# Theory of Cosmological Perturbations

## Part II

## — perturbations from inflation —

II.1. standard slow-roll inflation

# §1. Introduction

- Horizon problem

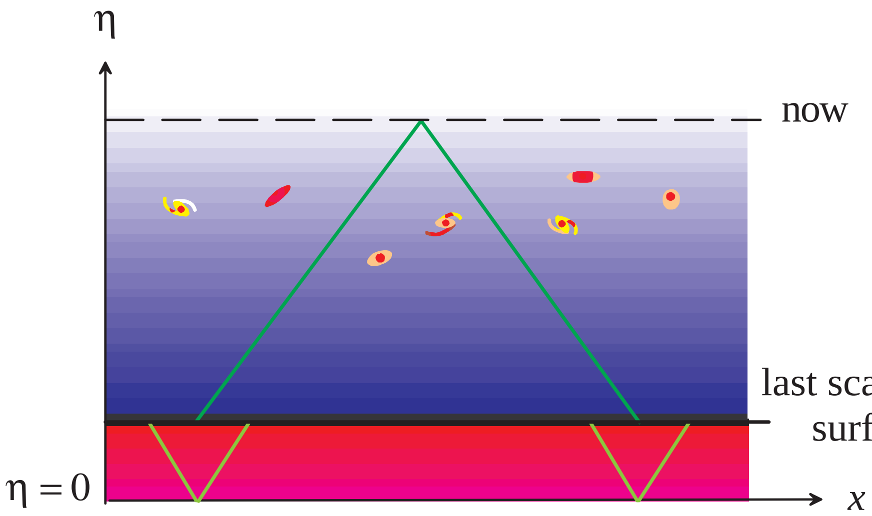
$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \qquad + \qquad \text{Einstein eqs.}$$

$$\Rightarrow \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad \boxed{\rho + 3p > 0 \quad \Leftrightarrow \quad \text{decelerated expansion}}$$

If $a \propto t^n$, then $n(n-1) < 0 \quad \Rightarrow \quad 0 < n < 1$

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\vec{x}^2\right), \quad d\eta = \frac{dt}{a}.$$

($\eta$: conformal time $\cdots$ maintains causality)




$d\eta = \pm dx$ : light ray

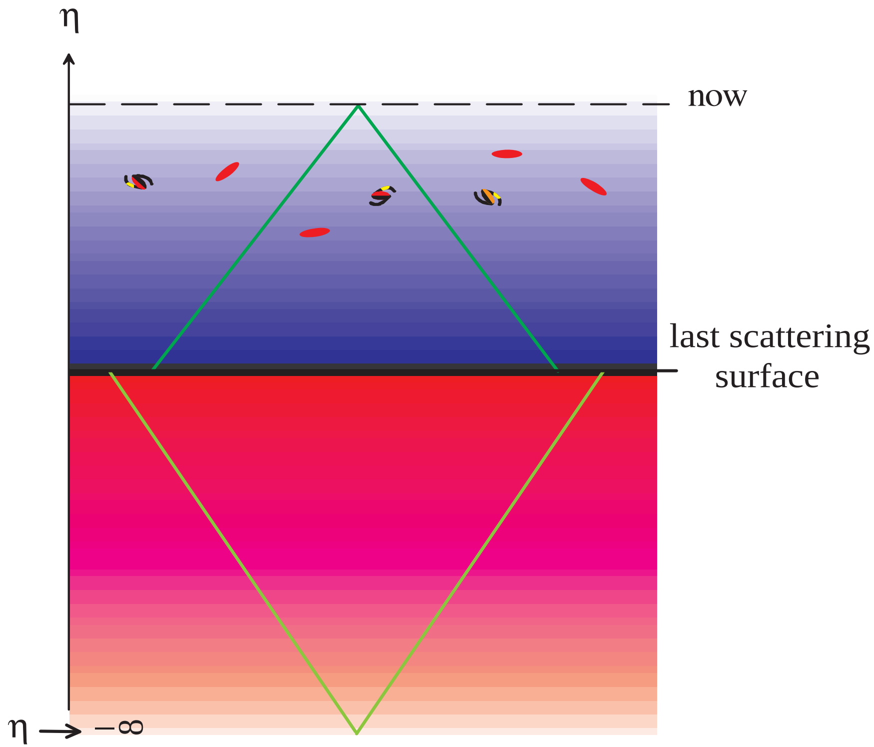
$$\eta = \int \frac{dt}{a} \to 0 \quad \text{for } t \to 0$$

- Solution to the horizon problem

Existence of a stage $a \propto t^n$ $\boxed{n > 1}$
in the early universe

$$\Leftrightarrow \quad \rho + 3p < 0$$

$$\Rightarrow \quad \int_0^t \frac{dt}{a} = \int d\eta = \infty \text{ !!}$$




- Entropy problem (= flatness problem)

Entropy within the curvature radius: $N_\gamma \sim$ conserved

$$N_\gamma = n_\gamma \left( \frac{a}{\sqrt{|K|}} \right)^3 \sim \left( \frac{T_0}{H_0} \right)^3 |1 - \Omega_0|^{-3/2} > \left( \frac{T_0}{H_0} \right)^3 \sim 10^{87}$$

$T_0 \sim 10^{-4}\text{eV} \quad H_0 \sim 10^{-33}\text{eV}$

Where does this big number come from?

"Huge entropy production in the early universe"

# §2. Single-field slow-roll inflation

Universe dominated by a scalar field:

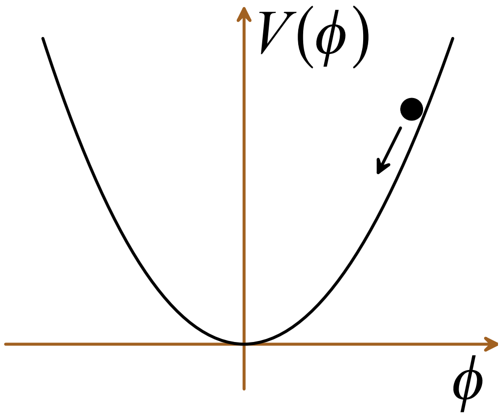
$$\begin{cases} \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{cases} \quad \Rightarrow \quad \rho + 3p = 2(\dot{\phi}^2 - V(\phi))$$

$$\text{if} \quad \dot{\phi}^2 < V(\phi) \quad \Longrightarrow \quad \frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3p) > 0\,; \quad M_P^2 = \frac{1}{8\pi G}$$

accelerated expansion

* Chaotic inflation (or Creation of Universe from nothing)

