Holographic Entanglement Entropy and Black Hole Entropy

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December 7, 2017

NCTS Annual Theory Meeting 2017: Particles, Cosmology and Strings

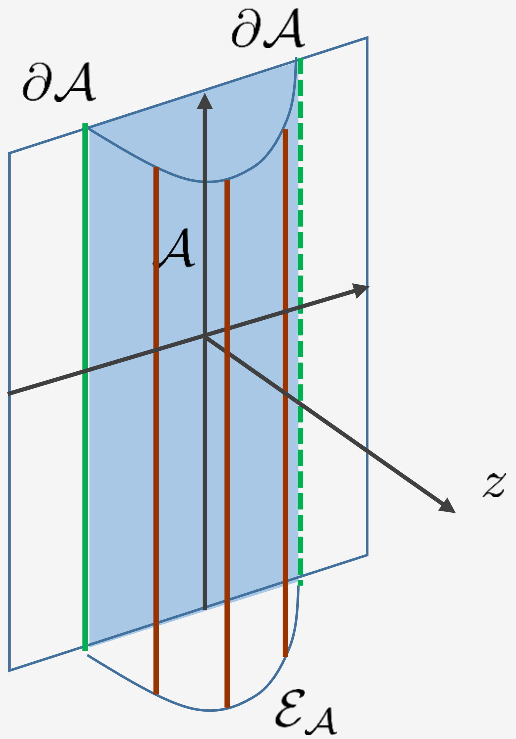
Outline

- Holographic Entanglement Entropy:
  - Ryu-Takayanagi formula
- Pure AdS Spacetime:
  - Strait Belt Boundary
- AdS-Schwarzschild Black Hole Spacetime
- Relation Between HEE and BH Entropy
- Summary

Holographic Entanglement Entropy

- We can calculate the entanglement entropy of an area $\mathcal{A}$ in a $(d+1)$-dimensional CFT by Ryu-Takayanagi formula:

$$S_{\mathcal{A}} = \frac{\mathrm{Area}(\mathcal{E}_{\mathcal{A}})}{4G_N^{(d+2)}}.$$



Reference: Rangamani, Takayanagi, arXiv: 1609.01287

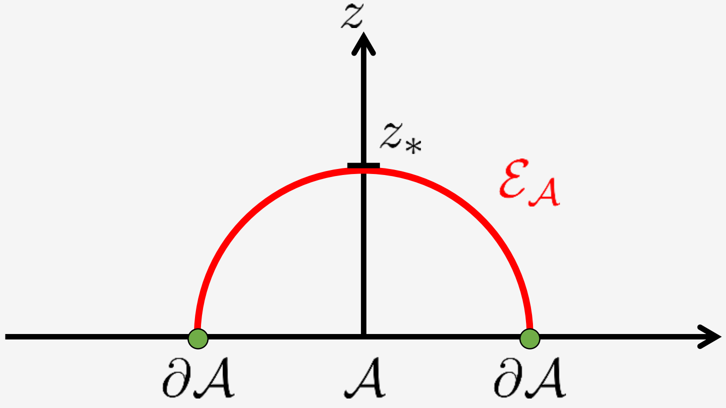
Pure AdS Spacetime

- Parametrize the surface by ξ^i and the metric can be expressed as:

$$ds^2_{\mathcal{M}} = g_{AB} dX^A dX^B.$$

Now $X^A(\xi^i)$ is the surface we are interested in, so

$$ds^2_{\mathcal{E}_\mathcal{A}} = h_{ij} d\xi^i d\xi^j = \left(g_{AB}(z,x) \frac{\partial X^A}{\partial \xi^i} \frac{\partial X^B}{\partial \xi^j} \right) d\xi^i d\xi^j.$$



Pure AdS Spacetime

- The entanglement entropy expressed in area functional for $\mathcal{E}_{\mathcal{A}}$ is

$$\mathcal{S}_{extremal} = \frac{1}{4G_N^{(d+2)}} \int d^d\xi \sqrt{\det(h_{ij})}.$$

