The infrared structure of nonlinear sigma models

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Infrared structure of QFT's is an old and central subject in physics.

- In both QED and Gravity, scattering amplitudes with one soft gauge boson factorize universally:

$$M_{n+1}(k_1, \cdots, k_n; q) = (S^{(0)} + S^{(1)} + \cdots) M_n(k_1, \cdots, k_n)$$

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QED: $$S^{(0)} \equiv \sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q},$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} e_i \frac{\varepsilon_{q\mu} q_\nu J_i^{\mu\nu}}{k_i \cdot q},$$

Gravity: $$S^{(0)} \equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q},$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q},$$

$$J_i^{\mu\nu} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right)$$

The universality of these expressions implies they must follow from some general prime principles.

Indeed, the leading soft factors of both QED and Gravity follow from the "on-shell gauge invariance":

$$q_\mu M_n^\mu(k_1, \cdots, k_{n-1}; q) = 0$$
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$$\sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q} \quad \xrightarrow{\varepsilon_q^\mu \to \varepsilon_q^\mu + q^\mu} \quad \sum_{i=1}^{n} e_i = 0$$

$$\sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q} \quad \xrightarrow{\varepsilon^{\mu\nu} \to \varepsilon^{\mu\nu} + q^\mu \Lambda^\nu} \quad \sum_{i=1}^{n} k_i^\mu = 0$$

- For scalar theories with a global symmetry, degenerate vacua lead to nonlinear sigma models and Nambu-Goldstone bosons, which has wide-ranging applications in physics.

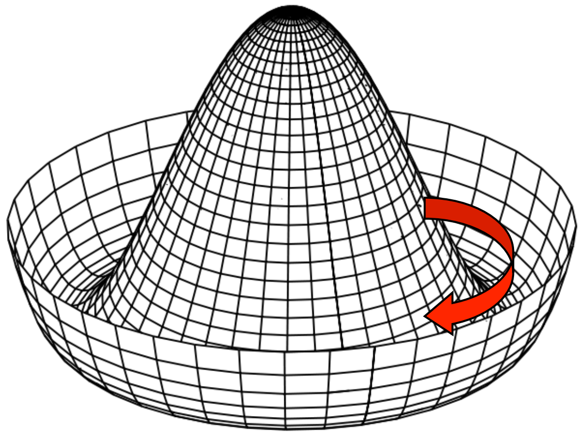
In this case, the "single soft" limit of Goldstone scattering amplitudes famously exhibits the "Adler's zero":

$$\lim_{\tau \to 0} M_{n+1}(k_1, \cdots, k_n; \tau q) = 0$$

This is a direct consequence of the presence of nontrivial vacua.

One simple argument is to recognize the different vacua are related by a rotation in the broken direction:

$$e^{i\theta}|\theta_0\rangle = |\theta_0 + \theta\rangle$$



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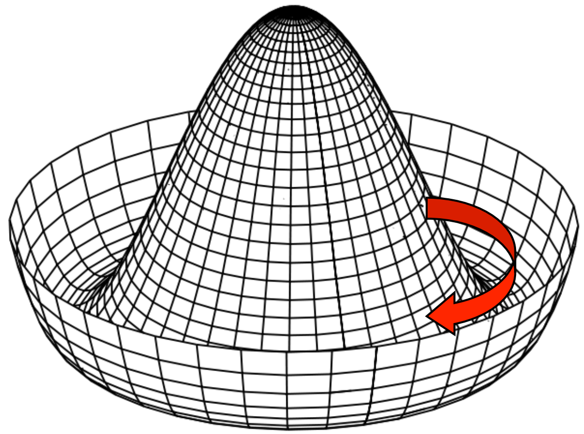

On the other hand, excitation along the broken direction gives the Goldstone boson,

$$e^{i(\rho(x)+\theta)}|\theta_0\rangle = e^{i\rho(x)}|\theta_0 + \theta\rangle$$

But the physics is invariant whether one chooses $|\theta_0\rangle$ or $|\theta_0 + \theta\rangle$

The "constant shift" is a symmetry of the NLSM, whose Ward identity implies the Adler's zero.