(Linde, Vilenkin, Hartle-Hawking, $\cdots$)




$$\rho_{\text{initial}} \lesssim M_P^4 \approx \left(10^{19}\,\text{GeV}\right)^4$$

$\cdots$ quantum gravitational

if $V''(\phi) \ll M_P^2$, then $\phi \gg M_P$

- Equations of motion:

$$\ddot{\phi} + \underbrace{3H\dot{\phi}}_{\text{friction}} + V'(\phi) = 0 \quad (H \lesssim M_P \text{ initially in chaotic inflation})$$

$$\Rightarrow \quad \boxed{\dot{\phi} \approx -\frac{V'}{3H}} \quad \text{(slow roll (1))} \quad \Leftrightarrow \quad \delta \equiv \frac{\ddot{\phi}}{H\dot{\phi}} = -\epsilon + \frac{\eta}{2}\,;\ |\delta| \ll 1$$

$$\begin{cases} \dot{H} = -\dfrac{1}{2M_P^2}(\rho + p) = -\dfrac{\dot{\phi}^2}{2M_P^2} \\ H^2 = \dfrac{1}{3M_P^2}\left(\dfrac{1}{2}\dot{\phi}^2 + V(\phi)\right) \end{cases} \qquad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon}$$

$$\Rightarrow \quad \boxed{H^2 \approx \frac{V(\phi)}{3M_P^2}} \quad \text{(potential dominated (2))} \quad \Leftrightarrow \quad \epsilon \equiv -\frac{\dot{H}}{H^2} \approx \frac{3\dot{\phi}^2}{2V(\phi)} \ll 1$$

The slow-roll condition (1) is satisfied, provided that

$$\eta \approx -2\eta_V + 4\epsilon_V\,,\ \eta_V \equiv \frac{M_P^2 V''}{V}\,;\ |\eta_V| \ll 1\,,\ \epsilon_V \equiv \frac{M_P^2 V'^2}{2V^2} \ll 1$$

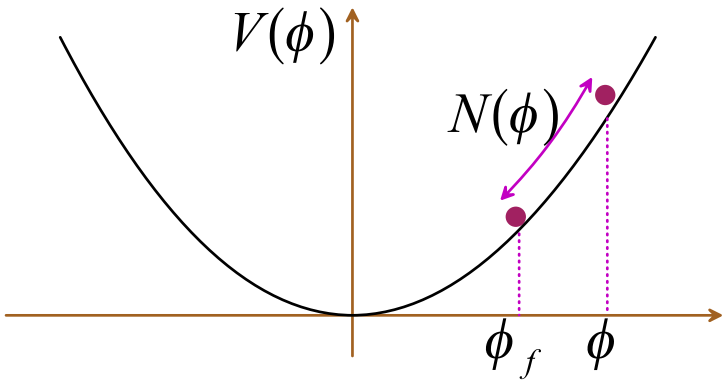
· Slow-roll inflation assumes that the above two are fulfilled.
(Note that these are not necessary but sufficient conditions.)

· There are models that violate either or both of the above two conditions.
(Need special care in the calculation of perturbations)

- $e$-folding number of inflation $\quad a \propto e^{-N}$

$$N(\phi) = \int_t^{t_f} H\,dt = \int_\phi^{\phi_f} \frac{H}{\dot{\phi}} d\phi \approx \frac{1}{M_P^2} \int_{\phi_f}^{\phi} \frac{V}{V'} d\phi = \frac{\phi^2 - \phi_f^2}{4M_P^2} \approx \left(\frac{\phi}{2M_P}\right)^2$$

$\Uparrow$ slow roll $\qquad \Uparrow$ $V = \frac{1}{2}m^2\phi^2$




For $V(\phi) \sim (10^{15}\mathrm{GeV})^4$, $N(\phi) \gtrsim 60$ solves horizon & flatness problems

$$N(\phi) \gtrsim 60 \quad \text{at} \quad \phi \gtrsim 15M_P \quad \text{for} \quad V = \frac{1}{2}m^2\phi^2$$

Slow roll ends at $\phi = \phi_f \sim \sqrt{2}M_P \quad \Rightarrow$ Reheating (entropy generation)

# §3. Generation of cosmological perturbations

Action: $S = \int d^4x \sqrt{-g} \left( \frac{M_P^2}{2} R - \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right)$ ; $M_P^2 = (8\pi G)^{-1}$ .

Cosmological perturbations are generated from quantum (vacuum) fluctuations of the inflaton $\phi$ and the metric $g_{\mu\nu}$.

- Scalar-type (density) perturbations
  - $g_{\mu\nu}$ and $\phi$:

$$ds^2 = a^2 \left[ -(1 + 2A) d\eta^2 - 2\partial_j B \, d\eta dx^j + \left( (1 + 2\mathcal{R}) \delta_{ij} + 2\partial_i \partial_j H_T \right) dx^i dx^j \right],$$

$$\phi(t, x^i) = \phi(t) + \chi(t, x^i)$$

$A$ : Lapse function ($\sim$ time coordinate) perturbation ($= A_k Y_k$)

$B$ : Shift vector ($\sim$ space coordinate) perturbation ($= k^{-1} B_k Y_k$)

Scalar perturbation has 2 degrees of coordinate gauge freedom.

$\mathcal{R}$ : Spatial curvature (potential) perturbation ($= \mathcal{R}_k Y_k$) $\left[ \delta \overset{(3)}{R} = -\frac{4}{a^2} \overset{(3)}{\Delta} \mathcal{R} \right]$

$H_T$ : Shear of the metric ($= k^{-2} H_{T,k} Y_k$)

No dynamical degree of freedom in the metric itself.

⋆ Action expanded to 2nd order

$$S_2 = \int d\eta d^3x \mathcal{L}_s = \frac{1}{2}\int d\eta\, d^3x\, a^2\Big[M_P^2\Big\{-6(\mathcal{R}' - \mathcal{H}A)^2 - 2\mathcal{R}\overset{(3)}{\Delta}\mathcal{R} - 4A\overset{(3)}{\Delta}\mathcal{R}\Big\}$$

$$+ (\chi' - A\phi')^2 + \chi(\overset{(3)}{\Delta} - a^2\partial_\phi^2 V)\chi - 6\phi'(\mathcal{R}' - \mathcal{H}A)\chi + 2A(\mathcal{H}\phi' - \phi'')\chi$$

$$-2\overset{(3)}{\Delta}(H_T' - B)\Big\{\phi'\chi + 2M_P^2(\mathcal{R}' - \mathcal{H}A)\Big\}\Big],$$

