Fractal Spacetime in Quantum & Stochastic Gravity from the Quantum Brownian Motion perspective

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-Peyresq, France, June 2016

Outline

0. Main theme: (backreaction) Effects of non-Gaussian noise from fluctuations of quantum fields on the spacetime near the Planck scale:

Does it imply Fractal (multiscale) spacetimes?

1. **Backreaction of Fluctuations of Quantum Fields on the background spacetime dynamics:** Quick review of **Stochastic Gravity** (1994 --)

Two pt function of Tmn as source driving the Einstein Equation. (Gaussian)

Einstein-Langevin Equation : **Hu** with Calzetta, Matacz, Sinha 1994-95; **Verdaguer** with Campos, Martin, Roura 1996-2000, Mazzitelli with Dalvit, Lombardo 1996

2. **Higher moments of Tmn** in Minkowski Spacetime: even at **low energy,** under test field conditions (no backreaction on spacetime) the higher moments **contribute significantly.** Fewster Ford Roman

3. **Near Planck scale**: **Dimensional Reduction.** Carlip, Ambjorn, Loll et al

spacetime at small length scales becomes **effectively 2 dim** ← UV, not IR

- **Spectral dimension** from Brownian motion.

  Use the **heat kernel expansion** for illustration ←QFTCST ' 70s.

4. **non-Gaussian noise** – a. Where do they arise?

- Simple case: interacting quantum fields will produce multiplicative colored noise. In QBM context, treated with perturbative techniques in Hu Paz Zhang PRD93,

  Example in S. A. Ramsey,* B. L. Hu,† and A. M. Stylianopoulos, *Nonequilibrium inflaton dynamics and reheating. II. Fermion production, noise, and stochasticity, Phys.Rev. D57, 6003-6021 (1998).*

b. **Anomalous Diffusion**: has very different spectral dimensions from Gaussian or normal diffusion: very different behavior at small scale vs large scale.

5. Fractal Spacetime: many theories of Quantum Gravity (`**Top Down**') predict the appearance of fractal dimensions, e.g. Ambjorn Loll / Reuter / Calcagni Oriti et al More later.

6. Does backreaction of nonGaussian noise bring forth a modification of the wave operator reflecting possible fractal spacetime dimension? ← **`Bottom-Up'**

I. Dimension Reduction / Fractal Spacetime

Some dimensional estimators

Following 5 slides are based on **S. Carlip's** 2016 Nijmegen talk. His recent review on dimensions and dimensional reduction: *Class. Quantum Grav.* **34** 193001 (2017) makes very good reading.

Geometric:

- Volume of a geodesic sphere
  - sphere of radius r, then $V \sim r^d$ defines a dimension
  - requires length, volume to be observables
  - may vary with scale

Topological: inductive and covering dimensions are integers. Box-counting and **Hausdorff** dimensions can be fractions

- Hausdorff dimension *d*_H
  - count number $N(r)$ of spheres of radius r needed to cover a region
- Spectral dimension $K(x,x';s) \sim (4\pi s)^{-d_S/2} e^{-\sigma(x,x')/2s}\left(1 + [a_1]s + [a_2]s^2 + \dots\right)$
  - random walk/diffusion described by heat kernel $K(x,x';s) \sim (4\pi s)^{-d/2} e^{-\sigma(x,x')/2s}(1+\mathcal{O}(s))$

    a_i HaMiDeW coefficients
  - "return probability" $K(x,x;s) \sim (4\pi s)^{-d_S/2}$ determines a dimension
  - no need for length or volume to be observables

$$d_S(s) = -2\frac{d \ln \langle K(x,x;s)\rangle_X}{d \ln s}$$

average over points *x* ε *X*.

- Walk dimension
  - start with same heat kernel
  - look at average distance as function of s

$$d_W = 2\left(\frac{d \ln \langle x^2 K(x,0;s)\rangle}{d \ln s}\right)^{-1}$$

*d*_W = 2*d*_H/*d*_S,

Physical processes:

- thermodynamic dimension
  - in d dimensions, free energy is $F/VT \sim T^{d-1}$
  - similar expressions for equation of state parameter, specific heat, etc.
- anomalous dimensions
  - in QFT, operators acquire anomalous scaling dimensions from quantum corrections
  - Greens function $G(x, x') \sim \sigma(x, x')^{-(d-2)/2}$ (or $\ln \sigma(x, x')$ in two dimensions)
  - anomalous dimensions flow with renormalization group: inherently scale-dependent

So what is the dimension of spacetime?

- High temperature string theory
  - Dimensional estimator: thermodynamic dimension
  - Reduction to $d_T = 2$ at high temperature
  - "...a lattice theory with a (1+1)-dimensional field theory on each lattice site" (Atick and Witten, 1988)
- Causal dynamical triangulations   Ambjorn J, Jurkiewicz J and Loll R 2005 *Phys. Rev. Lett.* **95** 171301
  - Dimensional estimator: spectral dimension
  - $d_S = 4$ at large scales, but drops to $d_S = 2$ at small scales ($\sim 100\ell_p$)
- Asymptotic safety   Weinberg 1979   more details in Next Slide
  - Dimensional estimator: anomalous dimension
  - If quantum gravity has UV fixed point, $d = 2$ at that scale
- Short distance Wheeler-DeWitt equation
  - Dimensional estimator: behavior of geodesics
  - In short distance limit, dominated by spacetimes in which geodesics "see" only two dimensions

Dimensional Reduction @ short distance

For the theory to remain valid at arbitrarily high energies or short distances, the renormalization group flow must have an ultraviolet fixed point. If it does, and if that fixed point has only a finite number of relevant directions, then the infinitely many couplings of the low energy theory would be determined by finitely many high energy parameters. Such a theory is 'asymptotically safe'.

What is the underlying physics?

Short answer: we don't know

But several interesting directions

Scale invariance

Asymptotic safety requires UV fixed point for RG flow

- Fixed point is necessarily scale invariant
- This guarantees $d = 2$ at fixed point (only in $d = 2$ is G_N dimensionless/scale free)

Perhaps one should reverse the argument:

- Demand symmetry at high energies
- Interpret $d = 4$ as symmetry-breaking phase

Dimensional Reduction in QG

G. ’t Hooft, Dimensional reduction in quantum gravity, in Salamfestschrift, A. Ali, J. Ellis and S. Randjbar-Daemi eds., World Scientific, Singapore (1993) [gr-qc/9310026]

J. Ambjørn, J. Jurkiewicz and R. Loll, Spectral dimension of the universe, Phys. Rev. Lett. 95 (2005) 171301

Astrid Eichhorn and Tim Koslowski, Towards **phase transitions between discrete and continuum quantum spacetime** from the renormalization group, PRD 90, 104039 (2014)