These are things we knew from half a century ago:

Bloch and Nordsieck (1937), F.E. Low (1954+…), Gell-Mann and Goldberger (1954), Weinberg (1965+…), Adler (1965+…) and etc.

Within the last decade, there are renewed efforts to understand better the infrared structure of QFT by the “amplitudes community” and “GR community”.

I will just cherry-pick the results I need.

One useful viewpoint is to think of the other vacua as coherent states of zero-momentum Goldstones:

$$e^{i\theta^a Q^a}|0\rangle = |0\rangle_\theta$$

$$= |0\rangle + i\theta^a\, Q^a|0\rangle - \frac{1}{2}\theta^a\theta^b\, Q^a Q^b|0\rangle + \cdots$$

$$\lim_{\tau\to 0}|\pi^a(\tau q)\rangle \qquad\qquad \lim_{\tau\to 0}|\pi^a(\tau p)\ \pi^b(\tau q)\rangle$$

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But different vacua form superselection sectors, one immediately sees the Adler's zero condition:

$$\lim_{p^\mu\to 0}\langle f+\pi^a(p)|i\rangle = 0$$

This also shows the “double soft” limit need not vanish.

Moreover, these amplitudes should probe the “structure” of the vacuum manifold:

$$\lim_{\tau \to 0} \langle \pi^a(\tau p)\ \pi^b(\tau q) | \pi^{a_1} \cdots \pi^{a_n} \rangle \propto [X^a, X^b]$$

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Resolution: the “double soft factor” must be anti-symmetric in the two soft momenta!

(Arkani-Hamed, Cachazo, Kaplan:0808.1446)

In fact, from these considerations one can guess the form of the double soft limit:

$$\lim_{\tau\to 0}\langle\pi^a(\tau p)\ \pi^b(\tau q)|\pi^{a_1}\cdots\pi^{a_n}\rangle \sim \sum_i \frac{(p-q)\cdot p_i}{(p+q)\cdot p_i}\langle\pi^{a_1}\cdots([X^a,X^b]\pi^{a_i})\cdots\rangle$$

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By now, many computations have been done for massless particles:

- the single and double soft limits for *massless* particles,
- the leading, subleading, subsubleading orders,
- in dimensionality ≠ 4,
- Yang-Mills, Gravity, NLSM, supersymmetric theories, string theories and other more exotic theories.

Schematically, they all look something like

$$M_{n+1}(k_1, \cdots, k_n; q) = (S^{(0)} + S^{(1)} + \cdots)M_n(k_1, \cdots, k_n)$$

$$M_{n+2}(k_1, \cdots, k_n; q_1, q_2) = (S_d^{(0)} + S_d^{(1)} + \cdots)M_n(k_1, \cdots, k_n)$$

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But here lies another puzzle:

For NLSM it is well-known that only even-point amplitudes exist—

$$M_{2n}(k_1, \cdots, k_{2n-1}; q) = (S^{(0)} + \overset{?}{S^{(1)}} + \cdots)\overset{?}{M_{2n-1}}(k_1, \cdots, k_{2n-1})$$

Is $S^{(1)}$ also zero? And if not, what is M_{2n-1} ??

(For half a century, the 1st question was never studied!)

Surprisingly, this question was answered only last year in a very elegant yet obscure fashion, by using the so-called CHY formulation of scattering equations.

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Cachazo, He, and Yuan proposed a compact and elegant formula for tree-level n-point scattering amplitudes of massless particles (spin-0, spin-1, spin-2).