- The $(d+2)$-dim AdS spacetime metric expressed by global coordinates:

$$ds^2 = \ell_{AdS}^2 \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_d^2 \right),$$

or in the Poincaré coordinates:

$$ds^2 = \frac{\ell_{AdS}^2}{z^2} \left(dz^2 - dx_0^2 + \sum_{i=1}^{d} dx_i^2 \right).$$

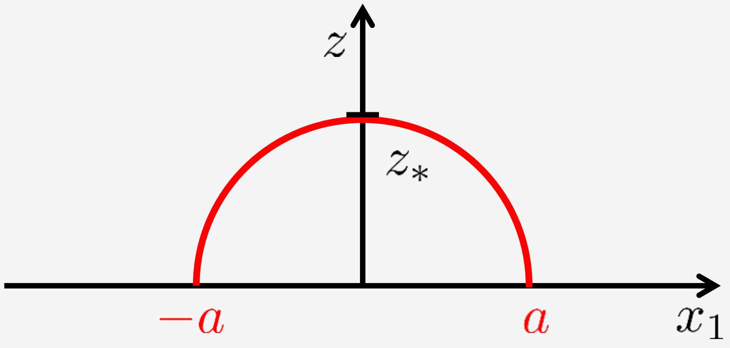
Pure AdS Spacetime

- The coordinates of a strait belt:

$$\mathcal{A}_S = \{\mathbf{x}_d \in \mathbb{R}^d \mid x_1 \in (-a, a), x_i \in \mathbb{R} \quad \text{for} \quad i = 2, 3, \cdots, d\}.$$

With translational symmetries in x_2 to x_d directions, $\rightarrow z = z(x_1)$, then the action(entanglement entropy) becomes

$$\mathcal{S}_{extremal} = \frac{\ell_{AdS}^d}{4G_N^{(d+2)}} \int dx_1 d^{d-1}\mathbf{x} \quad \frac{1}{z^d}\sqrt{\left(\frac{dz(x_1)}{dx_1}\right)^2 + 1}.$$



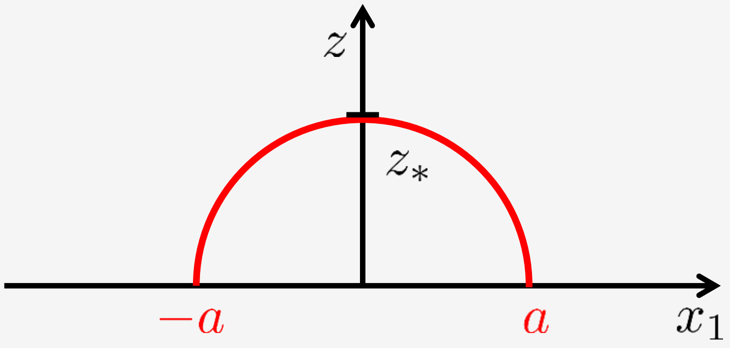
Pure AdS Spacetime

- Using canonical transformation to evaluate the Hamiltonian, we can find:

$$\frac{dz(x_1)}{dx_1} = \pm\frac{\sqrt{(z_*^{2d} - z^{2d})}}{z^d}.$$

- With $z(\pm a) = 0$, $\frac{dz(x_1)}{dx_1}\Big|_{z=z_*} = 0$, and change of variable $z = z_* y$,

$$a = \int_0^a dx_1 = \int_0^{z_*} z^d (z_*^{2d} - z^{2d})^{-1/2} dz = z_* \int_0^1 y^d \left(1 - y^{2d}\right)^{-1/2} dy,$$