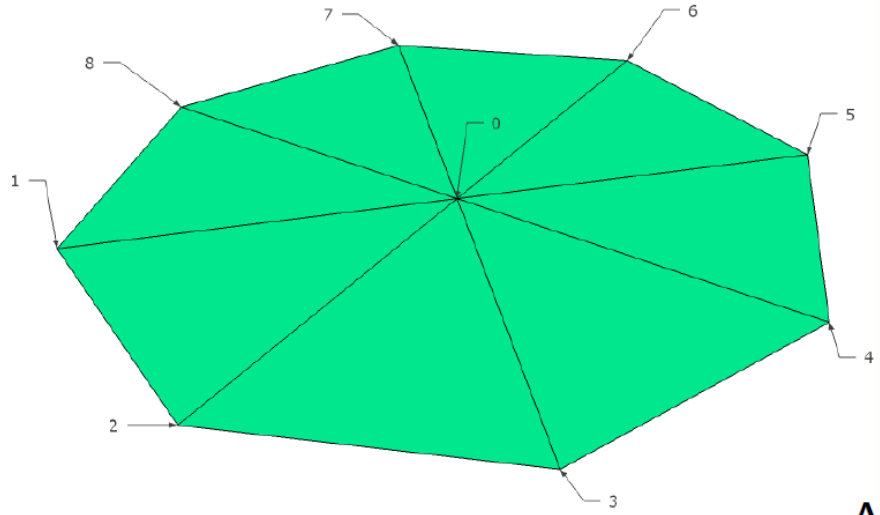
D. Benedetti and J. Henson, Spectral geometry as a probe of quantum spacetime, Phys. Rev. D 80 (2009) 124036 [arXiv:0911.0401]

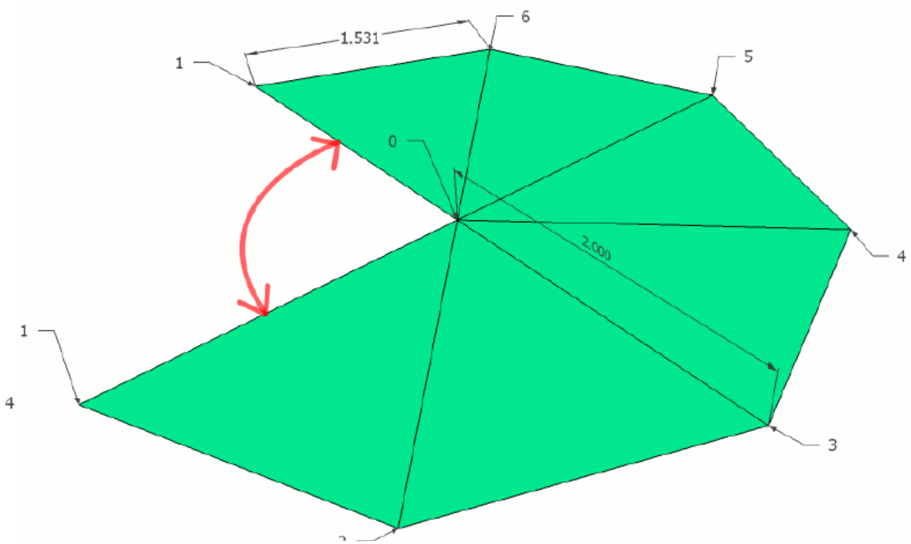
S. Carlip, Spontaneous dimensional reduction in short-distance quantum gravity?, AIP Conf. Proc. 1196 (2009) 72 [arXiv:0909.3329]

S. Carlip, The small scale structure of spacetime, in Foundations of Space and Time, G. Ellis, J. Murugan and A. Weltman eds., Cambridge University Press, Cambridge U.K. (2012) [arXiv:1009.1136]

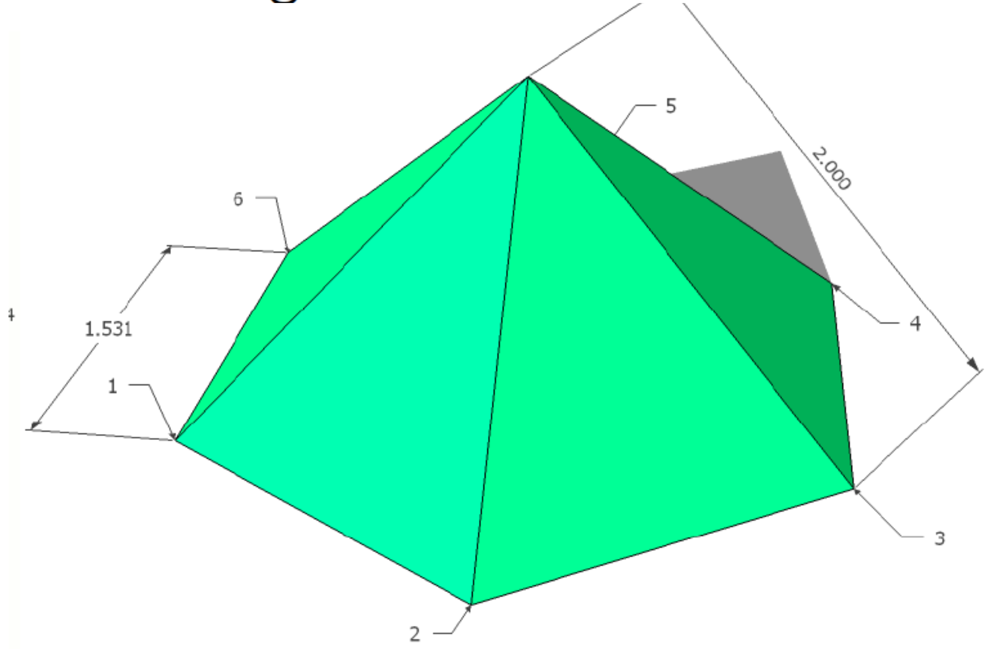
Fractal Spacetime

- O. Lauscher and M. Reuter, Fractal spacetime structure in **asymptotically safe gravity**, JHEP 10 (2005) 050
- G. Calcagni, Fractal universe and quantum gravity, Phys. Rev. Lett. 104 (2010) 251301 [arXiv:0912.3142]
- L. Modesto, Fractal structure of **loop quantum gravity**, Classical Quant. Grav. 26 (2009) 242002
- T.P. Sotiriou, M. Visser and S. Weinfurtner, Spectral dimension as a probe of the ultraviolet continuum regime of **causal dynamical triangulations**, Phys. Rev. Lett. 107 (2011) 131303 [arXiv:1105.5646]
- G. Calcagni, A. Eichhorn and F. Saueressig, Probing the quantum nature of spacetime by diffusion, Phys. Rev. D 87 (2013) 124028 [arXiv:1304.7247]
- G. Calcagni and G. Nardelli, Spectral dimension and diffusion in multiscale spacetimes, Phys. Rev. D 88 (2013) 124025, [arXiv:1304.2709].
- G. Calcagni, D. Oriti and J. Thürigen, Spectral dimension of **quantum geometries**, CQG 31 135014 (2014) **Group field theory**
- G. Calcagni, Multifractional theories: an unconventional review JHEP03(2017)138





