NCTS annual meeting, 8 Dec., 2017

Complexity and AdS/CFT

- Quantum Othello Game-

ArXiv:1707.03840(hep-th)

Koji Hashimoto (Osaka u)
w/ Norihiro Iizuka,
Sotaro Sugishita

Abelian gauge theories with

Complexity = exp[Entropy]

is maximally nonlocal.

1

Review

Complexity ∼ exp[Entropy] in gravity dual

6pp.

2

Complexity of Abelian Gauge Theories

For k-local Hamiltonians,

$$\begin{cases} k \ll S \ : \ \mathcal{C} \sim S^k \\ k \sim S \ : \ \mathcal{C} \sim e^S \end{cases}$$

6pp.

1

Review

Complexity $\sim$ exp[Entropy] in gravity dual

Eternal BH has growing extremal surface.

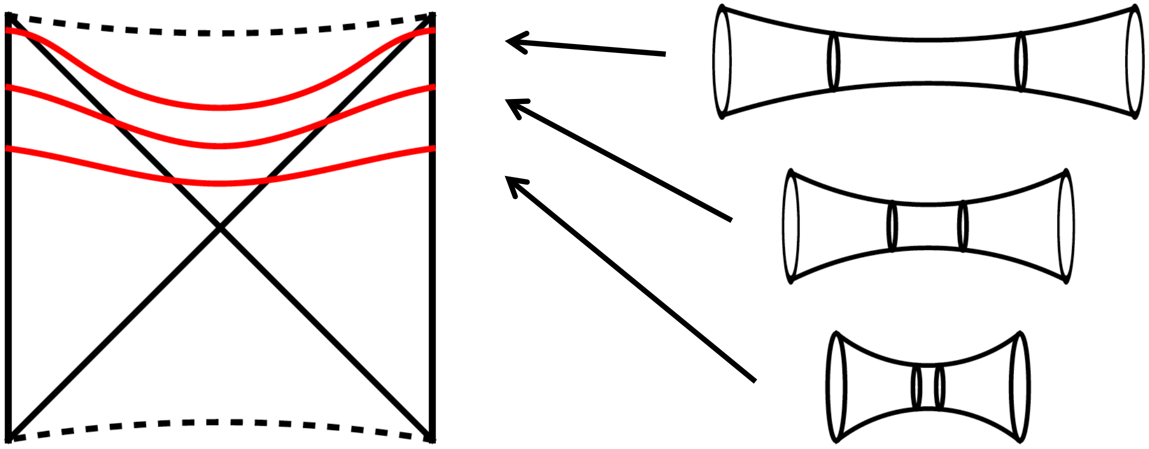
Susskind: Dual is Complexity.

How to calculate Complexity in QFTs.

Complexity = # gates for reconstructing e^{-iHt} .

Universal gate set.

Eternal BH has growing extremal surface.



Eternal black hole

Extremal surfaces

[Hartman, Maldacena]