Canonical momenta

$$P_\chi \equiv \frac{\partial \mathcal{L}_s}{\partial \chi'} = a^2(\chi' - A\phi')$$

$$P_\mathcal{R} \equiv \frac{\partial \mathcal{L}_s}{\partial \mathcal{R}'} = a^2\left(-6M_P^2(\mathcal{R}' - \mathcal{H}A) - 3\phi'\chi - 2M_P^2\overset{(3)}{\Delta}(H_T' - B)\right)$$

$$P_T \equiv \frac{\partial \mathcal{L}_s}{\partial H_T'} = -2\overset{(3)}{\Delta}[\phi'\chi + 2M_P^2(\mathcal{R}' - \mathcal{H}A)]$$

Solving the above for $\mathcal{R}'$, $H_T'$ and $\chi'$, the Hamiltonian is obtained from

$$\mathcal{H}_{s,\mathrm{tot}} = P_\mathcal{R}\mathcal{R}' + P_T H_T' + P_\chi \chi' - \mathcal{L}_s$$

$A$ and $B$ remain as Lagrange multipliers.

Action in the Hamiltonian form  Garriga, Montes, MS & Tanaka (1998)

$$S_2 = \int d\eta\, d^3x \mathcal{L}_s = \int d\eta\, d^3x \left( \sum_a P_a Q'_a - \mathcal{H}_s - A\, C_A - B\, C_B \right)$$

$$\mathcal{H}_s = \frac{1}{2a^2} P_\chi^2 - 4\pi G \phi' P_\mathcal{R} \chi + \cdots\,, \quad (\mathcal{H}_{s,\mathrm{tot}} = \mathcal{H}_s + AC_A + BC_B)$$

$$C_A = \phi'\, P_\chi + \mathcal{H} P_\mathcal{R} + 2M_P^2 a^2 \overset{(3)}{\Delta} \mathcal{R} + a^2(\mathcal{H}\phi' - \phi'')\chi \quad \text{(Hamiltonian constraint)}\,,$$

$$C_B = P_{H_T} \quad \text{(Momentum constraint)}\,,$$

$$Q_a = \{\mathcal{R}, H_T, \chi\}, \quad P_a = \{P_\mathcal{R}, P_T, P_\chi\}\,.$$

· Gauge transformation $[\xi^\mu = (T, \partial_i L)\,]$ is generated by $C_A$ and $C_B$:

$$\delta_g Q = \left\{ Q, \int (T C_A + L C_B)\, d^3x \right\}_{P.B.}$$

Up to total derivatives, $\mathcal{L}_2$ is gauge-invariant:

$$\delta_g \mathcal{L}_2 = 0 + (\text{total derivatives})$$

In particular, $C_A$ and $C_B$ are gauge-invariant by themselves.

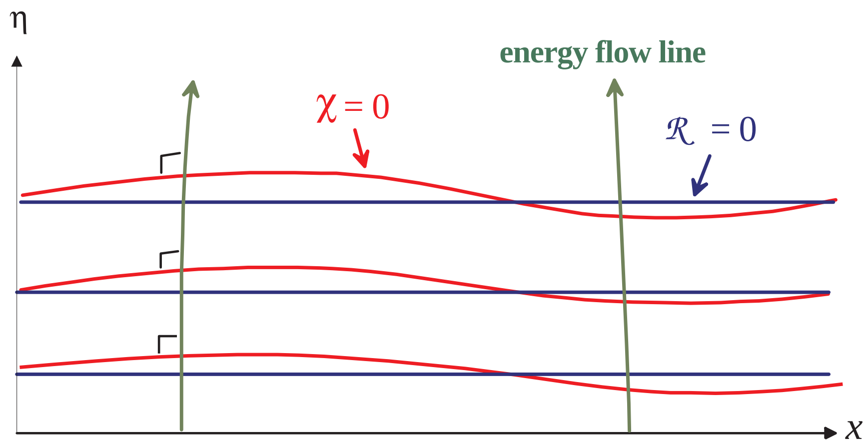
- Reduction to unconstrained variables *a lá* Faddeev-Jackiw (1988)

1. Solve $C_A = \phi' P_\chi + \cdots = 0$ for $P_\chi$ and insert it into $S_2$. Also insert $C_B = P_{H_T} = 0$.
2. The resulting $S_2$ is a functional of $\{P_\mathcal{R}, \mathcal{R}, \chi\}$: $\quad S_2^* = S_2^*[P_\mathcal{R}, \mathcal{R}, \chi]$
3. Since $C_A = C_B = 0$ are gauge-invariant, $S_2^*$ is still gauge-invariant. Hence it must be expressed solely in terms of gauge-invariant variables. Indeed, one finds

$$S_2^* = \int d\eta\, d^3x \left[ P_c \mathcal{R}_c' - \frac{2M_P^4 a^2}{\phi'^2} \left( \overset{(3)}{\Delta} \mathcal{R}_c + \frac{\mathcal{H}}{2M_P^2 a^2} P_c \right)^2 - a^2 M_P^2 \mathcal{R}_c \overset{(3)}{\Delta} \mathcal{R}_c \right]$$

$$P_c \equiv P_\mathcal{R} + \frac{2M_P^2 a^2}{\phi'} \overset{(3)}{\Delta} \chi \,, \quad \mathcal{R}_c \equiv \mathcal{R} - \frac{\mathcal{H}}{\phi'} \chi$$

This is in fact the same as choosing $\chi = 0$ gauge (called ‘comoving’ slicing).
i.e., $\mathcal{R}_c$ is the curvature perturbation on the comoving hypersurface.



⋆ $S_2^*$ in the 2nd order form:

$$S_2^* = \int d\eta\, d^3x\, \frac{z^2}{2}\left(\mathcal{R}_c'^2 - (\nabla\mathcal{R}_c)^2\right); \quad z \equiv \frac{a\phi'}{\mathcal{H}}, \quad \mathcal{H} \equiv \frac{a'}{a} = a\,H$$

⋆ can be generalized to the case of a non-trivial sound velocity $c_s^2 \neq 1$:

$$S_2^* = \int d\eta\, d^3x\, \frac{z^2}{2}\left(\mathcal{R}_c'^2 - c_s^2(\nabla\mathcal{R}_c)^2\right); \quad z \equiv \frac{a(\rho+p)^{1/2}}{c_s H}. \quad \text{(Garriga \& Mukhanov '99)}$$

Equation of motion (for Fourier modes: $\overset{(3)}{\Delta} \to -k^2$)

$$\mathcal{R}_c'' + 2\frac{z'}{z}\mathcal{R}_c' + c_s^2 k^2 \mathcal{R}_c = 0; \quad z \propto a\frac{(1+w)^{1/2}}{c_s} \propto a \text{ for slow-roll inflation}.$$

For $c_s k < \mathcal{H}$ ($\Leftrightarrow$ $c_s k/a < H$),

$$\mathcal{R}_c' \propto \begin{cases} z^{-1} & \sim \text{decaying mode} \\ 0 & \sim \text{growing mode} \end{cases}$$

· "growing" mode of $\mathcal{R}_c$ stays constant on super-(sound) horizon scales.