A deficit angle is introduced $\epsilon = 2\pi - 6\theta$



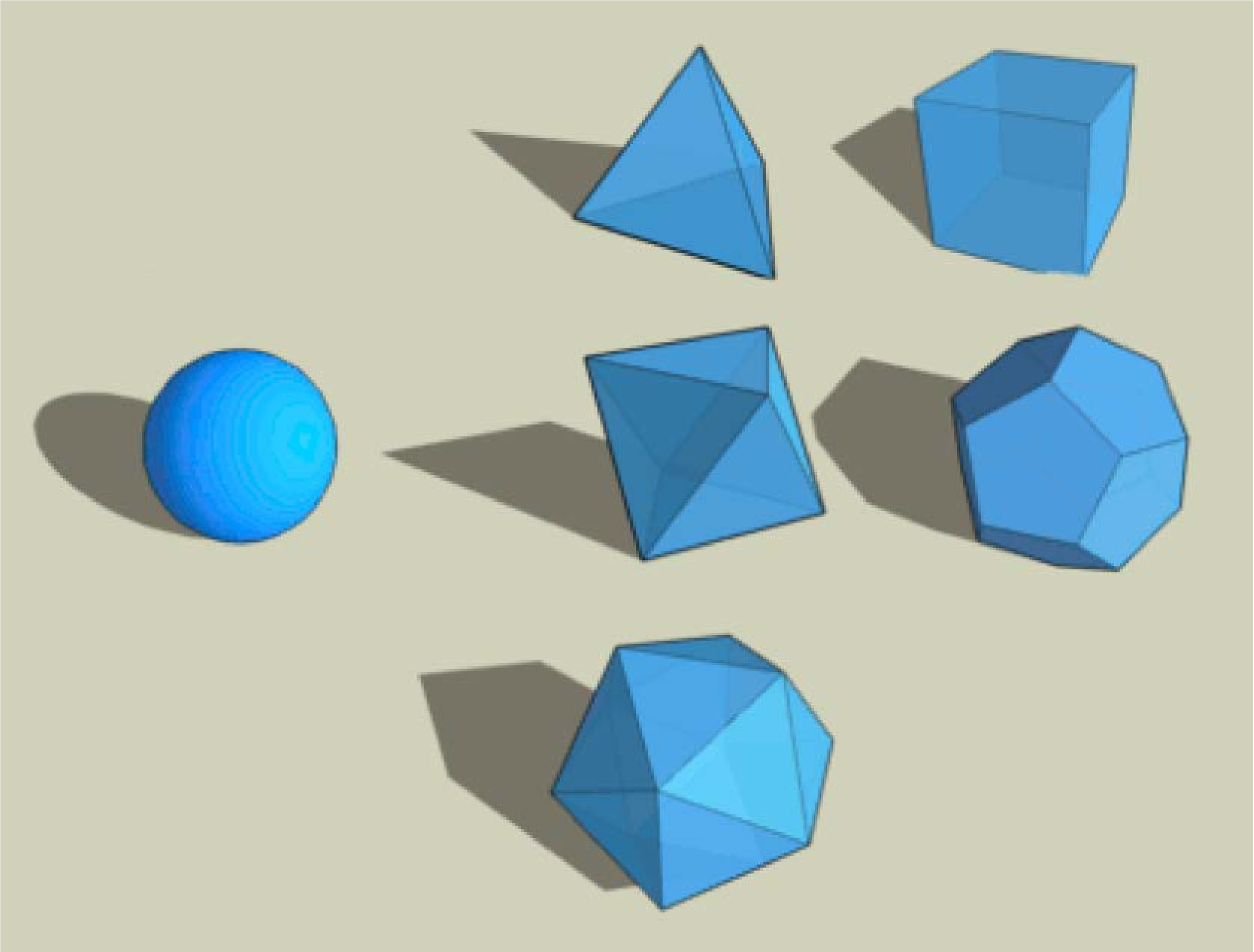
Example: Regge Calculus:

