SUSY LOCALIZATION AND SPHERE PARTITION FUNCTIONS

KAZUO HOSOMICHI (NTU)

INTRODUCTION

In recent years, many exact formulae for SUSY gauge theories have been found by SUSY localization principle.

- Define & calculate SUSY-protected observables using path-integrals
- Use SUSY to argue that the path integral localizes onto "saddle points"

$$\boxed{\infty\text{-dim path integral} \longrightarrow \text{finite-dim integrals or sums}}$$

Sphere Partition Function

In 2007, Pestun computed partition function of 4D $\mathcal{N}=2$ SUSY gauge theories on S^4.

$$\boxed{Z_{S^4} \equiv \int D(\text{fields}) \exp\left(-\text{Action}_{(S^4)}\right) = \int d^r a \cdot \Delta(a)}$$

where $r =:$ rank of gauge group

$\Delta(a) =:$ some known (though complicated) function.

depends on gauge coupling, mass of matters,

His argument was soon applied to many SUSY gauge theories in other dimensions.

Applications of the Exact Formulae

$Z_{S^4} \longrightarrow$ led to the discovery of AGT relation

= exact correspondences between observables

in 4D SUSY gauge theory $\Longleftrightarrow$ 2D CFTs

$Z_{S^3} \longrightarrow$ led to the proof of the conjecture

$F(N) \sim \text{const} \cdot N^{3/2}$ at large N

free energy of the system of N membranes in M-theory

$\boxed{Z_{S^2} \longrightarrow \text{Applications to Calabi-Yau compactifications}}$

What are Calabi-Yau manifolds?

Why are they important in superstring theory?

SUPERSTRING COMPACTIFICATIONS

We have five superstring theories, all 10-dimensional.

32 SUSY ·····	type IIA, type IIB
16 SUSY ·····	type I, Hetero $_{SO(32)}$, Hetero $_{E_8 \times E_8}$

··· We need to reduce the dimension and SUSY.

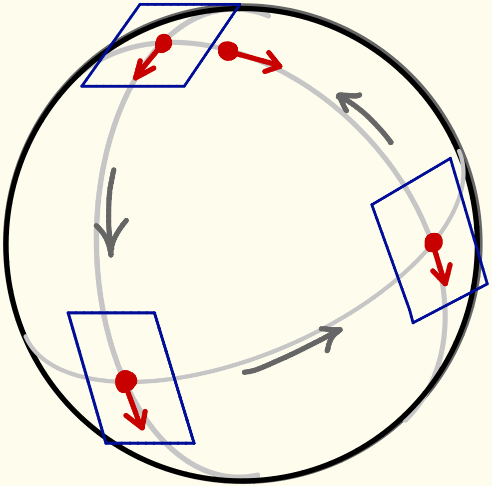
The six-torus T^6 (= product of six circles) is flat,

and does not reduce the SUSY.

··· We need curved 6-manifolds.

Holonomy

The curvature of a space can be measured by parallel-transporting a vector or a spinor along a closed path.



In a general 6-manifold,

- vectors receive SO(6) rotations.
- chiral spinors (4-component) receive SU(4) rotations.

CY3-FOLDS

(complex 3-dim Calabi-Yau manifolds)

… are 6-dimensional manifolds with $SU(3)$ holonomy.

(1 of the 4 spinor components does not feel curvature)

CY compactifications reduce the SUSY to 1/4.

IIA, IIB …… $32 \to 8$

I, Het ……… $16 \to 4$

CY3-FOLDS

(complex 3-dim Calabi-Yau manifolds)

The reduced holonomy $(SU(3) \subset SU(4))$ also implies

- CY3 satisfies Einstein's eq (Ricci-flatness)

$$R^{\lambda}{}_{\mu\lambda\nu} = 0 .$$

- CY3 is a Kähler manifold

$$ds^2 = g_{a\bar{b}} \cdot dz^a d\bar{z}^{\bar{b}} \qquad (a, \bar{b} = 1,2,3)$$

$$g_{a\bar{b}} = \frac{\partial}{\partial z^a} \frac{\partial}{\partial \bar{z}^{\bar{b}}} K(z, \bar{z})$$

Kähler potential

Let us discuss some general properties of the IIA or IIB superstrings on CY3.