· The existence of a constant mode is a general property of any cosmological model.

**But this does not mean that adiabatic $\mathcal{R}_c$ is constant on super-horizon scales.**

• Inflaton perturbation on flat slicing (assume $c_s = 1$ again)

Alternatively, in terms of $\chi$ on $\mathcal{R} = 0$ hypersurface (flat slicing),

$$\chi_F \equiv \chi - \frac{\phi'}{\mathcal{H}}\mathcal{R} = -\frac{\phi'}{\mathcal{H}}\mathcal{R}_c$$

$$S_2^* = S_2^*[\chi_F] = \int d\eta\, d^3x\, \frac{a^2}{2}\left({\chi_F'}^2 - (\nabla\chi_F)^2 - a^2 m_{eff}^2\, \chi_F^2\right);$$

$$m_{eff}^2 = -\frac{\left\{a^2\left(\phi'/\mathcal{H}\right)'\right\}'}{a^4\left(\phi'/\mathcal{H}\right)} = \partial_\phi^2 V + \frac{2}{M_P^2}\frac{d}{dt}\left(\frac{V}{H}\right)$$

$\chi_F \sim$ minimally coupled almost massless scalar in de Sitter space

$\because\ \partial_\phi^2 V \ll H^2$, $(2/M_P^2)(V/H)^{\dot{}} \approx 6\dot{H} \ll H^2$ for slow-roll inflation.

(N.B. the sufficient conditions for slow roll: $\partial_\phi^2 V \ll 3H^2$ & $\dot{H} \ll H^2$.)

· de Sitter approximation for the background:

$$H = \text{const.}\,, \quad a(\eta) = \frac{1}{-H\eta} \quad (-\infty < \eta < 0)$$

This is a good approximation for $k > \mathcal{H}$ (sub-horizon scale) modes

- Canonical quantization

$$\pi(\eta,\vec{x}) = \frac{\delta S_2^*[\chi_F]}{\delta \chi_F'(\eta,\vec{x})}\,, \quad [\chi_F(\eta,\vec{x}),\pi(\eta,\vec{x}')] = i\hbar\delta(\vec{x}-\vec{x}')$$

$$\Rightarrow \quad \hat{\chi}_F = \int \frac{d^3k}{(2\pi)^{3/2}}\left(\hat{a}_{\vec{k}}\,\chi_k(\eta)\,e^{i\vec{k}\cdot\vec{x}} + \text{ h.c.}\right); \quad [\hat{a}_{\vec{k}},\,\hat{a}^\dagger_{\vec{k}'}] = \hbar\delta(\vec{k}-\vec{k}')$$

$$\chi_k'' + 2\mathcal{H}\chi_k' + \left(k^2 + m_{eff}^2 a^2\right)\chi_k = 0\,; \quad \chi_{\vec{k}}\,\bar{\chi}_{\vec{k}}' - \chi_{\vec{k}}'\,\bar{\chi}_{\vec{k}} = \frac{i}{a^2}$$

$$\Leftrightarrow \quad \ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + m_{eff}^2\right)\chi_k = 0\,; \quad \chi_{\vec{k}}\,\dot{\bar{\chi}}_{\vec{k}} - \dot{\chi}_{\vec{k}}\,\bar{\chi}_{\vec{k}} = \frac{i}{a^3}$$

$$\text{slow roll} \quad \Rightarrow \quad m_{eff}^2 \ll H^2 \quad \sim \quad \text{massless}$$

de Sitter approximation:

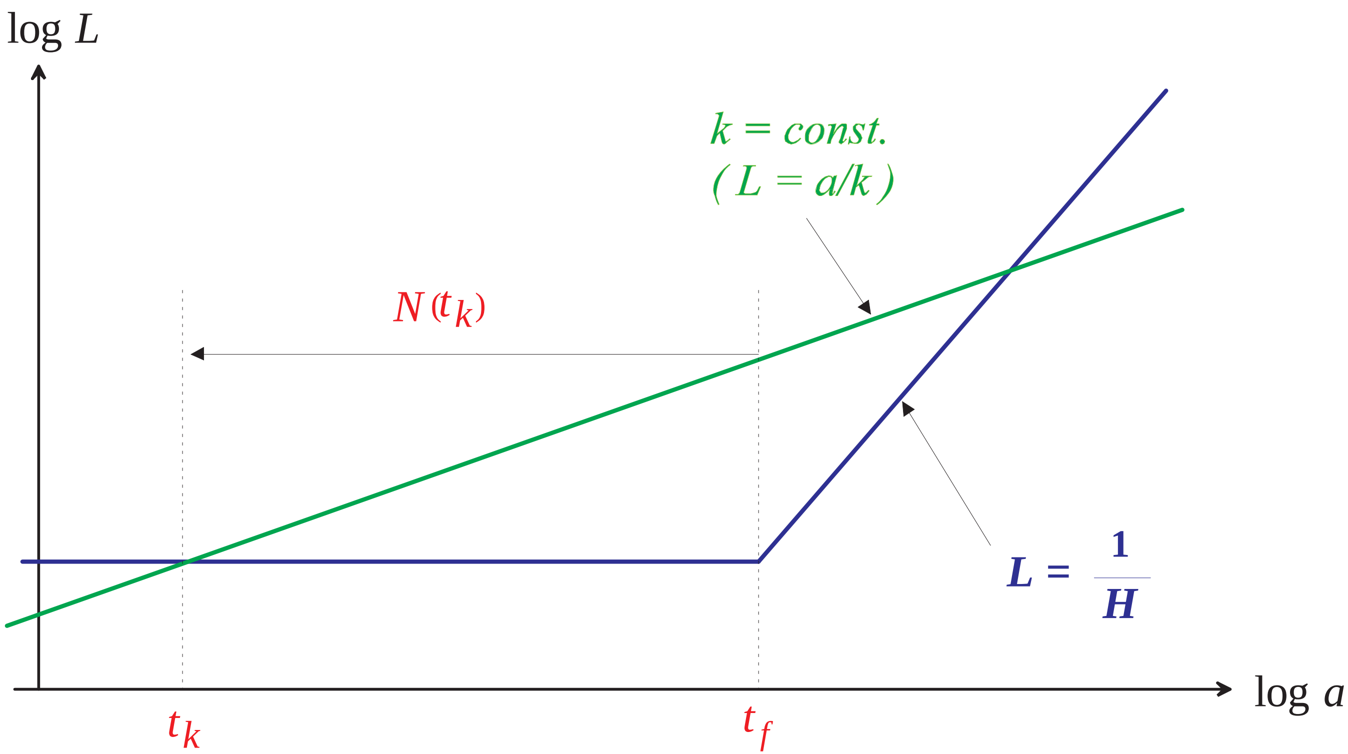
$$\Rightarrow \quad \chi_k \approx \frac{H}{(2k)^{3/2}}(i - k\eta)\,e^{-ik\eta} \quad \begin{cases} \xrightarrow[k/\mathcal{H}\to\infty]{} \dfrac{1}{\sqrt{2ka}}e^{-ik\eta} \\ \xrightarrow[k/\mathcal{H}\to 0]{} \dfrac{H}{\sqrt{2k^3}}e^{-i\alpha_k} \end{cases}$$