The proposal contains two parts:

- the kinematics

- the dynamics

(Cachazo, He, Yuan:1306.6575, 1306.2962,1307.2199,1309.0885)

Kinematics –

Scattering of *n* massless particles in an arbitrary dimension involves *n* null vectors satisfying total momentum conservation:

$$\{k_1^\mu, \ldots, k_n^\mu \mid \sum_{a=1}^{n} k_a^\mu = 0,\ k_1^2 = \ldots = k_n^2 = 0\}$$

CHY proposed a map from the null light-cone to the Riemann sphere with n-punctures:

$$k_a^\mu = \frac{1}{2\pi i} \oint_{|z-\sigma_a|=\epsilon} dz \frac{p^\mu(z)}{\prod_{b=1}^{n}(z-\sigma_b)}$$

$\{\sigma_1, \cdots, \sigma_n\}$ is the location of the punctures.

.

CHY proposed mapping the whole CP^1 to the light cone of (complexified) momentum by imposing

$$p^2(z) = 0$$

This constraint is embodied in a set of equations called the scattering equation:

$$p(\sigma_n) \cdot p'(\sigma_n) \propto \sum_{a \neq n} \frac{2k_n \cdot k_a}{\sigma_n - \sigma_a} = 0$$

The full CHY proposal looks like:

$$M_n = \int \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta\left(\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}}\right) \bullet$$

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Encode the dynamics

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$$\bullet = \quad \mathcal{I}_L(\{k,\epsilon,\sigma\})\,\mathcal{I}_R(\{k,\tilde{\epsilon},\sigma\})$$

	$\mathcal{I}_L$	$\mathcal{I}_R$
bi-adjoint scalar	$\mathcal{C}_n(\omega)$	$\mathcal{C}_n(\tilde{\omega})$
Yang-Mills	$\mathcal{C}_n(\omega)$	$\text{Pf}'\Psi_n$
Einstein gravity	$\text{Pf}'\Psi_n$	$\text{Pf}'\tilde{\Psi}_n$
Born-Infeld	$(\text{Pf}'\mathsf{A}_n)^2$	$\text{Pf}'\Psi_n$
Non-linear sigma model	$\mathcal{C}_n(\omega)$	$(\text{Pf}'\mathsf{A}_n)^2$
Yang-Mills-scalar	$\mathcal{C}_n(\omega)$	$\text{Pf}\,\mathsf{X}_n\,\text{Pf}'\mathsf{A}_n$
Einstein-Maxwell-scalar	$\text{Pf}\,\mathsf{X}_n\,\text{Pf}'\mathsf{A}_n$	$\text{Pf}\,\mathsf{X}_n\,\text{Pf}'\mathsf{A}_n$
Dirac-Born-Infeld (scalar)	$(\text{Pf}'\mathsf{A}_n)^2$	$\text{Pf}\,\mathsf{X}_n\,\text{Pf}'\mathsf{A}_n$
special Galileon	$(\text{Pf}'\mathsf{A}_n)^2$	$(\text{Pf}'\mathsf{A}_n)^2$

Cachazo, Cha, Mizera: 1604.03893

The CHY proposal embodies the Color-Kinematic duality:

	BS	NLSM	YM
BS	BS	NLSM	YM
NLSM	NLSM	SG	BI
YM	YM	BI	G

Fig. 1: Multiplication table of QFTs, including bi-adjoint scalar (BS) theory, the nonlinear sigma model (NLSM), Yang-Mills (YM) theory, the special Galileon (SG), Born-Infeld (BI) theory, and gravity (G).

Cheung: 1708.03872

CHY proposal has been checked extensively. In the case of YM it is verified to all orders using BCFW recursion. In the case of NLSM, it's been checked explicitly up to 8-pt analytically and 10-pt numerically.