(4D 8SUSY theories)

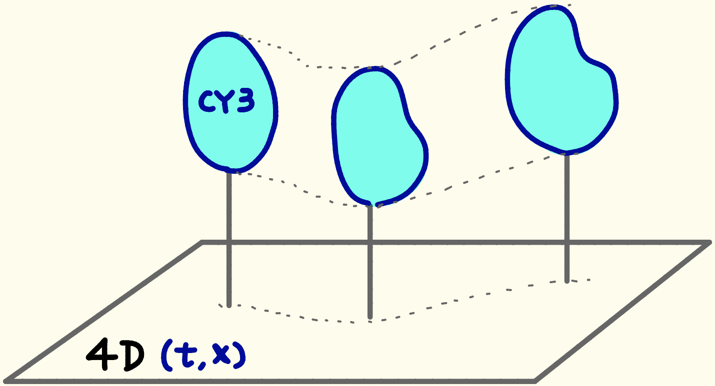
The 4SUSY theories are much more difficult.

Moduli

There are many CY3's of diffent topology known.

A CY3 of a given topology can change its shape & size while satisfying Einstein's eq.

... parametrized by a finite # of "moduli".



The shape of CY3 may vary as a function of (t, x).

Moduli are 4D massless scalars.

Moduli Space

The massless scalars $\phi^i(t,x)$ appear in the 4D effective Lagrangian as

$$\mathcal{L} = \cdots + \frac{1}{2} g_{i\bar{j}}(\phi) \cdot \partial_\mu \phi^i \partial^\mu \phi^{\bar{j}} + \cdots$$

ϕ^i are coordinates of the moduli space with metric $g_{i\bar{j}}(\phi)$.

- What's the dimension of the moduli space?
- How to determine the metric?

Dimension of the Moduli Space

… determined by the number of independent non-contractible cycles (closed submanifolds)

$\Leftrightarrow$ $h^{p,q} \equiv$ number of closed (p,q)-forms

$$\omega = \omega_{a..c\bar{d}..\bar{f}}(z,\bar{z})\, \underbrace{dz^a..dz^c}_{p}\, \underbrace{d\bar{z}^{\bar{d}}..d\bar{z}^{\bar{f}}}_{q}$$

$$\boxed{\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}}$$

\# of 2-cycles …. $h^{1,1}$

\# of 3-cycles ….. $2h^{2,1}+2$

\# of 4-cycles …. $h^{1,1}$

$(h^{1,1}=h^{2,2},\ h^{1,2}=h^{2,1})$

Product Structure

The moduli space takes the form $\boxed{\mathcal{M} = \mathcal{M}_K \times \mathcal{M}_c}$

$\mathcal{M}_K$ ···· moduli space of Kähler structures

$h^{1,1}$ - dimensional.

controls the sizes of even-dim cycles

$\mathcal{M}_c$ ····· moduli space of complex structures

$h^{2,1}$ -dimensional.

controls the size of 3-cycles

Both $\mathcal{M}_K$, $\mathcal{M}_c$ are "special Kähler manifolds".

Special Kähler Geometry

An n-dim special Kähler manifold has a set of

$2n+2$ holomorphic "functions" $(X^0, \cdots, X^n, \mathcal{F}_0, \cdots, \mathcal{F}_n)$

"special coordinates"

Kähler potential

$$K(z,\bar{z}) = -\log i\left(\bar{X}^I(\bar{z})\mathcal{F}_I(z) - X^I(z)\bar{\mathcal{F}}_I(\bar{z})\right)$$

When $\mathcal{F}_I$ are expressed as functions of X^I, they satisfy

$$\frac{\partial \mathcal{F}_I}{\partial X^J} = \frac{\partial \mathcal{F}_J}{\partial X^I}.$$

This implies there is a function $\mathcal{F}(X)$ called prepotential,

$$\text{s.t. } \mathcal{F}_I = \frac{\partial \mathcal{F}}{\partial X^I}.$$

K for CY moduli space

For $\mathcal{M}_K$, $(X^0, X^i; \mathcal{F}_i, \mathcal{F}_0)$ correspond to

the volumes of the 0,2,4,6-cycles.

