Theory of Cosmological Perturbations

Part II

- perturbations from inflation –

II.II. δN formalism

a (highly biased) list of references

linear

- M. Sasaki and E.D. Stewart,
A General analytic formula for the spectral index of the density perturbations produced during inflation,
Prog. Theor. Phys. 95, 71 (1996) [astro-ph/9507001].

quasi-nonlinear / separate universe approach

- M. Sasaki and T. Tanaka,
Superhorizon scale dynamics of multiscalar inflation,
Prog. Theor. Phys. 99, 763 (1998) [gr-qc/9801017].

nonlinear

- D.H. Lyth, K.A. Malik and M. Sasaki,
A General proof of the conservation of the curvature perturbation,
JCAP 0505, 004 (2005) [astro-ph/0411220].

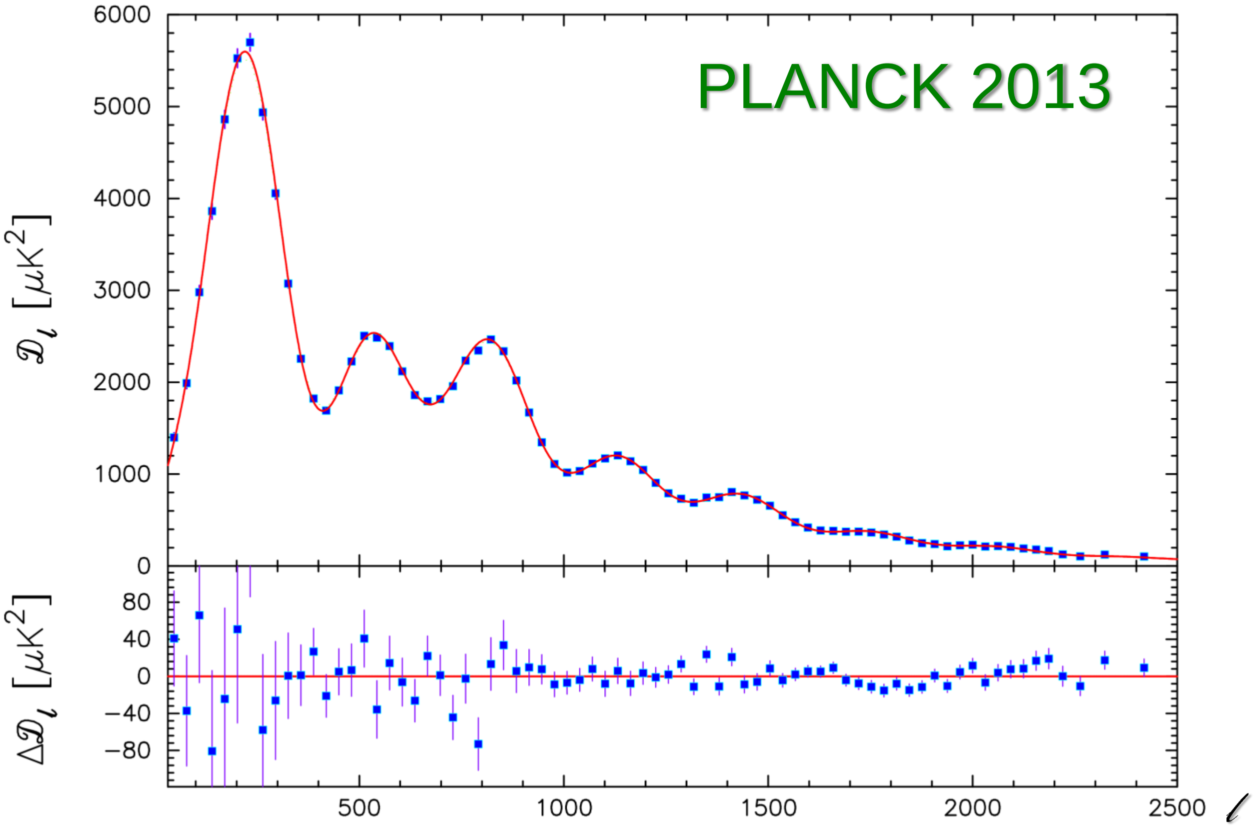
- A. Naruko and M. Sasaki,
Conservation of the nonlinear curvature perturbation in generic single-field inflation,
Class. Quant. Grav. 28, 072001 (2011) [arXiv:1101.3180 [astro-ph.CO]].

conformal frame (in)dependence

- J.-O. Gong, J.-c. Hwang, W.-I. Park, M. Sasaki and Y.-S. Song,
Conformal invariance of curvature perturbation,
JCAP 1109 (2011) 023 [arXiv:1107.1840 [gr-qc]].