Dual

Finite temperature CFT

Dual

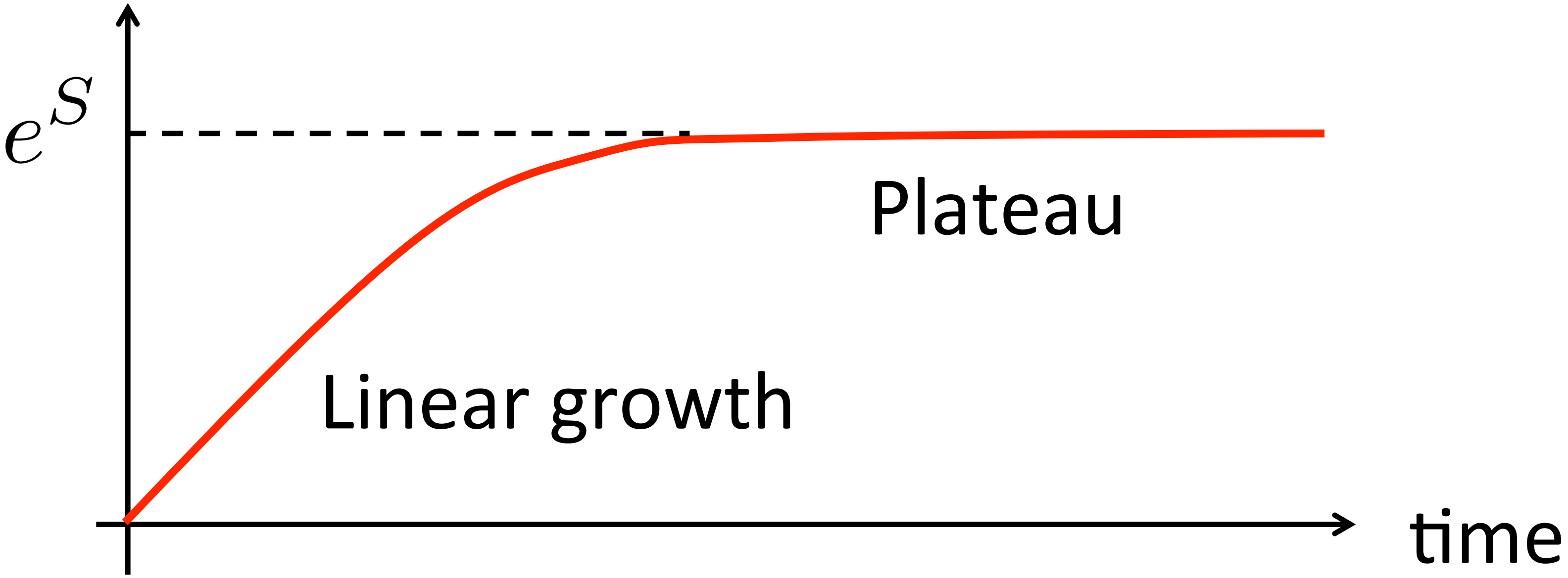
Complexity [Susskind]

[Susskind] [Stanford, Susskind][Brown, Roberts, Susskind, Swingle, Zhao]
and [Lehner, Myers, Poisson, Sorkin] [Couch, Fishler, Nguyen]
[Ben-Ami, Carmi] [Carmi, Myers, Rath], …

Susskind: Dual is Complexity.

[Susskind “Entanglement is not enough”]

$\mathcal{C}$: complexity



- Entanglement entropy grows till S (Ryu-Takayanagi surface, not go into horizon)
- Complexity grows till exp[S]

How to calculate Complexity in QFTs.

Our strategy: Discretize gauge theories.

U(1) gauge theory in 3d { U(1) → Z_N → Z_2
2d sp. → L x L lattice } L^2-qubit system

Cf) Z_2 case includes Kitaev's toric code. [Kitaev]

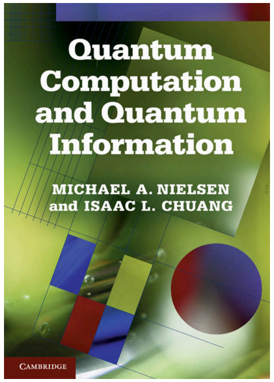
Cf) Free scalar theories

[Chapman, Heller, Morrochio, Pastawski]
[Jefferson, Myers] [Yang]

Cf) 1+1d information metric.

[Miyaji, Numasawa, Shiba, Takayanagi, Watanabe]

C = # gates for reconstructing e^{-iHt}.



Def Complexity $\mathcal{C}(U)$:

For unitary operator U , $\mathcal{C}(U)$ is a minimum # of gates U_i to satisfy

$$||U - U_1 U_2 \cdots U_{\mathcal{C}}|| < \epsilon$$

- $U_i \in$ Universal gate set
- Trace norm :

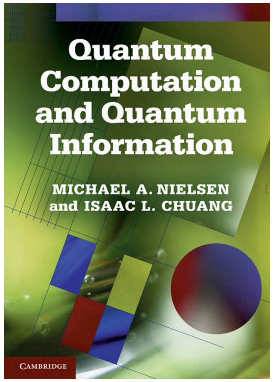
$$\Big|\Big| \; U - V \; \Big|\Big|^2 \equiv \frac{1}{\mathrm{Tr}(1)} \mathrm{Tr}\Big[(U - V)^\dagger (U - V)\Big]$$

- $\epsilon(\ll 1)$: Regularization cut-off

Universal gate set.