$$\langle\delta\phi^2\rangle_k\big|_{\text{on flat slice}} = \langle\chi_F^2\rangle_k \equiv \frac{4\pi k^3}{(2\pi)^3}|\chi_k|^2 \to \left(\frac{H}{2\pi}\right)^2 \quad \text{for} \quad k \lesssim \mathcal{H} \quad (\hbar = 1)$$

· de Sitter approximation breaks down at $k \ll \mathcal{H}$.
i.e., the time-variation of $\chi_k$ on super-horizon scales cannot be neglected.

· However, the corresponding mode of $\mathcal{R}_c$ becomes constant on super-horizon scales.

$$\Rightarrow \quad \mathcal{R}_{c,k}(\eta) \approx \mathcal{R}_{c,k}(\eta_k) = -\frac{\mathcal{H}}{\phi'}\chi_k(\eta_k) \approx \frac{H^2(t_k)}{\sqrt{2k^3}\,\dot{\phi}(t_k)}e^{-i\alpha_k}\,.$$




$t = t_k \quad \Leftrightarrow \quad \eta = \eta_k \quad \Leftrightarrow \quad k = \mathcal{H}(\eta_k)$ ⋯ horizon crossing time

Although not quite intuitive, one can quantize $\mathcal{R}_c$ from the beginning:

$$S_2^* = \int d\eta\, d^3x\, \frac{z^2}{2}\left(\mathcal{R}_c'^{\,2} - (\nabla\mathcal{R}_c)^2\right); \quad z \equiv \frac{a\phi'}{\mathcal{H}}, \quad \mathcal{H} \equiv \frac{a'}{a} = a\,H$$

$$P_c(\eta,\vec{x}) = \frac{\delta S_2^*}{\delta\mathcal{R}_c'(\eta,\vec{x})}, \quad [\mathcal{R}_c(\eta,\vec{x}), P_c(\eta,\vec{x}')] = i\hbar\delta(\vec{x}-\vec{x}')$$

$$\Rightarrow \quad \hat{\mathcal{R}}_c = \int \frac{d^3k}{(2\pi)^{3/2}}\left(\hat{a}_{\vec{k}}\, r_k(\eta)\, e^{i\vec{k}\cdot\vec{x}} + \text{ h.c.}\right); \quad [\hat{a}_{\vec{k}},\, \hat{a}^\dagger_{\vec{k}'}] = \hbar\delta(\vec{k}-\vec{k}')$$

$$r_k'' + 2\frac{z'}{z}r_k' + k^2 r_k = 0; \quad r_k\,\bar{r}_k' - r_k'\,\bar{r}_k = \frac{i}{z^2}$$

$$\Rightarrow \quad r_k \approx \begin{cases} \xrightarrow[k/\mathcal{H}\to\infty]{} \dfrac{H}{a\dot{\phi}}\dfrac{1}{\sqrt{2k}}e^{-ik\eta} \\[2ex] \xrightarrow[k/\mathcal{H}\to 0]{} \left.\dfrac{H^2}{\dot{\phi}}\right|_{\eta=\eta_k}\dfrac{1}{\sqrt{2k^3}}e^{-ik\eta_k}; \qquad -k\eta_k \approx 1 (k\approx\mathcal{H}) \end{cases}$$

$$\left\langle\mathcal{R}_c^2\right\rangle_k \equiv \frac{4\pi k^3}{(2\pi)^3}|r_k|^2 \to \left.\left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\right|_{k=\mathcal{H}} \quad \text{for} \quad -k\eta\to 0$$

- Curvature perturbation spectrum (say, at $\eta = \eta_f$)

$$\langle \mathcal{R}_c^2 \rangle_k \equiv \frac{4\pi k^3}{(2\pi)^3} P_{\mathcal{R}_c}(k;\eta) = \frac{4\pi k^3}{(2\pi)^3} |\mathcal{R}_{c,k}(\eta)|^2 = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \Bigg|_{t=t_k}$$

Since $dN = -Hdt$,

$$\frac{\partial N}{\partial \phi} = -\frac{H}{\dot{\phi}} \quad \Rightarrow \quad \langle \mathcal{R}_c^2 \rangle_k = \left(\frac{\partial N}{\partial \phi}\frac{H}{2\pi}\right)^2 \Bigg|_{t=t_k} = \left(\frac{\partial N}{\partial \phi}\delta\phi\right)^2 \Bigg|_{t=t_k} \text{ on flat slice}$$

That is, for single-field slow-roll inflation,

$$\mathcal{R}_c = \delta N|_{t=t_k} = \frac{\partial N}{\partial \phi}\delta\phi\Big|_{t=t_k} \quad (\delta\phi = \frac{H}{2\pi}) \quad \text{on flat slice}$$

Only the knowledge of the homogeneous background is sufficient
to predict the perturbation spectrum: $\delta N$-formula

If $\langle \mathcal{R}_c^2 \rangle_k \propto k^{n_s-1}$

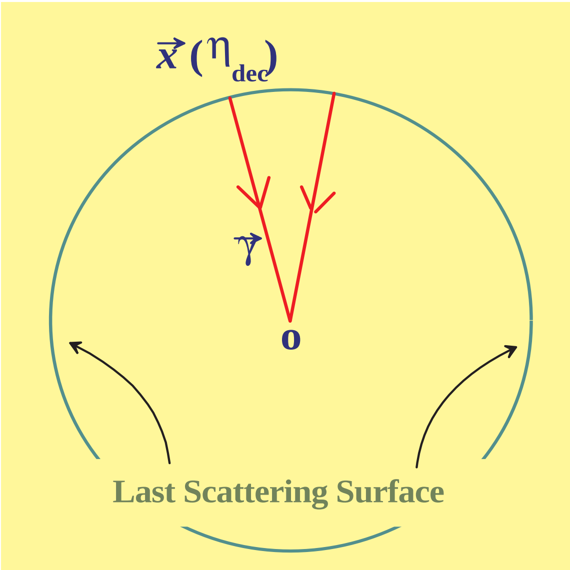
$n_s = 1$ : scale-invariant (Harrison-Zeldovich) spectrum

$n_s = 1 - \epsilon$ $(\epsilon \ll 1)$ for chaotic inflation $(V(\phi) \propto \phi^p)$.