2d simplicial complex

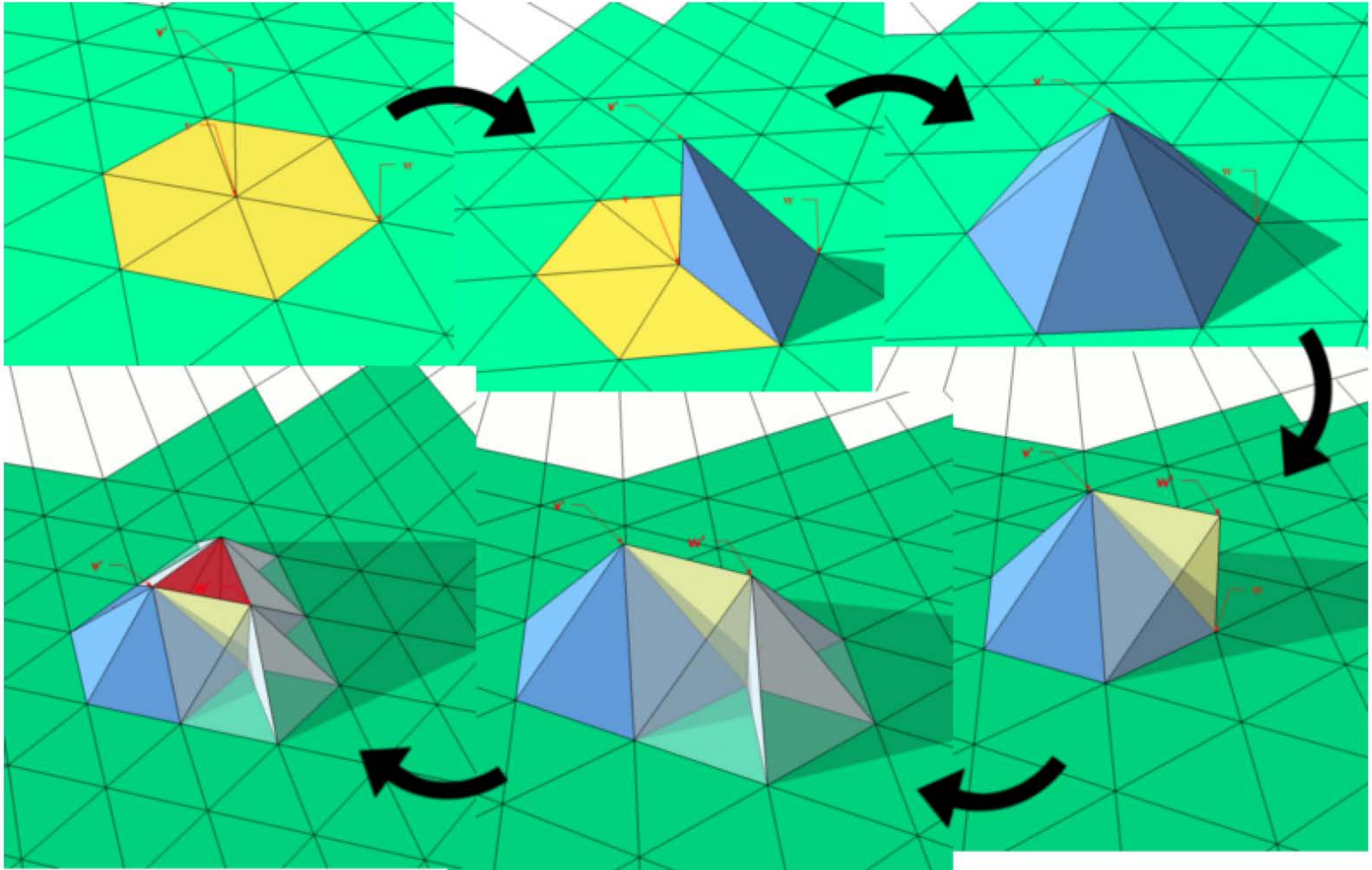
curvature from deficit angle

Picture credit: Marinelli and Immirzi 2013 @ Schwarzschild Meeting

3D simplicial complex: Constructing a 3-sphere



Setting up a spacetime tent: e.g., S3 x R



Example: Interesting results from the **Causal Dynamical Triangulation** program (since the 90s)

- Mostly fractal geometry of 2D: Dimensional Reduction / Fractal Spacetime
- Some (rare) evolves to 4D – existence of 4D spacetime is important!
- Amongst these there exist long-lived smooth structures
- After ~ 100 Planck time spacetime acquires semiclassical features
- Phase transition?

2. View from Bottom-up: Stochastic Gravity

Emergent / Quantum Gravity: need both `Top down' & `Bottom-up' Approaches

- B. L. Hu, Emergent /Quantum Gravity: Macro/Micro Structures of Spacetime in DICE2008 J. Phys. Conf. Ser. 174 (2009) 012015 [arXiv:0903.0878]

1) All candidate theories of Quantum Gravity – Theories for the Microscopic Structures of Spacetime – should **look at their commonalities** at low energy -- Planck scale lower limit, rather than differences at the trans-Planckian scale.
2) Quantum → Stochastic → Semiclassical → Classical in many physical systems.

- Focus on the Stochastic Regime: Metric Fluctuations ("spacetime foam")
- Stochastic Gravity: Metric Fluctuations induced by quantum matter field fluctuations, via Einstein-Langevin Equaton.
- In this stochastic regime it is more natural to anticipate fractal structures of spacetime in the transition from discrete to continuum.

We wish to approach this issue **from low energy (Bottom-) up**:

the real goal of Quantum Gravity. (`Top down' is Emergent Gravity)

1. **QFTCST** '70s
2. **Semiclassical Gravity** '80s
3. **Stochastic Gravity** '90s

Bottom Up approach is difficult, but not impossible. Some prior examples:

1. **Renormalization (regularization) of the stress energy tensor mandates the appearance of R2 terms** (α Box R, β Ricci2, γWeyl2, two out of three, thanks to the Gauss-Bonnet identity in 4d)
2. Inclusion of (Gaussian) **quantum field fluctuations mandates** the appearance of stochastic components -- **induced metric fluctuations**.
3. Inclusion of **non-Gaussian quantum field fluctuations** – does it open the door for **fractal spacetime structures**?

Part I. Semiclassical Gravity

Semiclassical Einstein Equation (Moller-Rosenfeld):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa\langle \hat{T}_{\mu\nu}\rangle_q + \kappa\,(T_{\mu\nu})_c$$

$\tilde{G}_{\mu\nu}$ is the Einstein tensor (plus covariant terms associated with the renormalization of the quantum field)

$\kappa = 8\pi G_N$ and G_N is Newton's constant

Free massive scalar field

$$(\Box - m^2 - \xi R)\hat{\phi} = 0.$$

$\hat{T}_{\mu\nu}$ is the stress-energy tensor operator

$\langle\,\rangle_q$ denotes the expectation value

Stochastic Gravity (recall role of QBM)

Einstein- Langevin Equation (schematically):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa\left(T^{\rm c}_{\mu\nu} + T^{\rm qs}_{\mu\nu}\right)$$

$T^{c}_{\mu\nu}$ is due to classical matter or fields

$$T^{\rm qs}_{\mu\nu} \equiv \langle \hat{T}_{\mu\nu} \rangle_{\rm q} + T^{\rm s}_{\mu\nu}$$

$T^{\rm qs}_{\mu\nu}$ is a new stochastic term

related to the quantum fluctuations of $T_{\mu\nu}$

NOISE KERNEL

- Exp Value of 2-point correlations of stress tensor: bitensor

$$N_{abc'd'}(x,x') = \frac{1}{2}\langle\{T_{ab}(x), T_{c'd'}(x')\}\rangle - \langle T_{ab}(x)\rangle\langle T_{c'd'}(x')\rangle$$

is real and positive semi-definite.

- measures quantum fluctuations of stress tensor

It can be represented by (shown via the Feynman-Vernon identity for Gaussian fields) a classical stochastic tensor source $\xi_{ab}[g]$

$$\langle \xi_{ab} \rangle_s = 0 \qquad \langle \xi_{ab}(x)\xi_{cd}(y)\rangle_s = N_{abcd}(x,y)$$

- **Symmetric, traceless** (for conformal field), **divergenceless**

Einstein-Langevin Equation

- Consider a weak gravitational perturbation h off a background g $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},$ The ELE is given by
  (The ELE is Gauge invariant)

$$G_{ab}[g+h] + \Lambda(g_{ab}+h_{ab}) - 2(\alpha A_{ab} + \beta B_{ab})[g+h]$$
$$= 8\pi G\left(\langle \hat{T}^R_{ab}[g+h]\rangle + \xi_{ab}[g]\right).$$

- Nonlocal dissipation and colored noise $\xi_{ab}[g]$

$$\langle \xi_{ab} \rangle_s = 0 \qquad \langle \xi_{ab}(x)\xi_{cd}(y)\rangle_s = N_{abcd}(x,y)$$

Stochastic Semiclassical Gravity

for early universe and black hole physics

- For problems in the early universe and black holes, one is interested in quantum processes related to the vacuum, e.g., particle creation, vacuum fluctuations, vacuum polarization (Hawking Effect).

- In analogous laboratory settings, with moving detectors, mirrors, e.g. Unruh Effect, dynamical Casimir Effect, etc.

B. L. Hu and E. Verdaguer, [Review]: *Liv. Rev. Rel. 11 (2008) 3*

----- [Monograph]: ***Semiclassical and Stochastic Gravity***

(Cambridge University Press, in preparation)

Noise and fluctuations in quantum field *induce metric fluctuations* of spacetime described by the *Einstein-Langevin Equation*.

Formulated with Tmn Trs taken wrt a vacuum state. Since it is a quadrature, Gaussian, one can introduce classical noise a la Feynman-Vernon to represent the Tmn two point function

New Question: For higher moments of the stress tensor (corr to NonGaussian noise in the QBM paradigm)

Does the induced metric fluctuations reveal a ***fractal structure of spacetime?***

3. Higher Moments of Stress Energy Tensor

IIA. Larry Ford: **Probability distribution for quantum stress tensor fluctuations**

Probability distribution for quantum stress tensor fluctuations of conformal fields in 1+1 D

C. Fewster, T. Roman & L. Ford, PRD 81, 121901 (R) (2010)

A skewed, non-Gaussian distribution.