Pure AdS Spacetime

- Require the volume of the $(d-1)$-dimensional space spanned by $(x_2, \cdots, x_d)$ on the minimal surface to be L^{d-1}:

$$\int d^{d-1}\mathbf{x} = L^{d-1}.$$

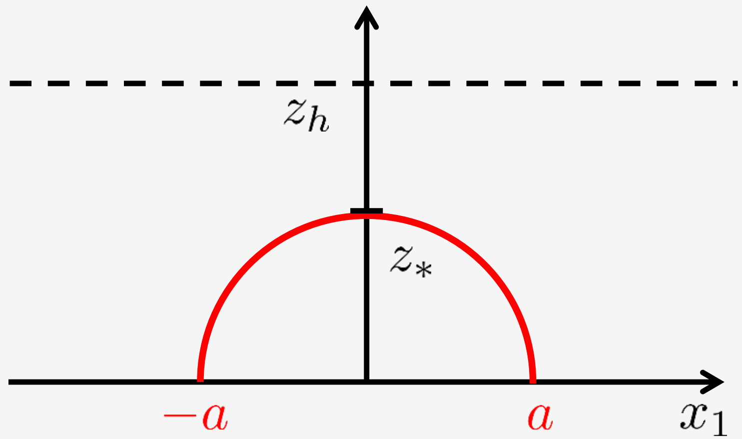
- The entanglement entropy:

$$\begin{aligned}
\mathcal{S}_{\mathcal{A}_S} &= \frac{l_{AdS}^d \cdot L^{d-1}}{2G_N^{(d+2)}} \cdot \frac{1}{z_*^{d-1}} \int_0^1 dy y^{-d}\left(1-y^{2d}\right)^{-1/2} \\
&= \frac{l_{AdS}^d}{2G_N^{(d+2)}} \cdot \frac{L^{d-1}}{z_*^{d-1}} \left\{ \lim_{\epsilon\to 0} \frac{1}{(d-1)\epsilon^{d-1}} - \frac{1}{(d-1)} \int_0^1 \frac{y^d}{\sqrt{1-y^{2d}}} dy \right\} \\
&= \frac{l_{AdS}^d}{2G_N^{(d+2)}} \cdot \frac{L^{d-1}}{z_*^{d-1}} \left\{ \lim_{\epsilon\to 0} \frac{1}{(d-1)\epsilon^{d-1}} - \frac{1}{(d-1)} \cdot \frac{a}{z_*} \right\}.
\end{aligned}$$

AdS-Schwarzschild Black Hole Spacetime

- The metric of an $(d+2)$-dimensional asymptotic AdS black hole is

$$ds^2 = \frac{l_{AdS}^2}{z^2}\left[-\left(1-\frac{z^{d+1}}{z_h^{d+1}}\right)dt^2 + \frac{dz^2}{\left(1-\frac{z^{d+1}}{z_h^{d+1}}\right)} + \sum_{i=1}^{d}(dx^i)^2\right].$$



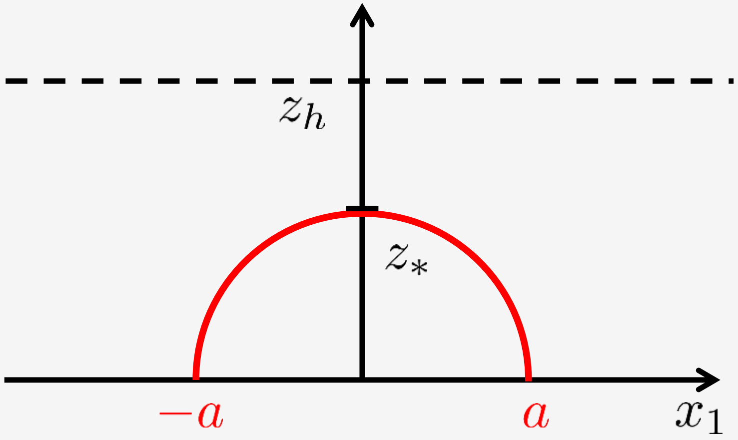
AdS-Schwarzschild Black Hole Spacetime

- The coordinates of a strait belt:

$$\mathcal{A}_S = \{\mathbf{x}_d \in \mathbb{R}^d \mid x_1 \in (-a, a), x_i \in \mathbb{R} \quad \text{for} \quad i = 2, 3, \cdots, d\}.$$