Def Universal gate set :

a set of unitary operators which can reconstruct any unitary operator



Ex) n-qubit system (2^n x 2^n unitary matrix)

Basis of states: $|0\rangle \otimes |1\rangle \otimes \cdots \otimes |0\rangle, \cdots$

Universal gate set: $\{U_i, U_{i,j}\}$

$$\begin{cases} U_i & \text{: single qubit gate acting on i-th qubit} \\ U_{i,j} & \text{: gate entangling i-th and j-th qubits} \end{cases}$$

Universal gate set.

U_i : single qubit gate acting on i-th qubit

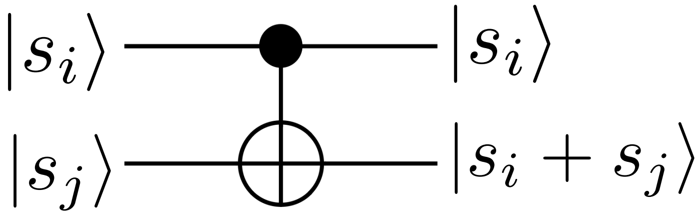
$$\alpha|0\rangle + \beta|1\rangle$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to U_i \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$U_{i,j}$: gate entangling i-th and j-th qubits

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

ex) CNOT gate



$$U^{\mathrm{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1

Review

Complexity ~ exp[Entropy] in gravity dual

Eternal BH has growing extremal surface.

Susskind: Dual is Complexity.

How to calculate Complexity in QFTs.

Complexity = # gates for reconstructing e^{-iHt} .

Universal gate set.

Review

1

Complexity ∼ exp[Entropy] in gravity dual

6pp.

2

Complexity of Abelian Gauge Theories

For k-local Hamiltonians,

$$\begin{cases} k \ll S \ : \ \mathcal{C} \sim S^k \\ k \sim S \ : \ \mathcal{C} \sim e^S \end{cases}$$

6pp.

2

Calculating Complexity

For k-local Hamiltonians,

$$\begin{cases} k \ll S \; : \; \mathcal{C} \sim S^k \\ k \sim S \; : \; \mathcal{C} \sim e^S \end{cases}$$

Discretization: Othello game board.

[Quantum, 2-site] Hamiltonian.

[Quantum, 2-site] Evolution of Complexity.

[Quantum, 2-site] Fastest Hamiltonian.

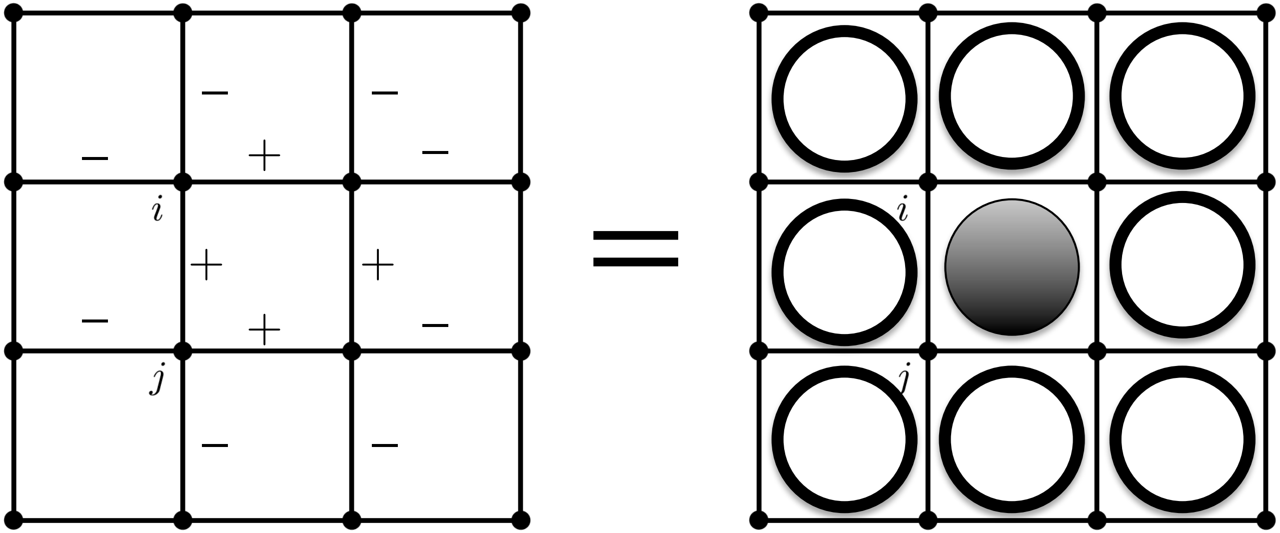
[Quantum, general] Z_2 , L^2 sites.