In general, the odd moments are non-zero.

In 4D, C. Fewster, T. Roman & L. Ford, PRD 85, 125038 (2012) showed that the distributions fall slower than exponentially, leading to the possible dominance of vacuum fluctuations over thermal fluctuations.

Implications for gravity and cosmology:
Large passive geometry fluctuations possible

(in our terms) **Induced metric fluctuations <=**
Stochastic Gravity theory requires they be determinedby the matter field fluctuations.
(in more poetic language) “**Spacetime Foam**”

IIB. Implications for the small scale structure of spacetime

S. Carlip, R. A. Mosna & J. P. M. Pitelli, Phys. Rev. Lett. 107, 021303 (2011).

(using the 2D probability distribution of Fewster Ford Roman 2010)

Asymptotic silence

Near spacelike singularity, spacetime becomes "asymptotically silent"

– light cones strongly focus, shrink to timelike lines
– "anti-Newtonian" limit (as if $c \to 0$)
– spatial points decouple; BKL behavior
– locally Kasner space

Some evidence for asymptotic silence as *generic* Planck scale behavior

– short distance approximation of Wheeler-DeWitt equation
– Planck scale focusing by non-Gaussian vacuum stress-energy fluctuations
(Nonlinearities: strong enough fluctuation $\Rightarrow$ runaway focusing)

III. Our Question: 1. *Can a smooth manifold (Laplace Beltrami Operator) admit non-Gaussian noise from a quantum field?*

Yes.

- S. A. Ramsey,* B. L. Hu,† and A. M. Stylianopoulos, *Nonequilibrium inflaton dynamics and reheating. II. Fermion production, noise, and stochasticity*, *PRD57, 6003-6021 (1998).*

A scalar field ϕ (the inflaton field) coupled to a spinor field ψ, via

$f \phi \bar{\psi} \psi$ Yukawa-interaction in a curved, dynamical, classical background spacetime metric *g*.

We derived the noises ξ_2 for $O(f^2)$ and ξ_4 for $O(f^4)$ interactions.

They are **multiplicative colored noises.**

Generalized Langevin Eq. (classical)

- E.g., R. Mankin, K. Laas and A. Sauga, “Generalized Langevin equation with multiplicative noise: Temporal behavior of the autocorrelation functions” PRE 83, 061131 (2011)

  - ----------------------------------

- H. T. Cho and B. L. Hu, “ **Non-Gaussian noise and nonlinear Langevin Equation in Quantum Brownian Motion**” (in preparation)

- A second paper with the contents of this talk is planned

  (with H. T. Cho and perhaps, Enric Verdaguer)

Q2: nonGaussian noise source from a quantum field **back-reacting** on a spacetime –

Does the requirement of **self-consistency** demand that we allow for the possibility of a **fractal spacetime**?

Possibly.

-- From anomalous diffusion processes considerations.

Note: we are talking about two different stochastic process:

- One of quantum matter (with non-Gaussian noise) moving in a given background space.
- The other referring to the spacetime dynamics governed by the Einstein- Langevin Eqn

4. Inspiration from stochastic processes

Part IV. Anomalous Diffusion

- The mean-squared displacement

For ‘normal’ diffusion: linear dependence on time: $\langle(\Delta\mathbf{r})^2\rangle = 2dK_1 t$

For anomalous diffusion: $\langle(\Delta\mathbf{r})^2\rangle = \langle(\mathbf{r} - \langle\mathbf{r}\rangle)^2\rangle = 2dK_\alpha t^\alpha$

Here, *d* is the (embedding) spatial dimension, and *K*1 and *Kα* are the normal and generalized diffusion constants of dimensions cm2 s−1 and cm2 s−*α*, respectively.

- The process will be categorized as **subdiffusive** (dispersive, slow) if 0 < α < 1, or **superdiffusive** (enhanced, fast) if α>1. Usually, the domain 1 < α < or = 2 is considered, α = 2 being the **ballistic limit** described by the wave equation, or its forward and backward modes (Landau and Lifshitz 1984).
- **Exception**: Unconfined **Le’vy flights**, for which we observe **a diverging mean squared displacement**

Table 1. Comparison of different anomalous diffusion models to normal Brownian motion (BM) (Lévy 1965, van Kampen 1981): PDFs of fractional Brownian motion (FBM) (Mandelbrot and van Ness 1968, Lim and Muniandy 2002, Lutz 2001b, Kolmogrov 1940), generalized Langevin equation with power-law kernel (GLE) (Kubo *et al* 1985, Lutz 2001b, Wang *et al* 1994, Wang and Tokuyama 1999); continuous time random walk (CTRW) of types subdiffusion (SD), Lévy flights (LF) and Lévy walks (Klafter *et al* 1987, 1996, Shlesinger *et al* 1993); as well as time-fractional dynamics (TFD), which covers both subdiffusion (in this case it corresponds to SD) and sub-ballistic superdiffusion (Metzler and Klafter 2000a, 2000d). The c_i are constants.

	PDF D=1	Comments
BM	$P(x,t) = (4\pi K t)^{-1/2} \exp(-x^2/(4Kt))$	
FBM[a]	$P(x,t) = (4\pi K_\alpha t^\alpha)^{-1/2} \exp(-x^2/(4K_\alpha t^\alpha))$	$0 < \alpha \leqslant 2$
GLE[b]	$P(x,t) = (4\pi K_\alpha t^\alpha)^{-1/2} \exp(-x^2/(4K_\alpha t^\alpha))$	$0 < \alpha < 2, \alpha \neq 1$
SD[c]	$P(x,t) \sim c_1 t^{-\alpha/2} \xi^{-(1-\alpha)/(2-\alpha)} \exp(-c_2 \xi^{1/(1-\alpha/2)})$, $\xi \equiv \|x\|/t^{\alpha/2} \therefore \psi(t) \sim \tau^\alpha/t^{1+\alpha}$	$0 < \alpha \leqslant 1$
LF[d]	$P(x,t) = \mathscr{F}^{-1}\{\exp(-K^\mu t\|x\|^\mu)\} \sim K^\mu t/\|x\|^{1+\mu}$	$0 < \mu \leqslant 2$
LW[e]	$P(k,u) = \frac{1}{u}\psi(u)/[1-\psi(k,u)] \therefore$ $\psi(x,t) = \frac{1}{2}\|x\|^{-\mu}\delta(\|x\| - v_\nu t^\nu)$	$\nu\mu > 1$
TFD[f]	$P(x,t) \sim c_1 t^{-\alpha/2} \xi^{-(1-\alpha)/(2-\alpha)} \exp(-c_2 \xi^{1/(1-\alpha/2)})$, $\xi \equiv \|x\|/t^{\alpha/2}$	$0 < \alpha < 2$

- Ralf Metzler and Joseph Klafter, "Anomalous transport by fractional dynamics" J. Phys. A: Math. Gen. 37 (2004) R161–R208

Anomalous Diffusion 2 (from Metzler and Klafter)

- **Two groups:** the **fractional dynamical equations** corresponding to

1. Dispersive transport (SD), L´evy flights (LF), or L´evy walks (LW) are **highly non-local**, and carry far-reaching **correlations in time and/or space**, represented in the **integro-differential nature** (with slowly decaying power-law kernels) of these equations.
2. In contrast, **fractional Brownian motion** (FBM) or **the generalized Langevin equation** (GLE) on the macroscopic level are **local in space and time**, and carry merely **time- or space dependent coefficients**.