1. Introduction

- Standard (single-field, slowroll) inflation predicts almost scale-invariant Gaussian curvature perturbations.



- CMB (WMAP,PLANCK,...) is consistent with the prediction.
- Linear perturbation theory seems to be valid.

However, nature may be a bit more complicated...

- Tensor perturbations (gravitational waves) may be detected in the near future.

  tensor-scalar ratio: $r<0.11$ (95%CL) PLANCK 2013

  N.B. BICEP2 claims $r \sim 0.2$

- Future CMB experiments may still detect non-Gaussianity...

  (-)gravitational potential: $\Phi=\Phi_{gauss}+ f_{NL}\Phi^2_{gauss}+ \cdots$

  $-8.9< f_{NL}<14.3$ (95%CL) PLANCK 2013

- Models need to be tested.

  multi-field, non-slowroll, string theory, vacuum bubbles, ...

$\boxed{\delta N \text{ formalism for curvature perturbations}}$

What is δN ?

- δN is the perturbation in # of e-folds counted backward in time from a fixed final time t_f

  therefore it is nonlocal in time by definition

- t_f should be chosen such that the evolution of the universe has become unique by that time.

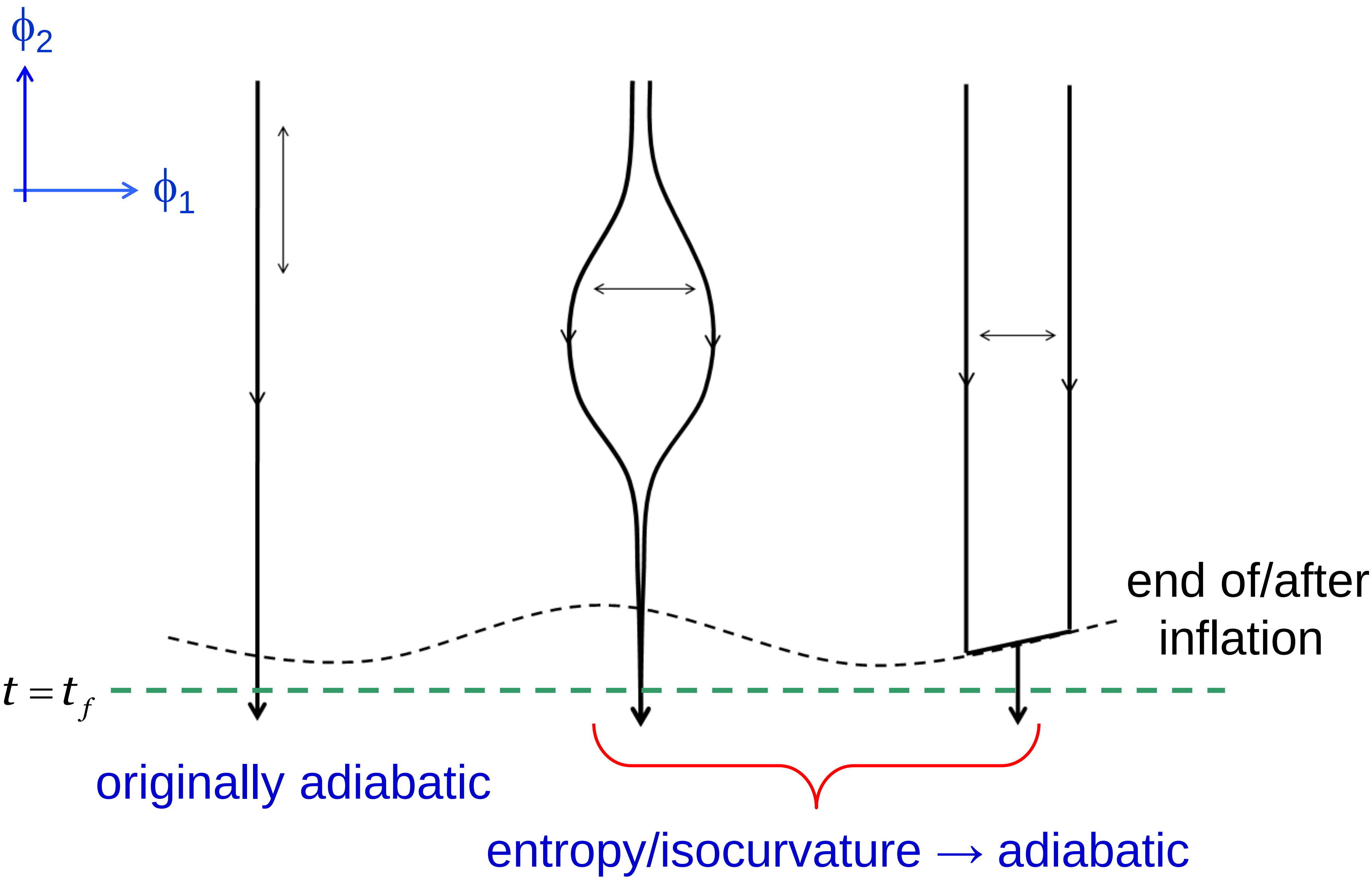
  "adiabatic limit"