- Large angle CMB anisotropy

$$\left(\frac{\delta T}{T}\right)(\vec{\gamma}, \eta_0) = (\zeta_{\rm r} + \Theta)(\eta_{\rm dec}, \vec{x}(\eta_{\rm dec})) + \int_{\eta_{\rm dec}}^{\eta_0} d\eta\, \partial_\eta \Theta(\eta, \vec{x}(\eta))$$

(Sachs-Wolfe) (Integrated Sachs-Wolfe)




$\zeta_{\rm r}$ : curvature perturbation on $\rho_{\rm photon} = $ const. surfaces

$\Theta \equiv \Psi - \Phi$

$\Psi =$ Newton potential

$\Phi =$ curvature pert. on Newton slice

For a dust-dominated universe at decoupling,

$$\text{SW: } \zeta_{\rm r} + \Theta = -\frac{1}{5}\mathcal{R}_{c*} - \frac{2}{5}S_{\rm dr} = \frac{1}{3}\Psi_* - \frac{2}{5}S_{\rm dr}\,, \quad \text{no ISW: } \partial_\eta\Theta \approx 0$$

$\mathcal{R}_{c*}$ : primordial adiabatic curvature perturbation; $\Phi_* = -\Psi_* = \frac{3}{5}\mathcal{R}_{c*}$

$$S_{\rm dr} = \frac{\delta\rho_{\rm d}}{\rho_{\rm d}} - \frac{3}{4}\frac{\delta\rho_{\rm r}}{\rho_{\rm r}} \quad \sim \text{ entropy perturbation}$$

- Observational implications of Large-angle CMB anisotropy
  COBE-DMR ('96); WMAP 9yr ('12); Planck ('13)

$$\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle \sim 10^{-10} \quad \text{at } \theta \sim 10^\circ. \qquad \frac{\delta T}{T} = \frac{1}{3}\Psi + \cdots \quad \text{for adiabatic perturbation}$$

$$\Downarrow$$

$$\left\langle \Psi^2 \right\rangle_k \sim 10^{-10} \quad \text{at} \quad \frac{k_0}{a_0} = H_0 \sim \frac{1}{\text{present horizon scale}} \qquad \left(H_0^{-1} \sim 3000\,\text{Mpc} \sim 10^{28}\text{cm}\right)$$

For $V = \frac{1}{2}m^2\phi^2$,

$$\left\langle \Psi^2 \right\rangle_{k_0} \approx \left(\frac{3}{5}\right)^2 \left\langle \mathcal{R}_c^2 \right\rangle_{k_0} = \left(\frac{3}{5}\right)^2 \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \Bigg|_{\frac{k_0}{a}=H} \approx \frac{m^2}{25 M_P^2} N^2(\phi) \Bigg|_{\frac{k_0}{a}=H}$$

$$\Rightarrow \quad \begin{cases} m \sim 10^{13}\text{GeV} \\ V \sim (10^{16}\text{GeV})^4 \end{cases}$$

- power-law index: $n_{\text{WMAP9yr}} = 0.9608 \pm 0.008, \quad n_{\text{Planck}} = 0.9603 \pm 0.0073$

  Blue or scale invariant spectrum ($n_s \geq 1$) is excluded at high CL!

- Tensor-type perturbations

$$ds^2 = -dt^2 + \ a^2(t)\,(\delta_{ij} + h_{ij})\,dx^i dx^j$$

$h_{ij}$ $\cdots$ Transverse-Traceless

$$\begin{aligned}\delta^2 S_G &= \frac{M_P^2}{8}\int d^4x\,a^3\left(\dot{h}_{ij}^2 - \frac{1}{a^2}(\nabla h_{ij})^2\right)\\ &= \frac{1}{2}\int d^4x\,a^3\left(\dot{\varphi}_{ij}^2 - \frac{1}{a^2}(\nabla \varphi_{ij})^2\right)\ ; \quad \varphi_{ij} := \frac{M_P}{2}h_{ij}\end{aligned}$$

$\varphi_{ij} \sim$ massless scalar (2 degrees of freedom)

$$\langle \varphi_{ij}^2 \rangle_k = \ 2 \times \left(\frac{H}{2\pi}\right)^2$$

$$\Rightarrow \ \frac{4\pi k^3}{(2\pi)^3} P_T(k) \equiv \langle h_{ij}^2 \rangle_k = 2 \times \frac{4}{M_P^2} \times \left(\frac{H}{2\pi}\right)^2 = \frac{2}{\pi^2}\frac{H^2}{M_P^2}$$

$\uparrow$ contribute to CMB anisotropy

$$r \equiv \frac{T}{S} = \frac{\text{tensor}}{\text{scalar}} \sim \frac{\langle h_{ij}^2 \rangle}{\langle \mathcal{R}_c^2 \rangle} \equiv \frac{P_T(k)}{P_S(k)} = 24\,\frac{\dot{\phi}^2}{V}\bigg|_{k_0 = aH} \qquad \text{slow roll} \quad \Rightarrow \quad r \ll 1\,.$$

$$r \sim 0.13 \quad \text{for} \quad V = \frac{1}{2}m^2\phi^2 \quad \Leftrightarrow \quad r_{\text{Planck}} < 0.11 \quad \text{(95\% CL)}$$

- Spectral index

∗ scalar-type (curvature) perturbation

$$n_S \equiv 1 + \frac{d\ln[P_{\mathcal{R}}(k)k^3]}{d\ln k}\,.$$

$$k = a(t_k)H \quad \rightarrow \quad d\ln k = \frac{da}{a} + \frac{dH}{H} \approx \frac{da}{a} = \left.\frac{d}{H\,dt}\right|_{t=t_k}.$$