※ have to be complexified and analytically continued.

have to incorporate "worldsheet instanton" correction

…… Difficult

For $\mathcal{M}_C$, $(X^I; \mathcal{F}_I)$ are the period integrals

$$X^I \equiv \int_{\alpha^I} \Omega, \quad \mathcal{F}_I \equiv \int_{\beta_I} \Omega$$

Ω … (3,0) form

α^I, β_I … 3-cycles

…… Easy (classical)

Mirror Symmetry

A traditional approach to solve $\mathcal{M}_K$ for a Calabi-Yau 3-fold "A" is:

$$\boxed{\begin{array}{l}\text{Find a mirror Calabi-Yau 3-fold "B"} \\ \text{such that} \quad h^{1,1}(A) = h^{2,1}(B), \quad h^{2,1}(A) = h^{1,1}(B)\end{array}}$$

$$\underbrace{\mathcal{M}_K(A)}_{\text{difficult}} = \underbrace{\mathcal{M}_C(B)}_{\text{easy}}, \quad \mathcal{M}_C(A) = \mathcal{M}_K(B)$$

Let us discuss the simplest example $h^{1,1}(A) = h^{2,1}(B) = 1$.

The Simplest Example

$A =:$ a quintic hypersurface $G(z_1, \cdots, z_5) = 0$

in $\mathbb{CP}^4 \equiv \dfrac{\{(z_1, \cdots, z_5) \neq (0, \cdots, 0)\}}{(z_1, \cdots, z_5) \sim (\lambda z_1, \cdots, \lambda z_5)}$

- size of $\mathbb{CP}^4$ ..... parametrizes $\mathcal{M}_K$

(

- coefficients of the quintic polynomial G

  ... parametrize $\mathcal{M}_C$

)

The Simplest Example

$B =:$ a quintic hypersurface

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 - 5\psi \cdot z_1 z_2 z_3 z_4 z_5 = 0$$

in $\mathbb{CP}^4/\Gamma$: $(z_1, \cdots, z_5) \sim (\lambda z_1, \cdots, \lambda z_5)$

$$(z_1, \cdots, z_5) \sim (\omega^{a_1} z_1, \cdots, \omega^{a_5} z_5)$$

$$\omega = e^{2\pi i/5}, \quad \sum a_i = 0 \bmod 5$$

ψ ···· coordinate on $\mathcal{M}_c$

The Simplest Example

The moduli space $\mathcal{M}_C(B) = \mathcal{M}_K(A)$ is easily solved.

$\left(X^0(\psi), X^1(\psi), \mathcal{F}_0(\psi), \mathcal{F}_1(\psi) \right)$ (special coordinates)

are period integrals of Ω over the basis 3-cycles,

where $\Omega \equiv \int \frac{dz^1 dz^2 dz^3}{\partial P / \partial z^4}$, $P = z_1^5 + z_2^5 + z_3^5 + z_4^5 + 1 - 5\psi z_1 z_2 z_3 z_4$

※ z_4 eliminated using $P(z_1 \cdots z_4) = 0$

They satisfy Picard-Fuchs equation, $(z \equiv \psi^{-5})$

$$\left\{ \left(z\frac{d}{dz}\right)^4 - z\left(z\frac{d}{dz}+\frac{4}{5}\right)\left(z\frac{d}{dz}+\frac{3}{5}\right)\left(z\frac{d}{dz}+\frac{2}{5}\right)\left(z\frac{d}{dz}+\frac{1}{5}\right) \right\} F(z) = 0$$

A more direct solution has been found.

Benini-Cremonesi '12

Doroud-Le Floch-Gomis-Lee '12

Jockers-Kumar-Lapan-Morrison-Romo '12

Gomis-Lee '12

A Direct solution

- realize the Calabi-Yau 3fold as a vacuum moduli space of a 2D $\mathcal{N}=(2,2)$ SUSY gauge theory (Witten '92)

The $U(1)$ gauge theory with matters

$$\begin{array}{lcccccc} & \phi_1, & \phi_2, & \phi_3, & \phi_4, & \phi_5, & P \\ (U(1)\text{ charge} & 1, & 1, & 1, & 1, & 1, & -5) \end{array}$$

with FI coupling $r>0$, Superpotential $W = P \cdot G_5(\phi)$

quintic polynomial

$\rightarrow$ vacua $P = G_5(\phi) = 0, \ |\phi_1|^2 + \cdots + |\phi_5|^2 = r \ / \ U(1)$

... quintic hypersurface in $\mathbb{CP}^4$ of size r.