- The entanglement entropy:

$$\mathcal{S}_{extremal} = \frac{\ell_{AdS}^d}{4G_N^{(d+2)}} \int dx_1 d^{d-1}\mathbf{x} \quad \frac{1}{z^d} \sqrt{\left(\frac{dz(x_1)}{dx_1}\right)^2 + 1}.$$



AdS-Schwarzschild Black Hole Spacetime

- Using canonical transformation to find out the Hamiltonian, we find:

$$\frac{dz(x_1)}{dx_1} = \pm \frac{\sqrt{\left(1 - \frac{z^{d+1}}{z_h^{d+1}}\right)(z_*^{2d} - z^{2d})}}{z^d}.$$

- With $z = z_* y$,

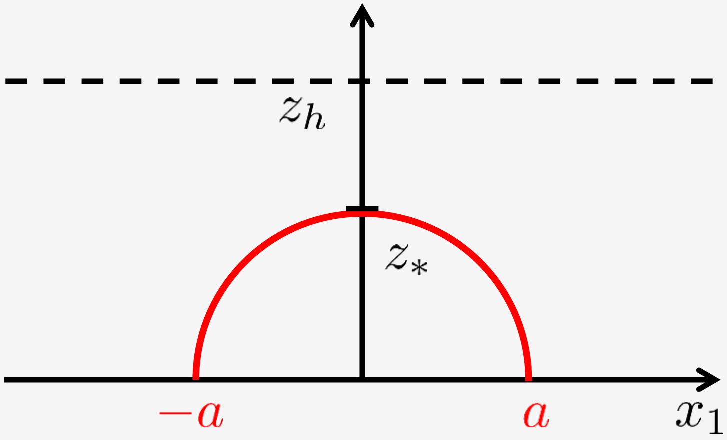
$$a = z_* \int_0^1 \frac{y^d}{\sqrt{(1 - y^{2d})\left(1 - \frac{z_*^{d+1}}{z_h^{d+1}} y^{d+1}\right)}} dy.$$

AdS-Schwarzschild Black Hole Spacetime

- The entanglement entropy:

$$\begin{aligned}\mathcal{S}_{\mathcal{A}_S} =& \frac{l_{AdS}^d}{2G_N^{(d+2)}} \cdot \frac{L^{d-1}}{z_*^{d-1}} \cdot \left[\lim_{\epsilon\to 0} \frac{1}{(d-1)} \cdot \frac{1}{\epsilon^{d-1}} + \frac{a}{z_*} \right. \\ &\left. + \frac{d}{(d-1)} \int_0^1 \frac{y(b^{d+1} - y^{d-1})}{\sqrt{(1-y^{2d})(1-b^{d+1}y^{d+1})}} - \frac{b^{d+1}}{2(d-1)} \int_0^1 \frac{(d+3)y\sqrt{1-y^{2d}}}{\sqrt{1-b^{d+1}y^{d+1}}} \right],\end{aligned}$$

here $b = \frac{z_*}{z_h}$.