Using CHY, it's simple and straightforward to derive the subleading single soft limit for NLSM:

$$A_n^{\rm NLSM}(\mathbb{I}_n) = -\tau \sum_{a=2}^{n-2} \hat{s}_{an} A_{n-1}^{{\rm NLSM}\oplus\phi^3}(\mathbb{I}_{n-1}\,|\, n-1, a, 1) + \mathcal{O}(\tau^2),$$

$$A_n^{{\rm NLSM}\oplus\phi^3}(\alpha|\beta) = \oint d\mu_n \left(\mathcal{C}_n(\alpha)\right) \left(\mathcal{C}(\beta)\,({\rm Pf}\,{\sf A}_{\bar{\beta}})^2\right).$$

The color factor of the original SU(N) flavor structure

The color factor of an alternative SU(N) flavor structure

$A_n^{{\rm NLSM}\oplus\phi^3}$: biadjoint cubic scalar interacting with NLSM

Cachazo, Cha, Mizera: 1604.03893

This turns out to be the case for all theories with vanishing single soft limit:

$$A_n^{\text{theory}_1} \xrightarrow{\text{soft limit}} \tau^p \, A_{n-1}^{\text{theory}_1 \oplus \text{theory}_2} + \mathcal{O}(\tau^{p+1})$$

$$A_n^{\text{sGal}} = \tau^3 \sum_{a=2}^{n-2} \sum_{\substack{c=2 \\ c\neq a}}^{n-1} \sum_{\substack{d=1 \\ d\neq a}}^{n-2} \hat{s}_{an}\hat{s}_{cn}\hat{s}_{dn} \, A_{n-1}^{\text{sGal}\oplus\text{NLSM}^2\oplus\phi^3}(a,c,1|n-1,d,a) + \mathcal{O}(\tau^4).$$

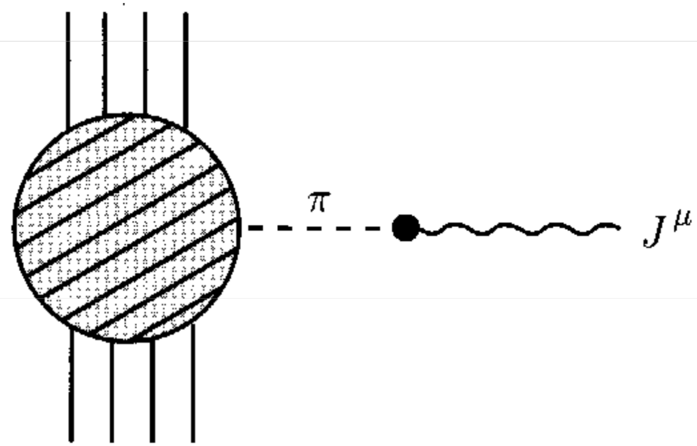
$$A_n^{\text{BI}} = \tau \sum_{a=2}^{n-2} \sum_{\substack{c=2 \\ c\neq a}}^{n-1} \hat{s}_{an}\hat{s}_{cn} \left(\frac{\epsilon_n \cdot k_{n-1}}{\hat{k}_n \cdot k_{n-1}} - \frac{\epsilon_n \cdot k_a}{\hat{k}_n \cdot k_a} \right) A_{n-1}^{\text{BI}\oplus\text{YM}}(a,c,1) + \mathcal{O}(\tau^2).$$

In QFT, let’s recall how Weinberg derives the Adler’s zero, starting from:

$$\langle 0|J^a_\mu|\pi^b(x)\rangle \sim \delta^{ab} f_\pi\, p_\mu\, e^{ip\cdot x}$$

which implies there’s a one-particle pole in the matrix element

$$\langle f|J^a_\mu|i\rangle \sim$$



$$\to i\frac{f_\pi q^\mu}{q^2} M_{fi} + N^\mu_{fi}$$

Figure from Weinberg, QFT Vol II

Current conservation then implies

$$M_{fi} = \frac{i}{f_\pi} q_\mu N^\mu_{fi} \to 0 \qquad \text{for} \qquad q \to 0$$