MULTIPLET ①

Vector multiplet for gauge group G

A_m ···· gauge field

σ, ρ ···· real scalars

D ········ auxiliary scalar

$\lambda = \begin{pmatrix} \lambda^+ \\ \lambda^- \end{pmatrix}$ ··· gaugino, R-charge $(+1)$

$\bar{\lambda} = \begin{pmatrix} \bar{\lambda}^+ \\ \bar{\lambda}^- \end{pmatrix}$ ··· gaugino, R-charge (-1)

SUSY localization (I)

The path integral over vector multiplet fields

on the sphere with metric $ds^2 = \ell^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

localizes onto saddle point configurations

$$\sigma = \frac{a}{\ell}\,, \quad D = -\frac{a}{\ell^2}\,,$$

$$\rho = -\frac{s}{\ell}\,, \quad A = s\cdot(\cos\theta \mp 1)\,d\varphi$$ ··· on N/S hemispheres

※ $a, s \in$ (Lie algebra), s is GNO quantized.

Multiplet ②

chiral multiplets

ϕ .... complex scalar, R-charge $2q$

$\psi \equiv \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$ .... Dirac fermion, $2q-1$

F .... complex aux. field $2q-2$

c.c.

$\bar{\phi}$

$\bar{\psi} \equiv \begin{pmatrix} \bar{\psi}^+ \\ \bar{\psi}^- \end{pmatrix}$

$\bar{F}$

furnishes a complex rep.
of the gauge group.

SUSY localization (II)

Path integral over chirals localizes to $\phi = \psi = F = 0$.

Gaussian approx. around there gives an exact result.

Take an $U(1)$ theory, and choose a saddle point (a, s).

Path integral over a single chiral of charge $+1$ gives

the "1-loop determinant"

$$Z_{1loop} = \frac{\Gamma(s + q - ia)}{\Gamma(s + 1 - q + ia)}$$

R-charge

Sphere Partition Function

For the $U(1)$ gauge theory with $5+1$ charged matters,

$$Z_{S^2} = \sum_{s\in\frac{1}{2}\mathbb{Z}} \int_{\mathbb{R}} \frac{da}{2\pi}\, e^{-it(a+is)-i\bar{t}(a+is)} \times \left[\frac{\Gamma(s-ia+q)}{\Gamma(1+s+ia-q)}\right]^5 \frac{\Gamma(1-5s+5ia-5q)}{\Gamma(-5s-5ia+5q)}$$

Here $t \equiv r+i\theta$ (θ… 2D theta angle) parametrizes $\mathcal{M}_K(A)$

※ $q>0$: regulator

Path integral over vector multiplet $\longrightarrow \sum_s \int \frac{da}{2\pi}$

Path integral over matter multiplet …… gaussian

Soon it was realized that

$$Z_{S^2}(t,\bar{t}) = \sum_{s\in\frac{1}{2}\mathbb{Z}} \int_{\mathbb{R}} \frac{da}{2\pi}\, e^{-it(a+is)-i\bar{t}(a+is)} \times \left[\frac{\Gamma(s-ia+q)}{\Gamma(1+s+ia-q)}\right]^5 \frac{\Gamma(1-5s+5ia-5q)}{\Gamma(-5s-5ia+5q)}$$

$$= \exp\left(-K(t,\bar{t})\right)$$

Kähler potential for $\mathcal{M}_K(A)$

An evidence of this can be seen

by closing the a-integration contour in LHP

and rewriting it into a residue-sum

Another formula for Z_{S^2}

$$Z_{S^2} = (\cdots)\cdot \oint_0 \frac{d\epsilon}{2\pi i}\left(\frac{5}{\epsilon^4} - \frac{400\,\zeta(3)}{\epsilon}\right) W(z,\epsilon)\, W(\bar{z},\epsilon)$$

where $z \equiv -5^5 e^{-t}$,

$$W(z,\epsilon) \equiv \sum_{k\geq 0} z^{k+\epsilon}\,\frac{\prod_{j=1}^{5k}(j+5\epsilon)}{5^{5k}\prod_{j=1}^{k}(j+\epsilon)^5}$$

$$\left\{\left(z\frac{d}{dz}\right)^4 - z\left(z\frac{d}{dz}+\frac{4}{5}\right)\left(z\frac{d}{dz}+\frac{3}{5}\right)\left(z\frac{d}{dz}+\frac{2}{5}\right)\left(z\frac{d}{dz}+\frac{1}{5}\right)\right\} W(z,\epsilon) = z^{\epsilon}\cdot\epsilon^4$$

$\cdots$ $Z_{S^2}(t,\bar{t})$ is a bilinear of the solutions to PF equation.