AdS-Schwarzschild Black Hole Spacetime

- When $b \to 1$, the boundary conditions gives

$$\frac{a}{z_h} = \int_0^1 \frac{y^d}{\sqrt{(1-y^{2d})(1-y^{d+1})}} dy.$$

- Entanglement entropy can be approximated as

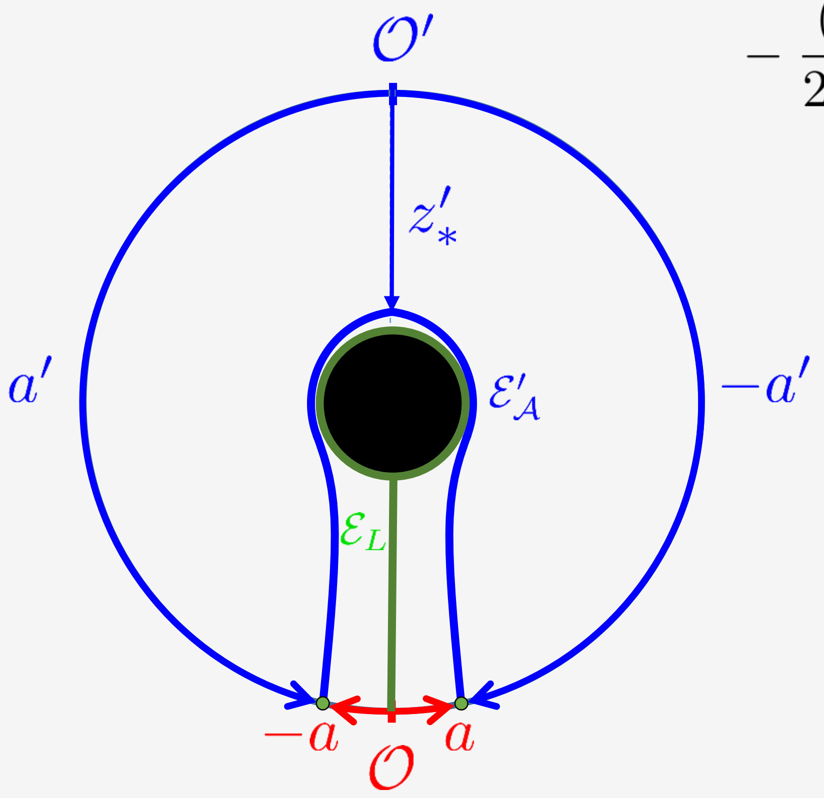
$$\mathcal{S}_{\mathcal{A}_S} = \frac{l_{AdS}^d}{2G_N^{(d+2)}} \cdot \frac{L^{d-1}}{z_h^{d-1}} \cdot \left\{ \lim_{\epsilon \to 0} \frac{1}{(d-1)\epsilon^{d-1}} + \frac{a}{z_h} \right.$$
$$\left. + \frac{d}{(d-1)} \int_0^1 \frac{y(1+y+\cdots+y^{d-2})}{g(y)} dy - \frac{(d+3)}{2(d-1)} \int_0^1 \frac{y\sqrt{1-y^{2d}}}{\sqrt{1-y^{d+1}}} dy \right\},$$

here $g(y) \equiv \sqrt{(1+y+\cdots+y^{2d-1}) \cdot (1+y+\cdots+y^d)}$.

- The first divergent term is the same as in pure AdS space and the second divergent term comes from the contribution of black hole entropy.