- Anomalous diffusion in terms of **non-linear Fokker–Planck equations** based on **non-extensive** statistical approaches.

Simplest Approach to Fractional Dynamics -- via a continuous time random walk (CTRW)

1. In a **standard random walk process** each step is of a fixed length in a random direction at each tick of a system clock. A process having *constant spatial and temporal increments*, Δx and Δt will give rise to the **standard diffusion process in the long-time limit**, i.e.,

After a sufficient number of steps the associated random variable $x(t) = N^{-1/2} \sum_i^N x_i$, where x_i is the position after the ith step, will be distributed by a **Gaussian** due to the central limit theorem.

2 In a **CTRW process** both jump length and waiting time are distributed according to **two PDFs, λ(x) and ψ(t).** The **propagator** for such a CTRW process in the absence of an external force is given in Fourier–Laplace space by

$$P(k,u) = \frac{1-\psi(u)}{u[1-\psi(k,u)]}$$

Subdiffusion

- Subdiffusion is classically described in terms of a CTRW with a long-tailed inverse power-law waiting time:

$$\psi(t) \sim \tau^{\alpha}/t^{1+\alpha} \quad \text{for } t \gg \tau. \qquad (0 < \alpha < 1), \qquad \text{Eq (2)}$$

- A waiting time PDF of this form is obtained under the Laplace expansion

$$\psi(u) \sim 1 - (u\tau)^{\alpha} \quad (u \ll \tau),$$

- It includes the **α = 1 (normal diffusion)** limit

$$\psi(u) = e^{-u\tau} \sim 1 - u\tau, \text{and } \psi(t) = \delta(t - \tau)$$

Fractional diffusion equation

- Fourier transform of Jump length *λ(x)* : after an analogous expansion of a short-range jump length PDF, we get *λ(k)* ~ 1 − *σk*2 *(k* → 0*)*

We obtain

$$P(k, u) \simeq \frac{1/u}{1 + u^{-\alpha} K_\alpha k^2}$$ Eq (4)

where $K_\alpha \equiv \sigma/\tau^\alpha$ is the anomalous diffusion constant.

- For **Brownian motion α =1** we obtain the normal diffusion equation

$$\partial P(x, t)/\partial t = K_1 \partial^2 P(x, t)/\partial x^2,$$

The **subdiffusive cases** $0 < \alpha < 1$ have a term of the form $u^{-\alpha} f(u)$

They obey the **fractional diffusion equation**: Eq (9)

$$\frac{\partial}{\partial t} P(x,t) = {}_0D_t^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} P(x,t).$$

where

$${}_0D_t^{1-\alpha} \equiv \frac{\partial}{\partial t}\, {}_0D_t^{-\alpha}$$

is a Riemann–Liouville fractional differential operator. and

$${}_0D_t^{-\alpha} f(t) \equiv \frac{1}{\Gamma(\alpha)} \int_0^t \mathrm{d}t' \frac{f(t')}{(t-t')^{1-\alpha}}$$

is the Riemann–Liouville fractional integral.

- It is important to keep track of the initial condition in the fractional diffusion equation (9).

Noticing that $${}_0D_t^{\alpha} 1 = t^{-\alpha} / \Gamma(1-\alpha)$$

We can retrieve the equivalent equation to (9) in the form

$${}_0D_t^{\alpha} P(x,t) - \frac{P_0(x)}{\Gamma(1-\alpha)} t^{-\alpha} = K_\alpha \frac{\partial^2}{\partial x^2} P(x,t).$$

Part V. Anomalous Diffusion as a probe into the Quantum Nature of Spacetime

- Gianluca Calcagni, Astrid Eichhorn and Frank Saueressig, Probing the quantum nature of spacetime by diffusion, PRD 87 (2013) 124028
- G. Calcagni and G. Nardelli, Spectral dimension and diffusion in multiscale spacetimes, Phys. Rev. D 88 (2013) 124025.
- Gianluca Calcagni, Daniele Oriti, and Johannes Thürigen, Spectral dimension of quantum geometries Class. Quantum Grav. **31** 135014 (2014)
- Johannes Thürigen, Discrete quantum geometries and their effective dimension arXiv:1510.08706 PhD thesis, Humboldt-Universit\"at zu Berlin

Here, one uses **an anomalous diffusion equation on classical flat space** to mimic **the quantum structure of spacetime**

-- many technical and conceptual issues: e.g., time is not physical time, etc.

- A dynamical dimensional change is captured by a **modification of the Laplacian operator** appearing in the classical diffusion equation.
- The **effective metric** "seen" by the diffusing particle **depends on the momentum of the probe.**

- G. Calcagni, A. Eichhorn and F. Saueressig, Probing the quantum nature of spacetime by diffusion, Phys. Rev. D 87 (2013) 124028

- Expressing this metric through a **fixed reference scale** leads to a **modified diffusion equation** providing an effective description of the propagation of the probe particle on the quantum gravity background.

- A **multifractal structure with spectral dimension**

$$d_S = \frac{2d}{2+\delta}.$$

The parameter δ captures the quantum effects and depends on the probed length scale.

3 characteristic regimes with different spectral dimensions

- Three characteristic regimes where the spectral dimension is approximately constant over many orders of magnitude:

$$d_S = \frac{2d}{2+\delta}.$$

1. At large distances, the **classical regime** where **δ = 0**

   The spectral dimension agrees with both the

   Hausdorff and topological dimension of the spacetime.

2. At smaller distances one encounters a **semiclassical regime** with **δ = d**, before entering into the fixed-point

3. **Quantum gravity regime** with **δ = 2**.

- Assume that the scale-dependent solutions of the equations of motion give rise to a power-law relation between the **effective metrics at scale *k*** and the IR reference scale k_0, $\langle g^{\mu\nu} \rangle_k \propto k^{\delta} \langle g^{\mu\nu} \rangle_{k_0}$

$$(\partial_\sigma - k^{\delta} \langle g^{\mu\nu} \nabla_\mu \nabla_\nu \rangle_{k_0}) P(x, x', \sigma) = 0,$$

where $g^{\mu\nu}$ is the **fixed IR reference metric**: the flat Euclidean metric.