[Quantum, general] Z_N , L^2 sites.

Discretization: Othello game board.

[Fradkin, Shenker ’79]

Gauge invariant states of Z_2 gauge theory
= States of Ising model (qubits) on plaquettes



[Quantum, 2-site] Hamiltonian.

Generic Hamiltonian

$$H = \cancel{a_{00}\left(\mathbf{1}\otimes\mathbf{1}\right)} + \sum_{i=1}^{3} a_{0i}\left(\mathbf{1}\otimes\sigma_i\right) + \sum_{i=1}^{3} a_{i0}\left(\sigma_i\otimes\mathbf{1}\right) + \sum_{i,j} a_{ij}\left(\sigma_i\otimes\sigma_j\right)$$

Strategy: Time dependence of complexity of e^{-iHt} comes from diagonalized eigenvalues

$$\exp[-iHt] = U_0^\dagger \exp[-i\Lambda t] U_0$$

$$U_0 H U_0^\dagger = \Lambda \qquad \Lambda \equiv \mathrm{diag}\{e_1, e_2, e_3, e_4\}$$

U_0 ~ State-dependent part

[Quantum, 2-site] Evolution of Complexity.

Decomposition by gates:

$$\exp[-i\Lambda t] = \underline{U_1(a)}\,\underline{U_2(b)}\,\underline{U_{\rm ent}(c)}$$

Single-qubit gates

Entangling gate

$$\begin{cases} U_1(a) \equiv \exp[-iat(\sigma_3 \otimes \mathbf{1})]\,, \\ U_2(b) \equiv \exp[-ibt(\mathbf{1} \otimes \sigma_3)]\,, \\ U_{\rm ent}(c) \equiv \exp[-ict(\sigma_3 \otimes \sigma_3)] = U_{12}^{\rm CNOT} U_2(c) U_{12}^{\rm CNOT} \end{cases}$$

$$\begin{pmatrix} e_1 = a+b+c\,, & e_2 = a-b-c\,, \\ e_3 = -a+b-c\,, & e_4 = -a-b+c \end{pmatrix}$$

Time-dependence of complexity :

$$\mathcal{C}(t) = \theta(|\sin at| - \epsilon) + \theta(|\sin bt| - \epsilon) + 3\theta(|\sin ct| - \epsilon)$$

[Quantum, 2-site] Fastest Hamiltonian.

Time to reach the maximum complexity

$$t_{\max} = \epsilon \times \max\{|a|^{-1}, |b|^{-1}, |c|^{-1}\}$$

Complexity growth rate

$$v \equiv \mathcal{C}_{\max}/t_{\max} = \frac{5}{\epsilon}\min\{|a|, |b|, |c|\}$$