  isocurvature perturbation that persists until $t=t_f$ must be dealt separately

- δN is equal to conserved NL comoving curvature perturbation $\mathcal{R}_{NL}$ on superhorizon scales at $t>t_f$

- δN formula is valid independent of theory of gravity

3 types of δN



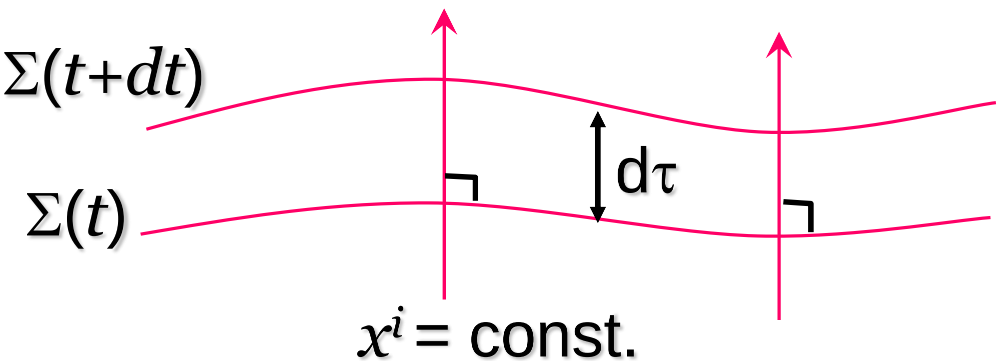
2. Linear perturbation theory

Bardeen '80, Mukhanov '80, Kodama & MS '84, ….

- metric (on a spatially flat background)

traceless

$$ds^2 = -(1+2A)dt^2 + a^2(t)\left[(1+2\mathcal{R})\delta_{ij} + H_{ij}\right]dx^i dx^j$$



$$\left(H_{ij}\right)_{\text{scalar}} = \left[\partial_i\partial_j - \frac{1}{3}\delta_{ij}\overset{(3)}{\Delta}\right]H_T$$

$$\left(H_{ij}\right)_{\text{tensor}} = \text{transverse-traceless}$$

- propertime along x^i = const.: $d\tau = (1+A)dt$
- curvature perturbation on $\Sigma(t)$: $\mathcal{R} \Leftrightarrow \overset{(3)}{R} = -\dfrac{4}{a^2}\overset{(3)}{\Delta}\mathcal{R}$
- expansion (Hubble parameter): $\tilde{H} = H(1-A) + \partial_t\left[\mathcal{R} + \dfrac{1}{3}\overset{(3)}{\Delta}H_T\right]$

Choice of gauge (time-slicing)

- comoving slicing $\quad T^{\mu}{}_{i} = 0 \quad \left(\phi = \phi(t) \text{ for a scalar field}\right)$

matter-based gauge

4-velocity $\perp$ t=const.

- uniform density slicing $\quad -T^{0}{}_{0} \equiv \rho = \rho(t)$

- uniform Hubble slicing

$$\tilde{H} = H(t) \;\Leftrightarrow\; -HA + \partial_t \left[\mathcal{R} + \frac{1}{3} \overset{(3)}{\Delta} H_T \right] = 0$$

geometry-based gauge

- flat slicing $\quad \overset{(3)}{R} = -\frac{4}{a^2} \overset{(3)}{\Delta} \mathcal{R} = 0 \;\Leftrightarrow\; \mathcal{R} = 0$

- Newton (shear-free) slicing

$$\partial_t \left(H_{ij} \right)_{\text{scalar}} = \left[\partial_i \partial_j - \frac{1}{3} \delta_{ij} \overset{(3)}{\Delta} \right] \partial_t H_T = 0 \;\Leftrightarrow\; \partial_t H_T = 0 \;\Leftrightarrow\; H_T = 0$$

comoving = uniform ρ = uniform H on superhorizon scales

Separate universe approach

(in linear perturbation theory)

$$G^{0}{}_{0} = 8\pi G T^{0}{}_{0} \quad \Rightarrow \quad 3\tilde{H}^2 - \frac{2}{a^2}\overset{(3)}{\Delta}\mathcal{R} + O\left(\varepsilon^4\right) = 8\pi G\rho$$

$$\varepsilon = \frac{\text{Hubble horizon scale}}{\text{wavelength}} \left(\ll 1 \text{ on superhorizon scales}\right)$$

at leading order in ε , Friedmann equation holds independent of time-slicing.