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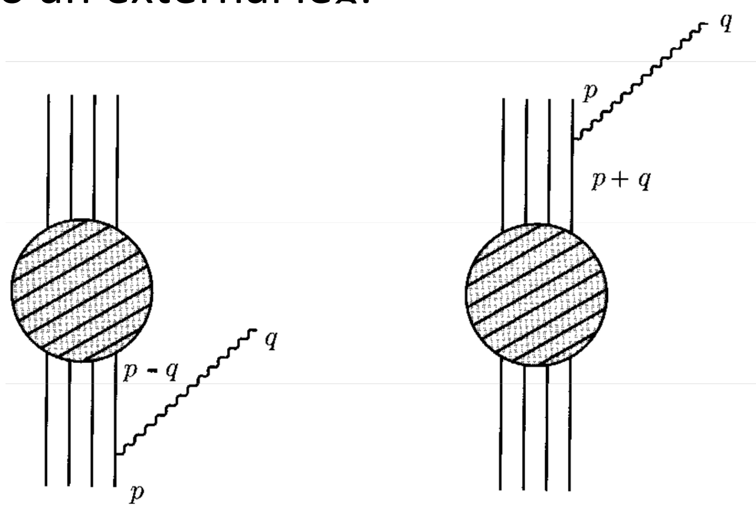

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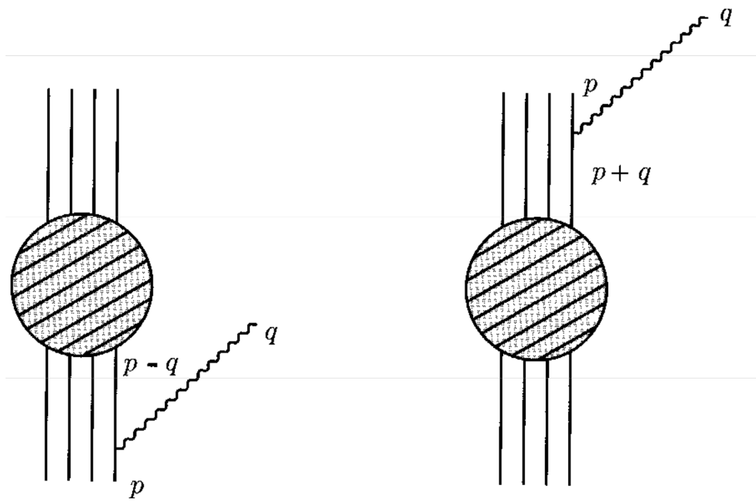

For pure pion scattering amplitudes, no such insertion exists.

It is not clear how to generalize this argument beyond the leading order.

It's useful to review the traditional approach of writing down the most general EFT for a nonlinearly realized global symmetry G, broken down to a linearly realized unbroken group H:

standard forms, which are described in detail. The mathematical problem is equivalent to that of finding all (nonlinear) realizations of a (compact, connected, semisimple) Lie group which become linear when restricted to a given subgroup. The relation between linear representations and nonlinear realizations is

Callan, Coleman, Wess and Zumino: 1968

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$$\xi = e^{i\Pi/f}, \quad \Pi = \pi^a X^a,$$

$$g\,\xi = \xi'\,U(g,\xi) \qquad \Pi' = \Pi'(\Pi, g)$$

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This is a complicated mess. So complicated that CCWZ didn't want to deal with it!

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$$\xi = e^{i\Pi/f}, \qquad \Pi =$$

This is the shift symmetry mentioned in the beginning.

$$g\,\xi = \xi'\,U(g,\xi) \qquad \Pi' = \Pi(\Pi, g)$$

However, we know that for

$$g = e^{i\varepsilon^a X^a}\,, \qquad \pi^{a\,\prime} = \pi^a + \varepsilon^a + \cdots$$

Instead, CCWZ looked for objects that have "simple" transformation properties under the action of G.