For slow-roll inflation,

$$n_S - 1 = \frac{d}{H\,dt}\ln[P_{\mathcal{R}}(k)k^3] = \frac{d}{H\,dt}\left(\ln H^4 - \ln\dot{\phi}^2\right) \approx \frac{2V''V - 3V'^2}{8\pi G V^2}\,.$$

For a chaotic type potential, $V \propto \phi^n$ $(n > 0)$,

$$n_S = 1 - \frac{n(n+2)}{8\pi G\phi^2} < 1 \quad \Leftarrow \quad \text{redder for larger } n$$

∗ tensor-type perturbation

$$n_T \equiv \frac{d\ln[P_T(k)k^3]}{d\ln k} = \frac{d}{H\,dt}\ln[P_T(k)k^3] = \frac{d}{H\,dt}\ln H^2 = 2\frac{\dot{H}}{H^2} = -\frac{8\pi G\dot{\phi}^2}{H^2}$$

$$\approx -3\frac{\dot{\phi}^2}{V} = -\frac{1}{8}\frac{P_T(k)}{P_S(k)} = -\frac{r}{8} \quad \Leftarrow \quad \text{consistency relation!}$$

- Model dependence

  * power-law inflation

$$V(\phi) \propto \exp[\lambda\phi/m_{pl}] \ \leftarrow \ \text{dilaton in string theories ?}$$

$$a \propto t^{\alpha} \quad (\alpha = \frac{16\pi}{\lambda^2})$$

$$\Rightarrow \quad n_S < 1\,, \qquad \frac{T}{S} \gtrsim 0.1$$

  * hybrid inflation ← supergravity-motivated ?

$$\text{e.g.,} \quad V(\phi,\psi) = \frac{1}{4\lambda}\left(M^2 - \lambda\psi^2\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}g^2\phi^2\psi^2$$

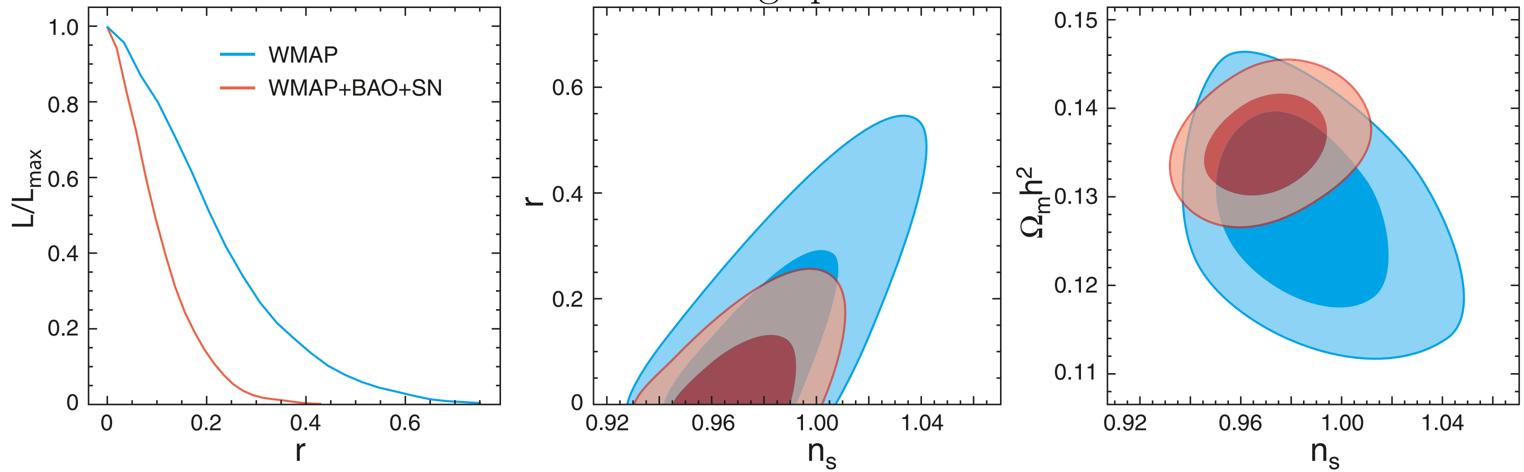
$$a \propto e^{Ht}\,, \quad H^2 \approx \frac{8\pi G}{3}V_0 \quad \text{when} \quad \psi = 0,\ \phi > M/g.$$

$$\Rightarrow \quad n_S > 1\,, \qquad \frac{T}{S} \ \text{can be large or small.}$$

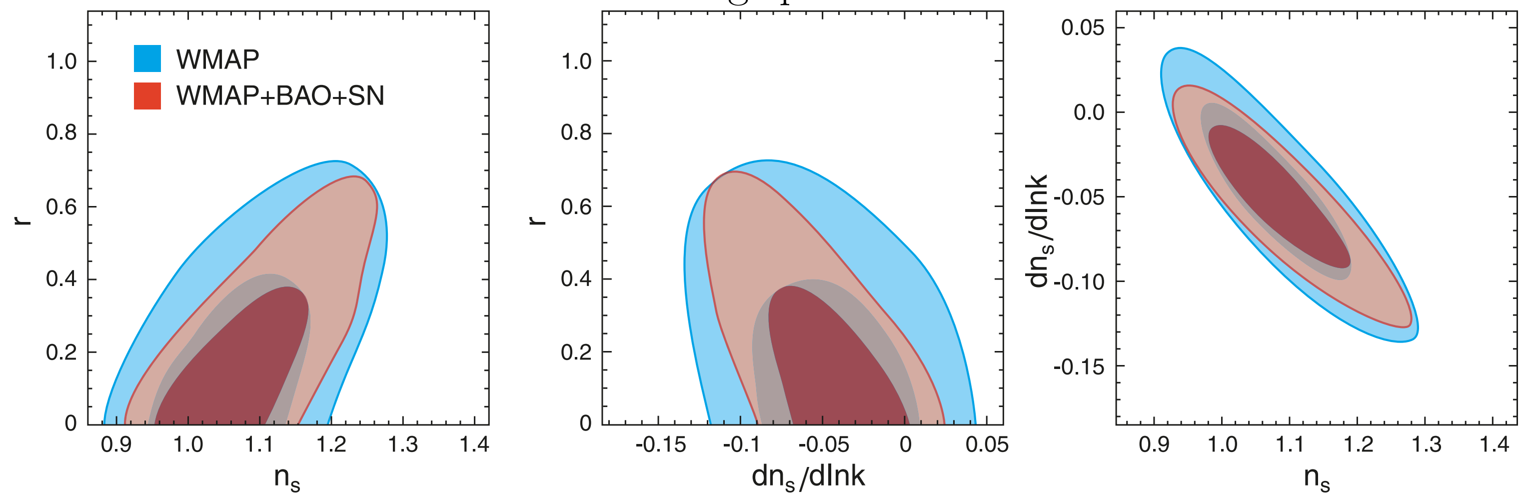
- Observational Constraints from WMAP & Planck

