asymptotic runaway direction along vector multiplets,
invalidating Q-exactness across $\xi = 0$

extra contributions from the continuum, interpolating across $\xi = 0$, which is dealt with by scaling up $e\xi$



this reminds us of typical subtleties that plague Witten index computation if an asymptotic direction is unavoidable

as we just saw, the twisted partition function need not even be integral if a flat asymptotic direction shows

localization does not compute Witten index
but rather a twisted partition function

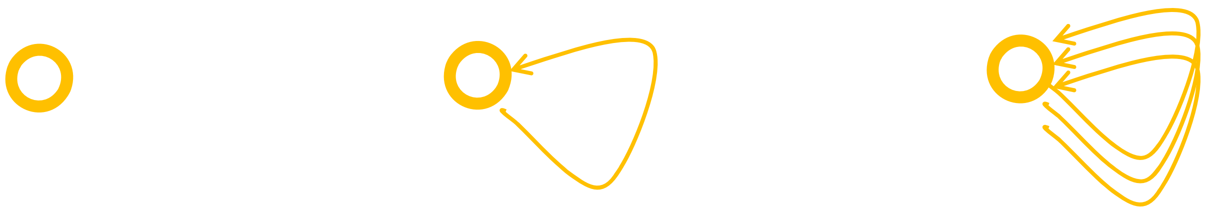
the two are not the same thing, in general

S.J. Lee + P.Y., 2016

back to the M-theory problems and their cousins

$$\mathcal{N} = 4, 8, 16$$

supersymmetric Yang-Mills quantum mechanics



S.J. Lee + P.Y., 2016

after rigorous applications of HKY procedure,

$$\Omega^{SU(2)}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{2}\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})}$$

$$\Omega^{SU(3)}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{3}\frac{1}{(\mathbf{y}^{-2}+1+\mathbf{y}^{2})}$$

$$\Omega^{SU(4)}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{4}\frac{1}{(\mathbf{y}^{-3}+\mathbf{y}^{-1}+\mathbf{y}+\mathbf{y}^{3})}$$

the fact that these features are not limited to
pure Yang-Mills quantum mechanics can be
inferred from the appearance of the rational invariant
in the refined wall-crossing formulae

$$\omega(\Gamma;\mathbf{y}) \equiv \sum_{p|\Gamma} \mathcal{I}(\Gamma/p;\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p-\mathbf{y}^{-p})}$$

Kontsevich+Soibelman 2008

Manschot+Pioline+Sen 2010
Kim+Park+Wang+P.Y. 2011

S.J. Lee + P.Y., 2016

other rank 2 examples

$$\Omega^{SO(4)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{4}\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}$$

$$\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{8}\left[\frac{2}{\mathbf{y}^{-2}+\mathbf{y}^2}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}\right]$$

$$\Omega^{G_2}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{12}\left[\frac{2}{\mathbf{y}^{-2}-1+\mathbf{y}^2}+\frac{2}{\mathbf{y}^{-2}+1+\mathbf{y}^2}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}\right]$$

S.J. Lee + P.Y., 2016

higher rank examples

$$\Omega^{SU(4)/SO(6)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{4}\frac{1}{(\mathbf{y}^{-3}+\mathbf{y}^{-1}+\mathbf{y}+\mathbf{y}^{3})}$$

$$\Omega^{SO(7)/Sp(3)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{48}\left[\frac{8}{\mathbf{y}^{-3}+\mathbf{y}^{3}}+\frac{6}{(\mathbf{y}^{-2}+\mathbf{y}^{2})(\mathbf{y}^{-1}+\mathbf{y})}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^{3}}\right]$$

$$\Omega^{SO(8)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{192}\left[\frac{32}{(\mathbf{y}^{-3}+\mathbf{y}^{3})(\mathbf{y}^{-1}+\mathbf{y})}+\frac{12}{(\mathbf{y}^{-2}+\mathbf{y}^{2})^{2}}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^{4}}\right]$$

S.J. Lee + P.Y., 2016

which are organized as

elliptic Weyl elements only

$0 \neq \mathrm{Det}\,(1 - w)$

$$\Omega^{G}_{\mathcal{N}=4}(\mathbf{y}) \quad = \frac{1}{|W_G|} \sum_{w \in W_G}{}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1} - \mathbf{y} \cdot w)}$$