- Encode the scaling effects in the diffusion time σ Eq. (12),

$$\left(k^{-\delta} \frac{\partial}{\partial \sigma} - \nabla_x^2 \right) P(x, x', \sigma) = 0.$$

dimensionally, $\sigma \sim k^{-\delta-2}$ $\qquad k = \sigma^{-\frac{1}{\delta+2}},$

- This results in a diffusion equation

$$\left(\frac{\partial}{\partial \sigma^{\beta}} - \nabla_x^2\right) P(x, x', \sigma) = 0.$$

in nonlinear time σ^{β} with

$$\beta = \frac{2}{\delta + 2},$$

- and a probability density which is semi-positive definite: $P(r, \sigma) = \frac{1}{(4\pi\sigma^{\beta})^{\frac{d}{2}}} e^{-\frac{r^2}{4\sigma^{\beta}}}.$
- $P[r, \sigma(k)] \propto k^d$ has the correct scaling behavior of a diffusion probability in *d* dimensions,
- Spectral dimension is independent of σ: $d_S = \frac{2d}{2 + \delta}.$

The diffusion equation can be written in terms of **fractional derivatives** (the "left Caputo" derivative) $\partial_\sigma^\beta \quad 0 < \beta \leq 1$

$$(\partial^\beta f)(\sigma) := \frac{1}{\Gamma(1-\beta)} \int_0^\sigma \frac{d\sigma'}{(\sigma-\sigma')^\beta} \partial_{\sigma'} f(\sigma'),$$

Resulting in

$$(\partial_\sigma^\beta - \nabla_x^2) P_\beta(x, x', \sigma) = 0,$$

- G. Calcagni, A. Eichhorn and F. Saueressig, Probing the quantum nature of spacetime by diffusion, Phys. Rev. D 87 (2013) 124028

$$P_1(r, s) = \frac{e^{-\frac{r^2}{4s}}}{(4\pi s)^{\frac{d}{2}}}$$

is the solution of the standard Brownian motion heat equation in d dimensions.

$$(\partial_\sigma - \nabla_x^2) P(x, x', \sigma) = 0.$$

with initial condition

$$P(x, x', 0) = \delta(x - x')$$

VI. Back to the Einstein-Langevin Eq.

- Consider a weak gravitational perturbation h off a background g $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},$ The ELE is given by

  (The ELE is Gauge invariant)

$$G_{ab}[g+h] + \Lambda(g_{ab}+h_{ab}) - 2(\alpha A_{ab} + \beta B_{ab})[g+h]$$
$$= 8\pi G\left(\langle \hat{T}^{R}_{ab}[g+h]\rangle + \xi_{ab}[g]\right).$$

- Nonlocal dissipation and colored noise $\xi_{ab}[g]$

$$\langle \xi_{ab} \rangle_s = 0 \qquad \langle \xi_{ab}(x)\xi_{cd}(y)\rangle_s = N_{abcd}(x,y)$$

Einstein- Langevin Eq with Gaussian noise

- Martin R and Verdaguer E Phys. Rev. D 61 (2000) 124024

solved the ELEq for the induced metric fluctuations off a **Minkowski spacetime** driven by **Gaussian noise** associated with the quantum fluctuations of a scalar field.

$$\langle h_{ab}(x)h_{cd}(y)\rangle_s = \langle h^0_{ab}(x)h^0_{cd}(y)\rangle_s \\ +\bar{\kappa}^2 \int d^4x' d^4y' \sqrt{g(x')g(y')} G^{ret}_{abef}(x,x') N^{efgh}(x',y') G^{ret}_{cdgh}(y,y').$$

De Sitter (Perez-Nadal G, Roura A and Verdaguer E 2010 *JCAP* 1005 036)

$$
\begin{aligned}
N_{abc'd'} \;=\;& P(\mu)n_a n_b n_{c'} n_{d'} + Q(\mu)(n_a n_b g_{c'd'} + n_{c'} n_{d'} g_{ab}) \\
&+R(\mu)(n_a n_{c'} g_{bd'} + n_b n_{d'} g_{ac'} + n_a n_{d'} g_{bc'} + n_b n_{c'} g_{ad'}) \\
&+S(\mu)(g_{ac'} g_{bd'} + g_{bc'} g_{ad'}) + T(\mu) g_{ab} g_{c'd'} .
\end{aligned}
$$

the unit vectors, $n_a(x; x_0)$ and $n_{a'}(x; x_0)$, tangent to the geodesic at the points x and x_0 respectively. $G_{ab'}(x; x_0)$ parallel-transports a vector from x_0 to x along the geodesic.

The (connected) symmetrized two-point function of the **Einstein tensor**:

$$
\langle G^{a\;(1)}_{\;b}(x)\, G^{c'\;(1)}_{\;d'}(x')\rangle_c = 64\pi^2 \bar{l}_P^4 N^{-1}\, N^{a\;\;c'}_{\;b\;\;d'}(x,x')
$$

Markus B. Froeb, Albert Roura and Enric Verdaguer, JCAP1007 (2014) 048

also calculated the **correlator of the Weyl tensor and Riemann tensor.**

Einstein-Langevin Eq with non-Gaussian noise

Stage 1. nonGaussian **Source**: For QBM of a weakly anharmonic oscillator, can use functional perturbative method *a la* Hu Paz Zhang PRD47 (1993) 157

$$S_{int}[x, \{q_n\}] = \int_0^t ds \sum_n v_n(x) q_n^k.$$

v_n(x) = λ C_nf(x)

$$P[\xi] = P^{(0)}[\xi] \left(C_0 + \int C_1 \xi + \int C_2 \xi\xi + \int C_3 \xi\xi\xi \right)$$

Gaussian QBM has only C_0 term

- *"Non-gaussian noise, modified fluctuation-dissipation relation, and non-linear Langevin equation"* with Prof. **Hing Tong Cho**, Tamkang Univ.

Stage 2: Backreaction on spacetime; Set up ELEq, solve for metric fluctuations

Summary

1. **Higher moments of Tmn** in Minkowski Spacetime: Fewster Ford Roman ..,
2. **Near Planck scale**: **Dimensional Reduction**. Carlip, Loll et al: Spacetime at small length scales becomes **effectively 2 dim** ← UV, not IR
3. **Spectral dimension** from Brownian motion. E.g., **heat kernel expansion** .
4. **Non-Gaussian noise / Anomalous Diffusion**: Very different spectral dimensions from Gaussian or normal diffusion: small vs large scale
5. **Fractal Spacetime**. e.g. Reuter / Ambjorn Loll / Calcagni Oriti et al
6. Does **backreaction of nonGaussian noise** bring forth a modification of the wave operator reflecting possible fractal spacetime dimension?
7. Solutions of **the Einstein-Langevin Equation** with non-Gaussian noise source involving higher moments of Tmn. *Stay Tuned!*

Thank You!

Happy New Year!