WMAP 5yr: arXiv:0803.0547 [astro-ph]

without running spectral index

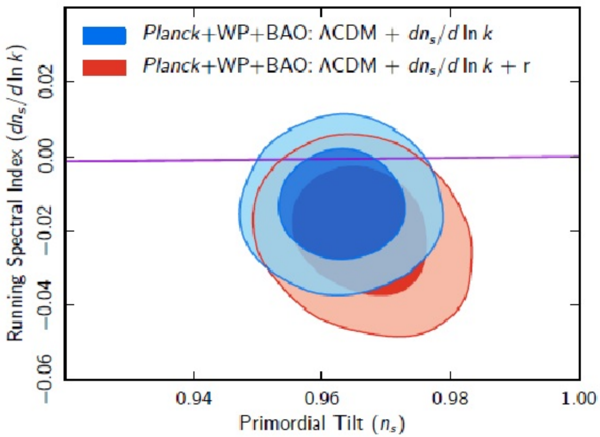



with running spectral index



- Planck constraints with/without running spectral index

Planck: arXiv:1303.5082 [astro-ph.CO]



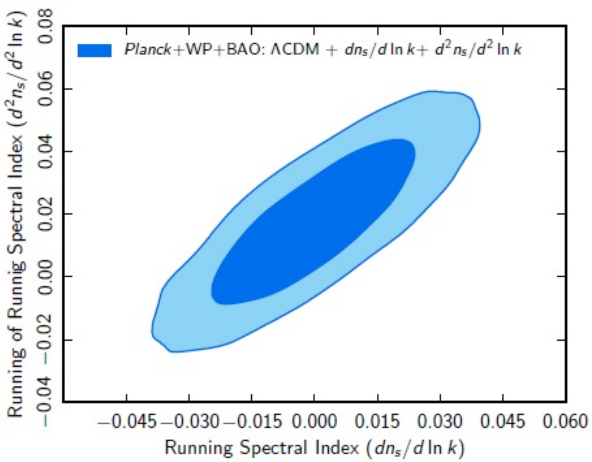




**Fig. 3.** Marginalized joint 68% and 95% CL regions for $(d^2n_s/d\ln k^2\,,\,dn_s/d\ln k)$ using *Planck*+WP+BAO.

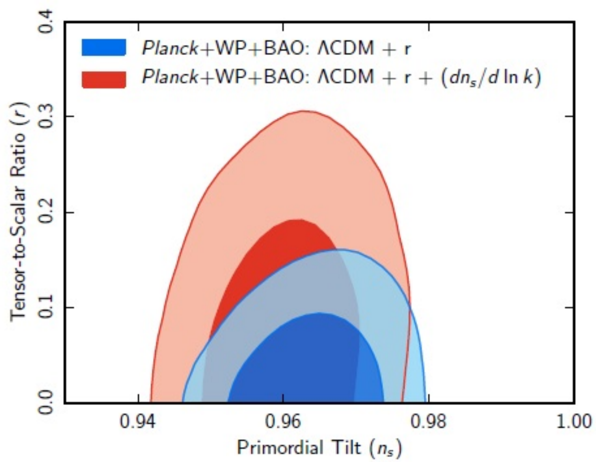
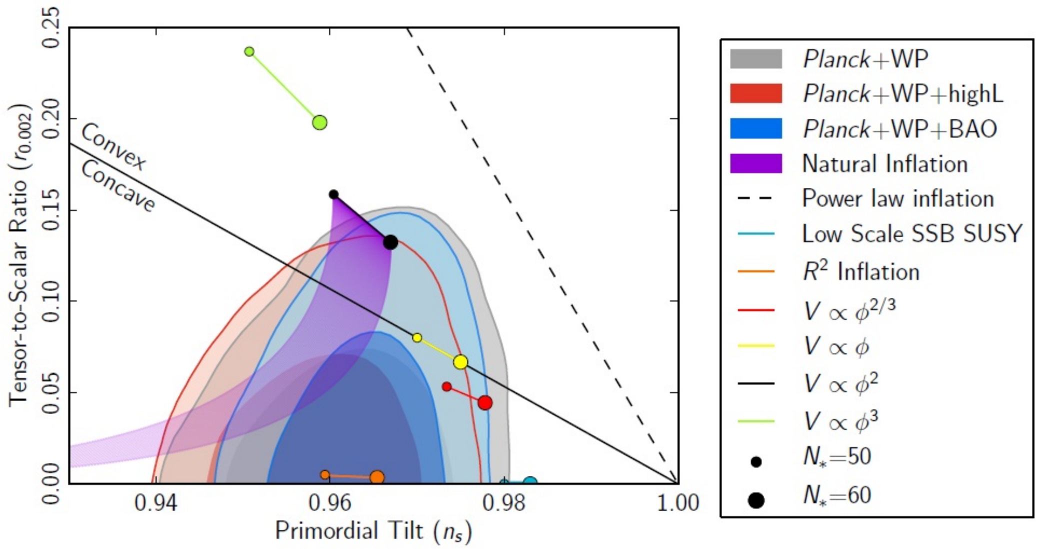



**Fig. 4.** Marginalized joint 68% and 95% CL regions for $(r\,,n_s)$, using *Planck*+WP+BAO with and without a running spectral index.

# Constraints on single-field slow-roll inflation

Constraints are becoming tighter:
Single-field models may be excluded in the near future.




Multi-field models? Non-Gaussianity? Other discriminators?

# §4. Summary of single-field slow-roll inflation

- The growing mode of the curvature perturbation on comoving slices $\mathcal{R}_c$ stays constant super-horizon scales.
  - · $\mathcal{R}_c \approx \Delta N$ in the slow-roll case.
  - · $\mathcal{R}_c$ may vary in time if the slow-roll condition is violated.
  - · Slow-roll models predict almost scale-invariant spectrum, but other spectral shapes are possible.
- Tensor perturbations may be non-negligible.

Recent BICEP2 result suggests $r \sim 0.2$.

On-going and future observations

– BigBOSS, Euclid $\cdots$ $\sim 5 \times 10^7$ galaxies, up to $z \lesssim 2$

– Planck, LiteBIRD, PRISM $\cdots$ high resolution CMB polarization map

$\Downarrow$

Inflaton potential may be determined

$\Downarrow$

Understanding of physics of the early universe ($\approx$ extreme high energy physics)