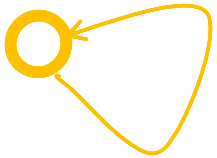
Weyl group

elliptic Weyl elements for some classical groups

G	W	Elliptic Weyl Elements
$SU(N)$	S_N	$(123\cdots N)$
$SO(4)$	$Z_2 \times S_2$	$(\dot{1})(\dot{2})$
$SO(5)/Sp(2)$	$(Z_2)^2 \times S_2$	$(1\dot{2}), \quad (\dot{1})(\dot{2})$
$SO(6)$	$(Z_2)^2 \times S_3$	$(1\dot{2})(\dot{3})$
$SO(7)/Sp(3)$	$(Z_2)^3 \times S_3$	$(\dot{1}\dot{2}\dot{3}), \quad (12\dot{3}), \quad (1\dot{2})(\dot{3}), \quad (\dot{1})(\dot{2})(\dot{3})$
$SO(8)$	$(Z_2)^3 \times S_4$	$(\dot{1}\dot{2}\dot{3})(\dot{4}), \quad (12\dot{3})(\dot{4}), \quad (1\dot{2})(3\dot{4}), \quad (\dot{1})(\dot{2})(\dot{3})(\dot{4})$

S.J. Lee + P.Y., 2016

pure $\mathcal{N}=8$ Yang-Mills quantum mechanics



elliptic Weyl elements only

$0 \neq \mathrm{Det}\,(1-w)$

$$\Omega^G_{\mathcal{N}=8}(\mathbf{y},x) = \frac{1}{|W_G|} \sum_{w\in W_G}{}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1}-\mathbf{y}\cdot w)} \cdot \frac{\mathrm{Det}\,\left(\mathbf{y}^{-1}x^{1/2}-\mathbf{y}x^{-1/2}\cdot w\right)}{\mathrm{Det}\,\left(x^{1/2}-x^{-1/2}\cdot w\right)}$$

Weyl group

why?

$$\mathcal{I} \quad = \quad \mathcal{I}_{\text{bulk}} \quad + \quad \delta\mathcal{I}$$

$$\lim_{\beta\to 0} \text{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$$\Omega \equiv \lim_{e^2\to 0} \text{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

→ very simple formulae for continuum sector contributions when no bound states are expected

$$\Omega^G_{\mathcal{N}=4,8} = \mathcal{I}^G_{\mathcal{N}=4,8;\text{bulk}} = -\delta\mathcal{I}^G_{\mathcal{N}=4,8}$$

$$= -\delta\mathcal{I}^{U(1)^r/W}_{\mathcal{N}=4,8}$$

$$= \mathcal{I}^{U(1)^r/W}_{\mathcal{N}=4,8;\text{bulk}}$$

P.Y. 1997

Green+Gutperle 1997
Kac+Smilga 1999

S.J. Lee + P.Y., 2016

$\mathcal{N}=16$ SU(N) theories, a.k.a. D0-brane bound state problem

$$\Omega^{SU(N)}_{\mathcal{N}=16}(\mathbf{y},x) \quad = \mathcal{I}^{SU(N)}_{\mathcal{N}=16} + \sum_{p|N;p>1} \mathcal{I}^{SU(N/p)}_{\mathcal{N}=16} \cdot \Delta^{SU(p)}_{\mathcal{N}=16}$$

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y},x) = \frac{1}{|W_G|}\sum_{w\in W_G}{}^{'} \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1}-\mathbf{y}\cdot w)} \cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\,\left(\mathbf{y}^{R_a-1}x^{F_a/2}-\mathbf{y}^{1-R_a}x^{-F_a/2}\cdot w\right)}{\mathrm{Det}\,\left(\mathbf{y}^{R_a}x^{F_a/2}-\mathbf{y}^{-R_a}x^{-F_a/2}\cdot w\right)}$$