Three Levels of Theoretical Structures

Einstein equation:

$$G_{\mu\nu}[g] = \kappa T_{\mu\nu}[g]$$

Semi-classical gravity (mean field):

$$G_{\mu\nu}[g] = \kappa \left(T_{\mu\nu}[g] + \left\langle T^{q}_{\mu\nu}[g] \right\rangle \right)$$

Stochastic gravity (including quantum fluctuations):

$$G_{\mu\nu}[g+h] = \kappa \left(T_{\mu\nu}[g+h] + \left\langle T^{q}_{\mu\nu}[g+h] \right\rangle + \xi_{\mu\nu}[g] \right)$$

to linear order in h, where $\xi_{\mu\nu}$ is the stochastic force induced by the quantum field fluctuations.

[slides courtesy Prof Hing Tong Cho]

the stochastic force $\xi_{\mu\nu}$, with the correlation

$$\langle \xi_{\mu\nu}(x)\xi_{\alpha\beta}(y)\rangle_s = N_{\mu\nu\alpha\beta}(x,y)$$

$N^{\mu\nu\alpha\beta}$ is the noise kernel

$$N^{\mu\nu\alpha\beta}(x,y) = \frac{1}{2}\left\langle \left\{ t^{\mu\nu}(x), t^{\alpha\beta}(y) \right\} \right\rangle$$

where $t^{\mu\nu}(x) = T^{\mu\nu}(x) - \langle T_{\mu\nu}(x)\rangle$.

Stochastic Gravity from the open system paradigm: **Gravity** as the system, **Quantum Field** as environment

The corresponding CTP effective action is

$$
\begin{aligned}
e^{i\Gamma[g_+,g_-]} &= e^{iS_g[g_+]-iS_g[g_-]} \int_{CTP} D\phi_+ D\phi_- e^{iS_m[\phi_+,g_+]-iS_m[\phi_-,g_-} \\
&= e^{iS_g[g_+]-iS_g[g_-]+iS_{IF}[g_+,g_-]}
\end{aligned}
$$

where S_g and S_m are the gravity and the quantum field actions, respectively. S_{IF} is the influence action due to the quantum field.

The influence action for stochastic gravity S_{IF}

$$
\begin{aligned}
= \; & \frac{1}{2}\int d^4x\sqrt{-g(x)}\,\langle T^{\mu\nu}(x)\rangle\,\Delta h_{\mu\nu}(x) \\
& -\frac{1}{8}\int d^4x\,d^4y\sqrt{-g(x)}\,\sqrt{-g(y)}\,\Delta h_{\mu\nu}(x) \\
& \qquad \left[K^{\mu\nu\alpha\beta}(x,y)+H_A^{\mu\nu\alpha\beta}(x,y)+H_S^{\mu\nu\alpha\beta}(x,y)\right]\Sigma h_{\alpha\beta}(y) \\
& +\frac{i}{8}\int d^4x\,d^4y\sqrt{-g(x)}\,\sqrt{-g(y)}\,\Delta h_{\mu\nu}(x)N^{\mu\nu\alpha\beta}(x,y)\Delta h_{\alpha\beta}(y)
\end{aligned}
$$

See, e.g., Martin and Verdaguer 2000 PRD

$$
\langle T_{\mu\nu}(x)\rangle = \frac{2}{\sqrt{-g(x)}}\frac{\delta S_{IF}}{\delta g_{+\mu\nu}(x)}\bigg|_{g_+=g_-}
$$

← As in Semiclassical Gravity

- Need to average the operator over a finite spacetime region

- Expect the vacuum probability distribution to have a lower cutoff at the quantum inequality bound on expectation values in an arbitrary state - lowest eigenvalue of the averaged operator.

Discussion: Several questions need to be addressed: (L Ford)

- Generalizations to curved spacetime?
- What selects the sampling function?
- Effects of spatial averaging?

Let $u = \int T_{tt}\, g(t,\tau)\, dt$ averaged energy density

A result for conformal field theory (2 spacetime dimensions)

C. Fewster, T. Roman & L. Ford, PRD 81, 121901 (R) (2010)

$$u = \frac{1}{\sqrt{\pi}\tau} \int_{-\infty}^{\infty} T_{tt}(\vec{x}, t)\, e^{-t^2/\tau^2}\, dt$$

$$x = u\,\tau^2$$

$$P(x) = \frac{\pi^{c/24}}{\Gamma(c/24)}\, (x + x0)^{\frac{c}{24}-1}\, e^{-\pi(x+x_0)}$$

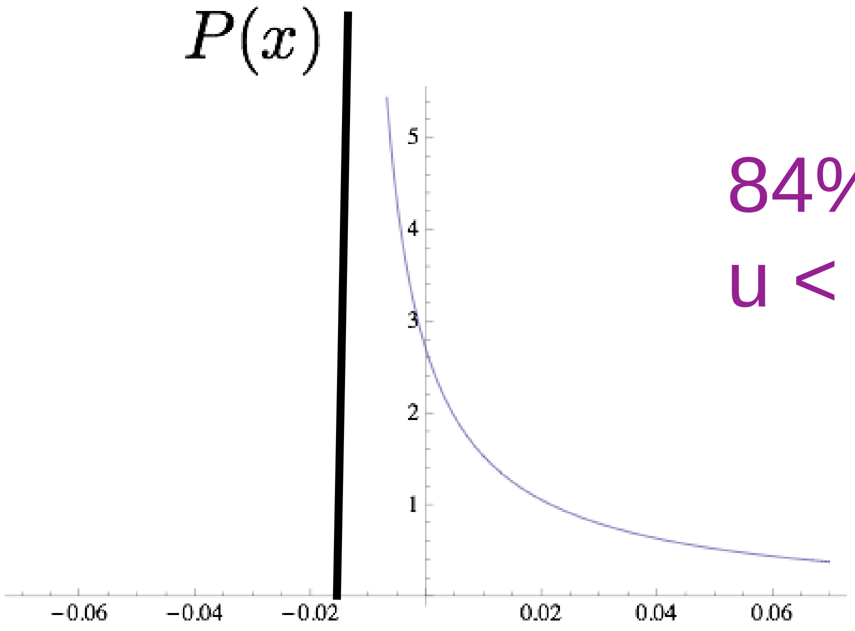
$P(x) = 0 \quad x < -x_0$ $\qquad -x_0$ = quantum inequality bound

c = central charge $\qquad c = 24$ for string theory

A massless scalar field in two dimensions: c=1

$$P(x) = \frac{\pi^{1/24}}{\Gamma(1/24)} \left(x + \frac{1}{24\pi} \right)^{-23/24} e^{-\pi(x+1/24\pi)}$$

$$P(x) = 0 \quad x < -\frac{1}{24\pi}$$



$x = \tau^2 u$

84% chance of finding u < 0

P(u) is a shifted Gamma distribution

Negative energy is more likely than positive energy

Positive fluctuations tend to be larger in magnitude

Negative fluctuations cannot be more negative than the quantum inequality bound on the expectation value

Hamburger Moment Theorem

There exists a unique P(x) for a given set of moments, provided that M_n grow no faster than $n!$ as $n \to \infty$

Satisfied by the shifted gamma distribution.