These are contained in the Cartan-Maurer one-form:

$$\xi^\dagger \partial_\mu \xi \; = \; i\mathcal{D}^a_\mu X^a + i\mathcal{E}^i_\mu T^i \equiv i\mathcal{D}_\mu + i\mathcal{E}_\mu$$

They are the "Goldstone covariant derivative" and the "associated gauge field",

$$\mathcal{D}_\mu \to U\mathcal{D}_\mu U^{-1} \; , \quad \mathcal{E}_\mu \to U\mathcal{E}_\mu U^{-1} - (\partial_\mu U)U^{-1}$$

upon which the complete effective lagrangian can be built (apart from the topological terms)

$$\mathcal{L}_{eff} = \frac{f^2}{2}\mathrm{Tr}\mathcal{D}_\mu \mathcal{D}^\mu + \cdots$$

In this fashion, CCWZ circumvents the problem of working out how the pions transform under the broken G:

$$\xi = e^{i\Pi/f}, \qquad g\,\xi = \xi' \, U(g,\xi)$$

$$\Pi' = \Pi'(\Pi, g)$$

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Not needed for writing down the EFT.

CCWZ is extremely powerful, but it adopts a “top-down” perspective, which requires knowing ahead of time what the broken group “G” is in the UV.

It also obscures the fact that Goldstone bosons are infrared degrees of freedoms that connect different vacua.

It turns out there is an alternative to CCWZ. It dictates that interactions of Goldstone bosons are determined by specifying

- the Adler's zero condition
- The unbroken symmetry H

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An EFT for the NLSM can be derived without mentioning the broken group G, up to the normalization of decay constant f:

$$\mathcal{L}^{(2)} = \frac{1}{2}\langle \mathcal{D}_\mu \pi | \mathcal{D}^\mu \pi \rangle \ , \quad |\mathcal{D}_\mu \pi\rangle = \frac{\sin\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} |\partial_\mu \pi\rangle \qquad \mathcal{T} \equiv \frac{1}{f^2} |T^i \pi\rangle\langle \pi T^i|$$

where the generators of H are anti-symmetric and purely imaginary:

$$(T^i)^* = -T^i = (T^i)^T$$

Low: 1412.2145
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Moreover, one can derive the nonlinear shift in general:

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It is

$$|\pi\rangle \to |\pi\rangle + F_1(\mathcal{T})|\epsilon\rangle \ , \quad F_1(\mathcal{T}) = \sqrt{\mathcal{T}} \cot \sqrt{\mathcal{T}}$$

$$\mathcal{T} \equiv \frac{1}{f^2} |T^i \pi\rangle\langle \pi T^i|$$

under which the leading two-derivative lagrangian is invariant:

$$\mathcal{L}^{(2)} = \frac{1}{2} \langle \mathcal{D}_\mu \pi | \mathcal{D}^\mu \pi \rangle \ , \quad |\mathcal{D}_\mu \pi\rangle = \frac{\sin \sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} |\partial_\mu \pi\rangle$$

Given the knowledge of the general nonlinear shift, it is straightforward to work out the corresponding Ward identity:

$$
i\partial_\mu \langle 0| \left\{ [F_4(\mathcal{T})]_{ab}\, \partial^\mu \pi^b \right\}(x) \prod_{i=1}^{n} \pi^{a_i}(x_i)|0\rangle
= \sum_{r=1}^{n} \langle 0|\pi^{a_1}(x_1)\cdots [F_1(\mathcal{T})]_{a_r a}(x_r)\delta^{(4)}(x-x_r)\cdots \pi^{a_n}(x_n)|0\rangle .
$$

$$
F_4(\mathcal{T}) = \frac{\sin\sqrt{\mathcal{T}}\cos\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} \qquad F_1(\mathcal{T}) = \sqrt{\mathcal{T}}\cot\sqrt{\mathcal{T}}
$$

Zhewei Yin and IL: 1709.08639

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$$F_4(\mathcal{T}) = \frac{\sin\sqrt{\mathcal{T}}\cos\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} \qquad\qquad F_1(\mathcal{T}) = \sqrt{\mathcal{T}}\cot\sqrt{\mathcal{T}}$$

Why is this Ward identity interesting?