S.J. Lee + P.Y., 2016

$\mathcal{N}=16$ SU(N) theories, a.k.a. D0-brane bound state problem

$$\Omega^{SU(N)}_{\mathcal{N}=16}(\mathbf{y},x) \quad = \mathcal{I}^{SU(N)}_{\mathcal{N}=16} + \sum_{p|N;p>1} \mathcal{I}^{SU(N/p)}_{\mathcal{N}=16} \cdot \Delta^{SU(p)}_{\mathcal{N}=16}$$

$\mathbf{y} \to 1$

$$\to \sum_{p|N} 1 \times \frac{1}{p^2}$$

$$\mathcal{I}^{SU}_{\mathcal{N}=16} = 1$$

P.Y. / Sethi,Stern 1997

Green,Gutperle1997 + Nekrasov, Moore, Shatashvili 1998

S.J. Lee + P.Y., 2016

generalizes to other $\mathcal{N} = 16$ theories

$$\Omega^{G}_{\mathcal{N}=16}(\mathbf{y}, x) \quad = \mathcal{I}^{G}_{\mathcal{N}=16} + \sum_{G' \subset G; G' \neq G} \# \cdot \Delta^{G'}_{\mathcal{N}=16}$$

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y}, x) = \frac{1}{|W_G|} \sum_{w \in W_G}{}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} \cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\,\left(\mathbf{y}^{R_a - 1} x^{F_a/2} - \mathbf{y}^{1-R_a} x^{-F_a/2} \cdot w\right)}{\mathrm{Det}\,\left(\mathbf{y}^{R_a} x^{F_a/2} - \mathbf{y}^{-R_a} x^{-F_a/2} \cdot w\right)}$$

S.J. Lee + P.Y., 2016

or more definitely,

$$\begin{aligned}
\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=16} &= 1 + 2\Delta^{SO(3)/Sp(1)}_{\mathcal{N}=16} + \Delta^{SO(5)/Sp(2)}_{\mathcal{N}=16} \\
\Omega^{G_2}_{\mathcal{N}=16} &= 2 + 2\Delta^{SU(2)}_{\mathcal{N}=16} + \Delta^{G_2}_{\mathcal{N}=16} \\
\Omega^{SO(7)}_{\mathcal{N}=16} &= 1 + 3\Delta^{SO(3)}_{\mathcal{N}=16} + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16} \\
\Omega^{Sp(3)}_{\mathcal{N}=16} &= 2 + 3\Delta^{Sp(1)}_{\mathcal{N}=16} + \left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + \Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16} \\
\Omega^{SO(8)}_{\mathcal{N}=16} &= 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^3 + 3\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(8)}_{\mathcal{N}=16} \\
\Omega^{SO(9)}_{\mathcal{N}=16} &= 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(3)}_{\mathcal{N}=16}\cdot\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16} + \Delta^{SO(9)}_{\mathcal{N}=16} \\
\Omega^{Sp(4)}_{\mathcal{N}=16} &= 2 + 5\Delta^{Sp(1)}_{\mathcal{N}=16} + 2\left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(1)}_{\mathcal{N}=16}\cdot\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16} + \Delta^{Sp(4)}_{\mathcal{N}=16}
\end{aligned}$$

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y},x) = \frac{1}{|W|}\sum_{w}{}' \frac{1}{\mathrm{Det}\left(\mathbf{y}^{-1} - \mathbf{y}\cdot w\right)}\cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\left(\mathbf{y}^{R_a-1}x^{F_a/2} - \mathbf{y}^{1-R_a}x^{-F_a/2}\cdot w\right)}{\mathrm{Det}\left(\mathbf{y}^{R_a}x^{F_a/2} - \mathbf{y}^{-R_a}x^{-F_a/2}\cdot w\right)}$$

even without the full understanding of the recursive structure for the continuum contributions, the results suffice for reading off the Witten index $\mathcal{I}^{G}_{\mathcal{N}=16}$ from the unique integral part

$$\begin{aligned}
\mathcal{I}^{SO(5)=Sp(2)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{G_2}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(7)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{Sp(3)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{Sp(8)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(9)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{Sp(4)}_{\mathcal{N}=16} &= 2 \\
&\vdots
\end{aligned}$$