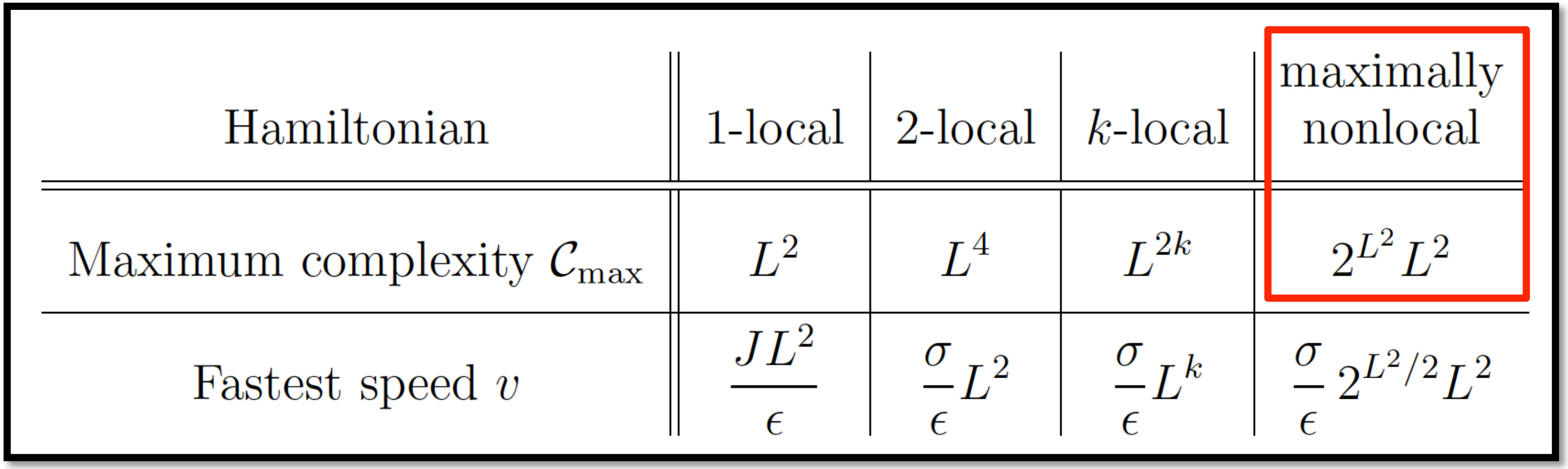
Fixing variance of eigenvalues, $\sigma^2 = \frac{1}{4}\sum_{k=1}^{4} e_k^2 = a^2 + b^2 + c^2$

the fastest Hamiltonian is with $|a| = |b| = |c| = \frac{\sigma}{\sqrt{3}}$,

$$H = U_0^\dagger \left[J(\sigma_3 \otimes \mathbf{1}) + J(\mathbf{1} \otimes \sigma_3) \pm J(\sigma_3 \otimes \sigma_3)\right] U_0$$

Fastest = All entangling, all equal weight

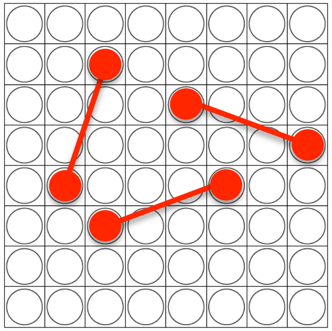
[Quantum, general] Z_2, L^2 sites.



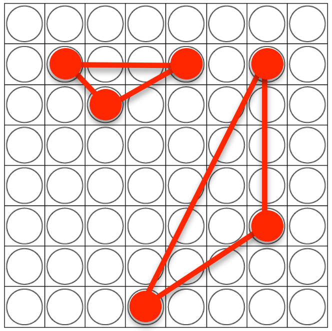
Hamiltonian	1-local	2-local	k-local	maximally nonlocal
Maximum complexity $\mathcal{C}_{\max}$	L^2	L^4	L^{2k}	$2^{L^2} L^2$
Fastest speed v	$\frac{JL^2}{\epsilon}$	$\frac{\sigma}{\epsilon} L^2$	$\frac{\sigma}{\epsilon} L^k$	$\frac{\sigma}{\epsilon} 2^{L^2/2} L^2$

Def "k-local Hamiltonian" has terms entangling at most k qubits.