Zhewei Yin and IL: 1709.08639

Given the knowledge of the general nonlinear shift, it is straightforward to work out the corresponding Ward identity:

$$i\partial_\mu \langle 0| \left\{ [F_4(\mathcal{T})]_{ab}\, \partial^\mu \pi^b \right\}(x) \prod_{i=1}^{n} \pi^{a_i}(x_i)|0\rangle$$
$$= \sum_{r=1}^{n} \langle 0|\pi^{a_1}(x_1)\cdots [F_1(\mathcal{T})]_{a_r a}(x_r)\delta^{(4)}(x-x_r)\cdots \pi^{a_n}(x_n)|0\rangle.$$

$$F_4(\mathcal{T}) = \frac{\sin\sqrt{\mathcal{T}}\cos\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}} \qquad F_1(\mathcal{T}) = \sqrt{\mathcal{T}}\cot\sqrt{\mathcal{T}}$$

Why is this Ward identity interesting?

Upon LSZ reduction, this is a new representation of scattering amplitudes of Goldstone bosons, that is different from the Feynman diagrams.

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The amplitudes in NLSM can then be written as

$$M^{a_1 \cdots a_{n+1}} \to \frac{1}{\sqrt{Z}} \sum_{k=1}^{\infty} \frac{-(-4)^k}{(2k+1)!}$$

$$\times \; \tau \, \langle 0| \int d^4x \, [\mathcal{T}^k(x)]_{ab} \; ip_{n+1} \cdot \partial \, \pi^b(x) |\pi^{a_1} \cdots \pi^{a_n} \rangle$$

This is valid at the quantum level, and to all orders in 1/f.

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This momentum comes
from the ∂_μ in $\partial_\mu J^\mu$

The RHS is already directly proportional to the soft momentum and the Adler's zero is manifest!

To compare with CHY, we only need to go to the tree-level and flavor-ordered amplitudes. Recall that

$$M_{n+1}^{\mathrm{nl\sigma m}}(\mathbb{I}_{n+1}) = \tau \sum_{i=2}^{n-1} s_{n+1,i}\; M_n^{\mathrm{nl\sigma m}\oplus\phi^3}(\mathbb{I}_n|1,n,i) + \mathcal{O}(\tau^2)$$

For comparison we focus on the tree-level and flavor-ordered amplitudes:

$$M(\mathbb{I}_{n+1}) = \sum_{k=1}^{[n/2]} \frac{-(-4)^k}{(2k+1)!f^{2k}} \sum_{\{l_m\}} \sum_{j=1}^{2k-1} \left[\begin{pmatrix} 2k \\ j \end{pmatrix} (-1)^j - 1 \right] p_{n+1} \cdot q_{l_{j+1}}$$
$$\times \prod_{m=1}^{2k+1} J(l_{m-1}+1, \cdots, l_m) \; ,$$

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$$\times \prod_{m=1}^{2k+1} J(l_{m-1}+1,\cdots,l_m) \ ,$$

Match the coefficient we obtain $M_n^{\mathrm{nl}\sigma\mathrm{m}\oplus\phi^3}(\mathbb{I}_n|1,n,i)$

Some Remarks:

- The “extended theory” in CHY arises from the matrix elements of

$$\tilde{O}_k^a(q) = \int d^4x\, e^{-iq\cdot x}\partial_\mu \left\{\left[\mathcal{T}^k(x)\right]_{ab}\partial^\mu \pi^b(x)\right\} \qquad q \to 0$$

In this case, the matrix elements can be interpreted as “amplitudes”:

$$M^{1234} = \tau\,\frac{2}{3f^2}(T^i)_{4r}(T^i)_{sb}\langle 0|\int d^4x\,\pi^r\pi^s\,iq\cdot\partial\,\pi^b|1234\rangle$$

$$M_{n+1}^{\mathrm{nl}\sigma\mathrm{m}}(\mathbb{I}_{n+1}) = \tau\sum_{i=2}^{n-1} s_{n+1,i}\; M_n^{\mathrm{nl}\sigma\mathrm{m}\oplus\phi^3}(\mathbb{I}_n|1,n,i)$$