The new approach to CY compactification

$\oplus$ exact formulae in SUSY gauge theories

allows us to study wider class of Calabi-Yau 3folds,
especially those constructed from
non-abelian 2D gauge theories.

INCLUSION OF DEFECTS

Sungjay Lee (KIAS), Takuya Okuda (U Tokyo) and KH,

work in progress

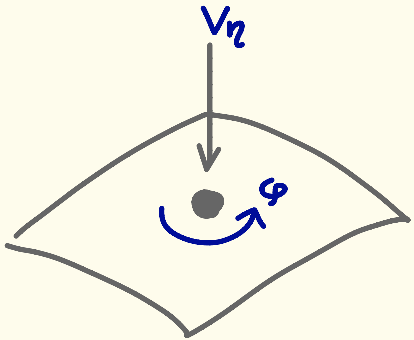
Vortex Defects

In 2D gauge theories we can consider the defects defined by the "singular boundary condition"

$$A \simeq \eta \cdot d\varphi$$

$\eta \in$ (Gauge symmetry Lie algebra)

φ … angle coordinate around the defect



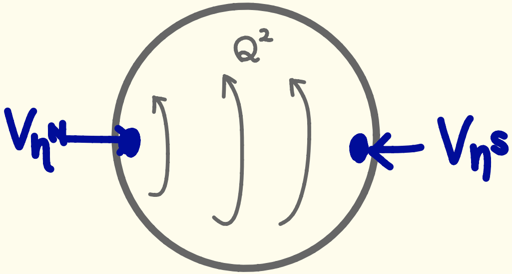
For SUSY-preserving defects, one can compute correlators.

Our Work

① Define the 2D $\mathcal{N}=(2,2)$ SUSY theory of vector & chiral multiplet on S^2

② Introduce vortex defects V_{η^N}, V_{η^S} at NP & SP so that Q is preserved

$\Rightarrow$ compute correlators



General Properties

(for $U(1)$ case)

Assuming charge quantization,

$A \simeq \eta\, d\varphi \longrightarrow A \simeq (\eta+1)\, d\varphi$ is a gauge symmetry.

The behavior of matters around V_η depends on $\eta \bmod \mathbb{Z}$.

An exact shift relation: $V_{\eta+1} = e^{-t}\, V_\eta$

Defect in the GLSM for quintic CY :

We found a defect V_η satisfying

$$\eta\in(0,\tfrac{1}{5}] \qquad \langle V_\eta(\mathrm{NP})\rangle = -\tfrac{1}{5}\left(z\tfrac{d}{dz}\right)^4 Z_{S^2} \qquad (z=-5^5e^{-t})$$

$$\eta\in(-\tfrac{1}{5},0] \qquad \langle V_\eta(\mathrm{NP})\rangle = Z_{S^2}$$

$$\eta\in(-\tfrac{2}{5},-\tfrac{1}{5}] \qquad \langle V_\eta(\mathrm{NP})\rangle = \left(1+5z\tfrac{d}{dz}\right) Z_{S^2}$$

$$\eta\in(-\tfrac{3}{5},-\tfrac{2}{5}] \qquad \langle V_\eta(\mathrm{NP})\rangle = \left(1+5z\tfrac{d}{dz}\right)\left(2+5z\tfrac{d}{dz}\right) Z_{S^2}$$

$$\eta\in(-\tfrac{4}{5},-\tfrac{3}{5}] \qquad \langle V_\eta(\mathrm{NP})\rangle = \left(1+5z\tfrac{d}{dz}\right)\left(2+5z\tfrac{d}{dz}\right)\left(3+5z\tfrac{d}{dz}\right) Z_{S^2}$$

$$\eta\in(-1,-\tfrac{4}{5}] \qquad \langle V_\eta(\mathrm{NP})\rangle = \left(1+5z\tfrac{d}{dz}\right)\left(2+5z\tfrac{d}{dz}\right)\left(3+5z\tfrac{d}{dz}\right)\left(4+5z\tfrac{d}{dz}\right) Z_{S^2}$$

Combining with the shift relation one recovers PF equation.

With more exact & powerful formulae,
SUSY localization may help us understanding
the physics & math of CY compactification even better.