Relation Between HEE and BH Entropy

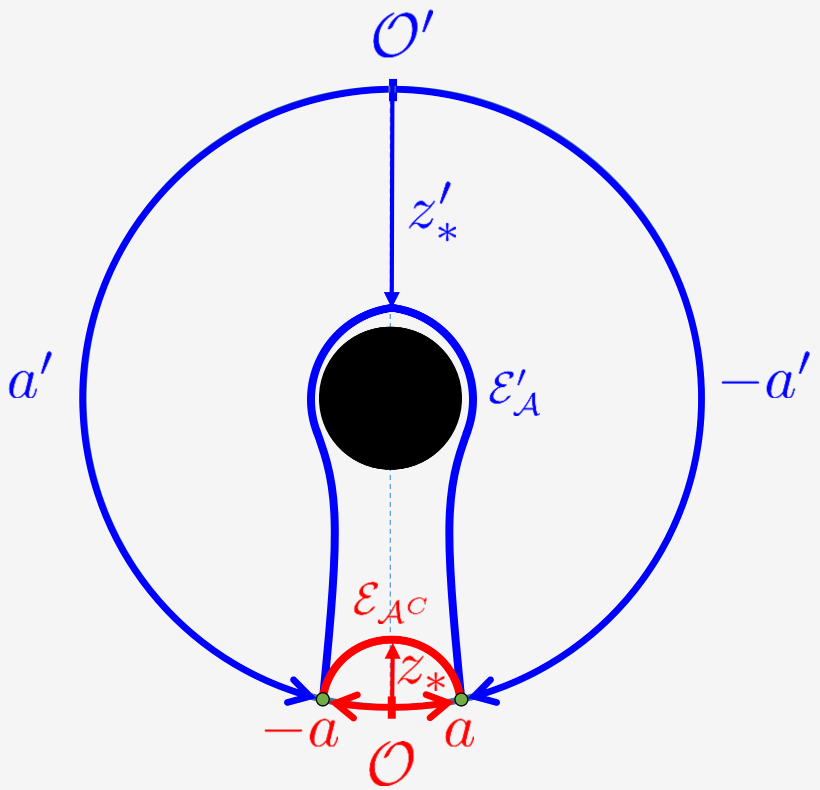
$$\mathcal{E}'^{(b\to 1)}_{\mathcal{A}} - \mathcal{E}_L = \frac{2l^d_{AdS}\cdot L^{d-1}}{z_h^{d-1}}\left\{\frac{a'}{z_h} - \frac{d}{(d-1)}\int_0^1 \frac{y^{2d}(1-y)}{\sqrt{1-y^{d+1}}\cdot\sqrt{1-y^{2d}}}dy - \frac{(d-3)}{2(d-1)}\int_0^1 \frac{y\left(1-\sqrt{1-y^{2d}}\right)}{\sqrt{1-y^{d+1}}}dy\right\}$$

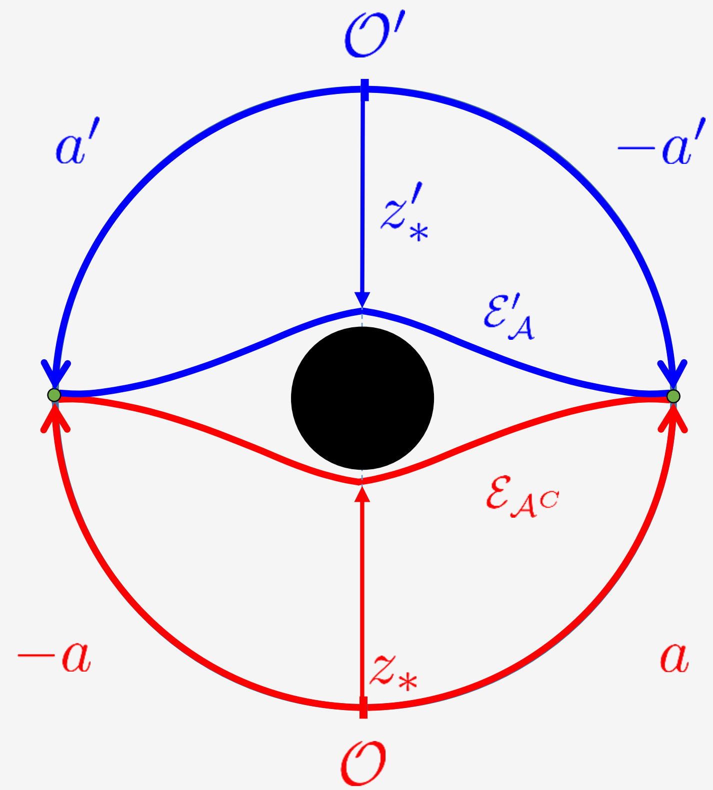


Relation Between HEE and BH Entropy

- Araki-Lieb Inequality:

$$|\mathcal{S}_{\mathcal{A}} - \mathcal{S}_{\mathcal{A}^C}| \leq \mathcal{S}_\rho, \quad \rho \text{is the entire density matrix.}$$





Summary

- The holographic entanglement entropy of ground state (d+1)-dimensional CFT.
- The entanglement holographic entropy of thermal-excited state (d+1)-dimensional CFT.
- Holographic interpretation of Araki-Lieb inequality.
- Phase transition between two kinds of entropy?
- The boundary effects when considering BCFT?

Thank you