- CHY only provided "amplitudes" for the extended theory,

$$A_4(1^\Sigma, 2^\Sigma, 3^\phi, 4^\phi) = A_4(1^\Sigma, 2^\phi, 3^\Sigma, 4^\phi) = -s_{24}$$

$$A_5(1^\phi, 2^\phi, 3^\phi, 4^\Sigma, 5^\Sigma) = \frac{s_{34} + s_{45}}{s_{12}} + \frac{s_{45} + s_{15}}{s_{23}} - 1$$

$$A_5(1^\phi, 2^\phi, 3^\Sigma, 4^\phi, 5^\Sigma) = \frac{s_{34} + s_{45}}{s_{12}} - 1$$

while we get a little bit more information,

$$V^{\mathrm{nl\sigma m}\oplus\phi^3}(\mathbb{I}_{2k+1}|1, 2k+1, j) = \frac{i}{2}\frac{-(-4)^k}{(2k+1)!f^{2k}}\left[\begin{pmatrix} 2k \\ j-1 \end{pmatrix}(-1)^{j-1} - 1\right]$$

But a concrete formulation remains somewhat elusive.

- This approach also gives a unified treatment to theories with "enhanced" Adler's zeros.

Dirac-Born-Infeld:

$$\mathcal{L} = -F^d\sqrt{1-\frac{(\partial\phi)^2}{F^d}} + F^d \qquad \delta\phi = \theta_\mu\left(x^\mu - F^{-d}\phi\partial^\mu\phi\right)$$

$$\begin{aligned} & i\partial_\mu\partial_\nu\langle\left[g^{\mu\nu}\phi\sqrt{1-\frac{(\partial\phi)^2}{F^d}} + \frac{F^{-d}\phi\partial^\mu\phi\partial^\nu\phi}{\sqrt{1-(\partial\phi)^2/F^d}}\right](x)\prod_{i=1}^{n}\phi(x_i)\rangle \\ = & \sum_{r=1}^{n}\langle\phi(x_1)\cdots\left\{\delta^{(4)}(x-x_r)\right. \\ & \left.+\partial_\mu\left[F^{-d}\phi(x)\partial^\mu\phi(x)\delta^{(4)}(x-x_r)\right]\right\}\cdots\phi(x_n)\rangle, \end{aligned}$$

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Special Galileon:

$$\pi \;\to\; \pi + \theta^{\mu\nu}(\alpha^2 x_\mu x_\nu - \partial_\mu\pi\partial_\nu\pi)$$

$$\begin{aligned}
\partial_\rho J^{(\mu\nu)\rho} \;&=\; (\alpha^2 X^{T\mu\nu} + \partial^\mu\pi\partial^\nu\pi)\left(\mathcal{L}_1^{\rm TD} - \frac{1}{6\alpha^2}\mathcal{L}_3^{\rm TD} + \frac{1}{120\alpha^4}\mathcal{L}_5^{\rm TD}\right) \\
&=\; \alpha^2\partial_\rho\left[X^{T\mu\nu}\overleftrightarrow{\partial_\lambda}\pi T_1^{\rho\lambda}\right] - \frac{1}{6}\partial_\rho\left[X^{T\mu\nu}\overleftrightarrow{\partial_\lambda}\pi T_3^{\rho\lambda}\right] \\
&\quad + \frac{2}{3}\partial_\rho\left[\pi\partial^\mu\pi T_2^{\nu\rho}\right] - \frac{1}{15\alpha^2}\partial_\rho\left[\pi\partial^\mu\pi T_4^{\nu\rho}\right] \\
&\quad + \frac{1}{3}\partial_\rho\left[\partial_\mu\pi\partial_\nu\pi\partial_\lambda\pi T_1^{\rho\lambda}\right] + \frac{1}{10\alpha^2}\left[\partial_\mu\pi\partial_\nu\pi\partial_\lambda\pi T_3^{\rho\lambda}\right],
\end{aligned}$$

Zhewei Yin and IL: to appear

- It is clear that the infrared structure of NLSM, and QFT in general, is much richer than we knew.

- The shift symmetry perspective seems particularly well-suited to study the soft limit of scattering amplitudes.

We have just begun our exploration!