$$\Omega^G_{\mathcal{N}=4,8,16}(\mathbf{y},x)\Big|_{\mathbf{y}\to 1}$$

$$= \mathcal{I}^G_{\mathcal{N}=4,8;16}\Big|_{\text{bulk};\mathbf{y}\to 1}$$

	$\mathcal{N}=4,8$	$\mathcal{N}=16$
$SU(N)$	$\frac{1}{N^2}$	$\sum_{p\vert N}\frac{1}{p^2}$
$SO(4)$	$\frac{1}{16}$	$\frac{25}{16}$
$SO(6)=SU(4)$	$\frac{1}{16}$	$\frac{21}{16}$
$SO(8)$	$\frac{59}{1024}$	$\frac{3755}{1024}$
$SO(5)$	$\frac{5}{32}$	$\frac{53}{32}$
$SO(7)$	$\frac{15}{128}$	$\frac{267}{128}$
$SO(9)$	$\frac{195}{2048}$	$\frac{7555}{2048}$
$Sp(2)$	$\frac{5}{32}$	$\frac{53}{32}$
$Sp(3)$	$\frac{15}{128}$	$\frac{395}{128}$
$Sp(4)$	$\frac{195}{2048}$	$\frac{8067}{2048}$
G_2	$\frac{35}{144}$	$\frac{395}{144}$

cf) Moore, Nekrasov, Shatashvili 1998
Kac, Smilga 1999
Staudacher 2000
Pestun 2002

general answer for $\mathcal{N} = 16$ with other classical groups

$$\sum_N \mathcal{I}^{SU(N)}_{\mathcal{N}=16} t^N = \frac{1}{1-t}$$

$$\sum_N \mathcal{I}^{SO(N)}_{\mathcal{N}=16} t^N = \prod_{n=1}^{\infty} (1 + t^{2n-1})$$

$$\sum_N \mathcal{I}^{Sp(N)}_{\mathcal{N}=16} t^{2N} = \prod_{n=1}^{\infty} (1 + t^{2n})$$

and, for *O(N)*, after some more similar procedure that take the parity projection into account,

$$\sum_N \mathcal{I}^{SU(N)}_{\mathcal{N}=16} t^N = \frac{1}{1-t}$$

$$\sum_N \mathcal{I}^{O(N)}_{\mathcal{N}=16} t^N = \prod_{n=1}^{\infty} (1 + t^{2n-1})$$

$$\sum_N \mathcal{I}^{Sp(N)}_{\mathcal{N}=16} t^{2N} = \prod_{n=1}^{\infty} (1 + t^{2n})$$

S.J. Lee + P.Y., to appear

the latter two count threshold bound states
of D-particles at an $\mathcal{R}^9/Z_2$ orientifold point

$$\sum_N \mathcal{I}^{SU(N)}_{\mathcal{N}=16} t^N = \frac{1}{1-t}$$

$$\mathbf{S}^1 \times \mathcal{R}^{9+1}$$

$$\sum_N \mathcal{I}^{O(N)}_{\mathcal{N}=16} t^N = \prod_{n=1}^{\infty} (1+t^{2n-1})$$

$$\sum_N \mathcal{I}^{Sp(N)}_{\mathcal{N}=16} t^{2N} = \prod_{n=1}^{\infty} (1+t^{2n})$$

$$\mathbf{S}^1 \times \mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$$

S.J. Lee + P.Y., to appear

finally, back to M theory on a circle, with an Orientifold point

M on $\mathbf{S}^1 \times \mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$ = IIA on $\mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$

anomaly cancelation requires a single chiral fermion supported on $\mathbf{S}^1 \times \mathcal{R}^{0+1}$