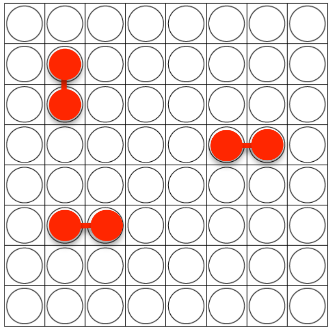
Ex) 2-local



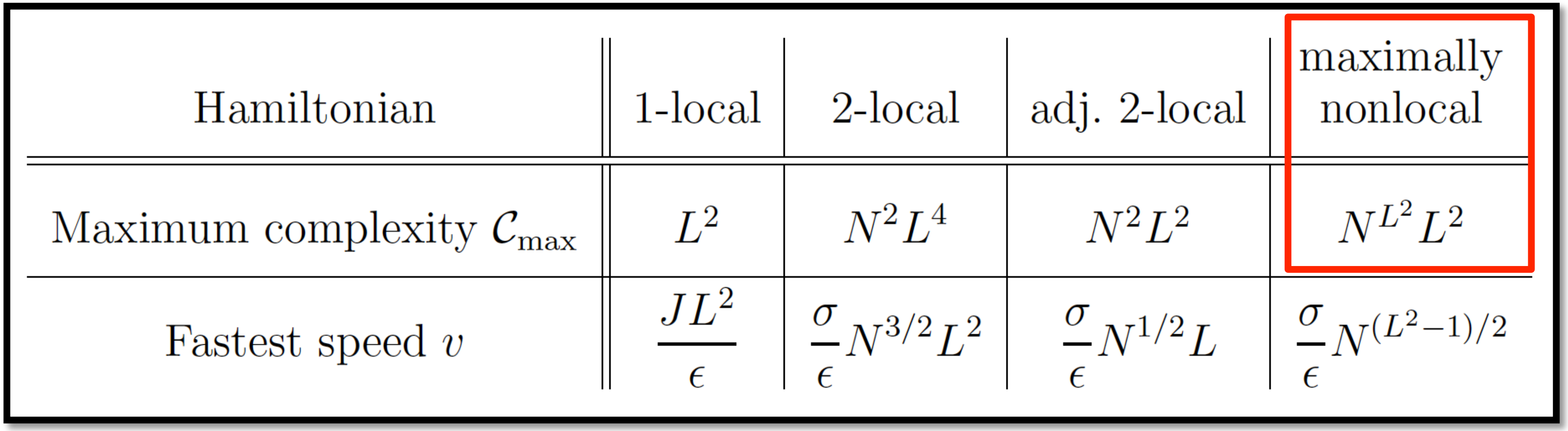
3-local



Adjacent 2-local



[Quantum, general] Z_N , L^2 sites.



Hamiltonian	1-local	2-local	adj. 2-local	maximally nonlocal
Maximum complexity $\mathcal{C}_{\max}$	L^2	N^2L^4	N^2L^2	$N^{L^2}L^2$
Fastest speed v	$\frac{JL^2}{\epsilon}$	$\frac{\sigma}{\epsilon}N^{3/2}L^2$	$\frac{\sigma}{\epsilon}N^{1/2}L$	$\frac{\sigma}{\epsilon}N^{(L^2-1)/2}$

General result for k-local Hamiltonian

$$\begin{cases} k \ll S \ : \ \mathcal{C} \sim S^k \\ k \sim S \ : \ \mathcal{C} \sim e^S \end{cases}$$

2

Calculating Complexity

For k-local Hamiltonians,
$$\begin{cases} k \ll S \;:\; \mathcal{C} \sim S^k \\ k \sim S \;:\; \mathcal{C} \sim e^S \end{cases}$$

Discretization: Othello game board.

[Quantum, 2-site] Hamiltonian.

[Quantum, 2-site] Evolution of Complexity.

[Quantum, 2-site] Fastest Hamiltonian.

[Quantum, general] Z_2, L^2 sites.

[Quantum, general] Z_N, L^2 sites.

Review

1

Complexity ~ exp[Entropy] in gravity dual

6pp.

2

Complexity of Abelian Gauge Theories

For k-local Hamiltonians,

$$\begin{cases} k \ll S \ : \ \mathcal{C} \sim S^k \\ k \sim S \ : \ \mathcal{C} \sim e^S \end{cases}$$

6pp.

Abelian gauge theories with

Complexity = exp[Entropy]

is maximally nonlocal.