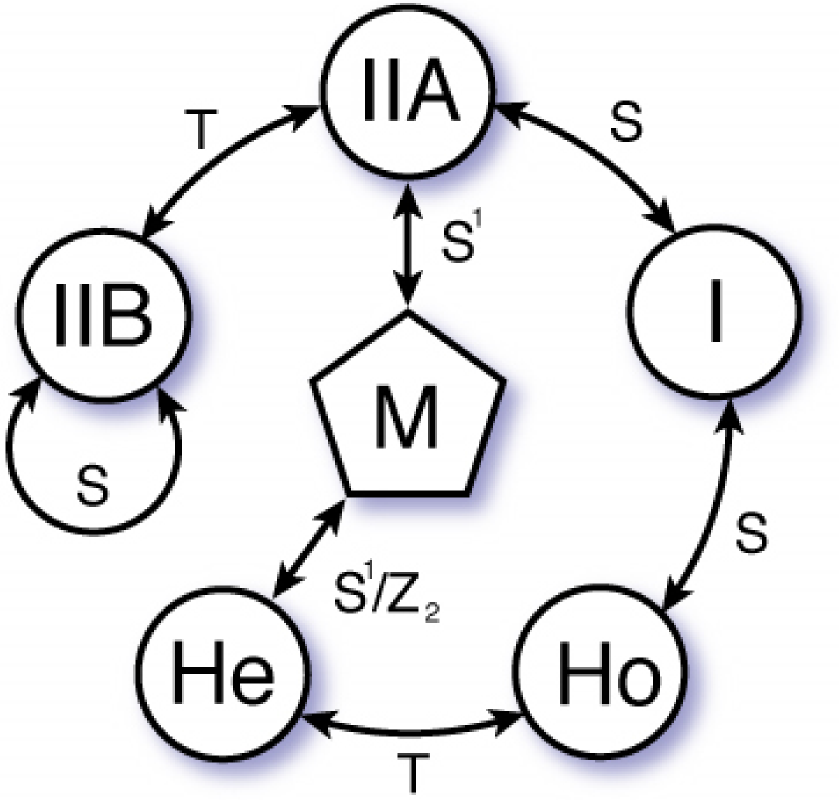
➡

Kaluza-Klein reduction generates two towers of fermionic harmonic oscillators, resulting in four Hilbert spaces whose partition functions constitute the two generating functions above

Dasgupta+Mukhi 1995

Kol+Hanany+Rajaraman 1999

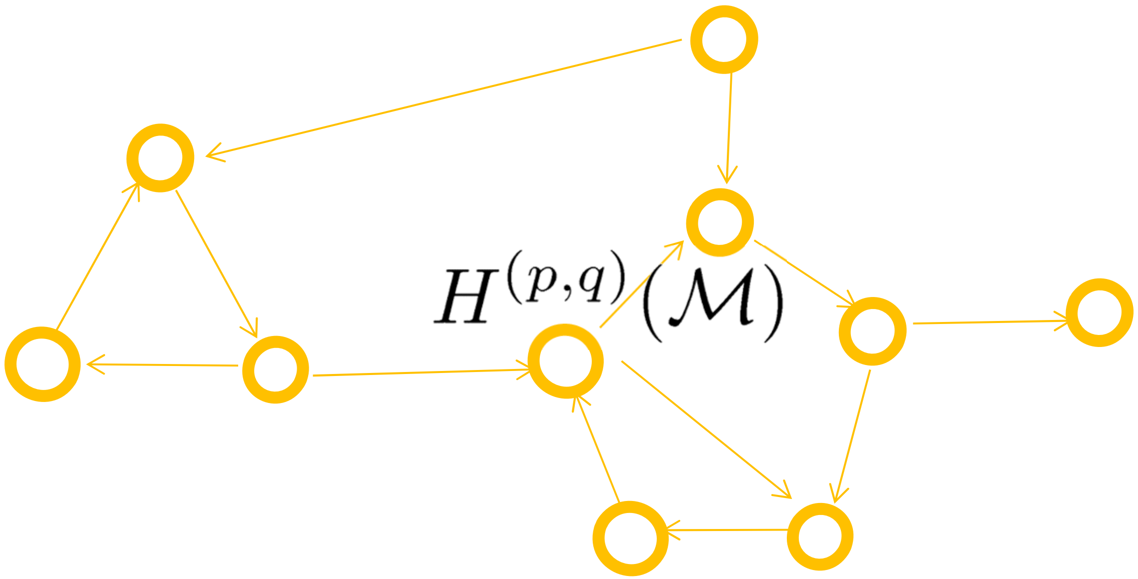
offering us a closure to a remarkable journey
that most directly explored M/String dualities



more importantly, this program is leading us to
the world of supersymmetric invariants
with rich physical and mathematical ramifications

witten index / scf index for gauge theories

Seiberg dualties / mutation



wall-crossing

quiver invariants

J=0 single center black holes