⇨ local ‘Hubble parameter’ given by $3\tilde{H}^2 = 8\pi G\rho + O\left(\varepsilon^2\right)$

‘local’ means ‘measured on scales of Hubble horizon size’ further, if $\mathcal{R}$ is time-independent,

Friedmann equation holds up through O(ε^2),

with local 'curvature constant' given by $K\left(x^i\right) = -\frac{2}{3}\overset{(3)}{\Delta}\mathrm{R}\left(x^i\right)$

$$3\tilde{H}^2 - \frac{2}{a^2}\overset{(3)}{\Delta}\mathcal{R} + O\left(\varepsilon^4\right) = 8\pi G\rho$$

$$\Longrightarrow \quad 3\tilde{H}^2 + \frac{K(x^i)}{a^2} + O\left(\varepsilon^4\right) = 8\pi G\rho$$

comoving curvature perturbation $\mathcal{R}_\mathrm{C}$ is conserved

in the adiabatic limit:

$$R_c'' + \frac{(z^2)'}{z^2}R_c' = O(\varepsilon^2); \quad z^2 \equiv \frac{a^2(\rho+P)}{H^2} \sim a^2$$

local Friedmann eq. holds up through O(ε^2),
for adiabatic perturbations (= adiabatic limit)
on comoving/uniform ρ/uniform H slices.

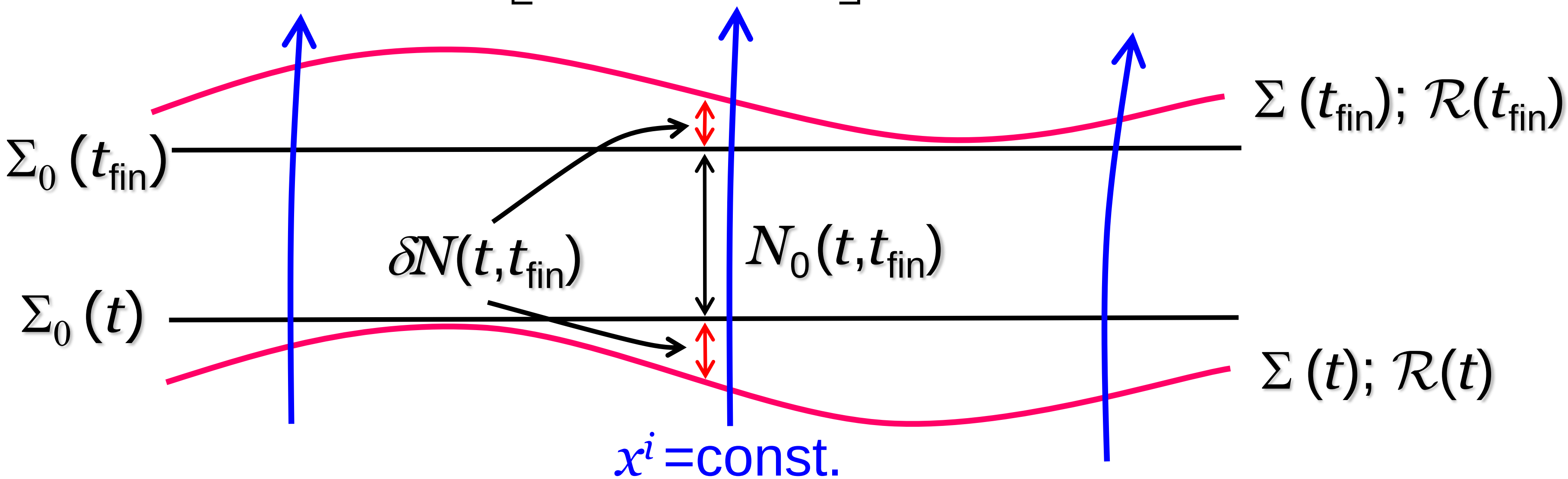
3. Linear δN formula

Starobinsky '85, MS & Stewart '96, ….

e-folding number perturbation between $\Sigma(t)$ and $\Sigma(t_{\rm fin})$:

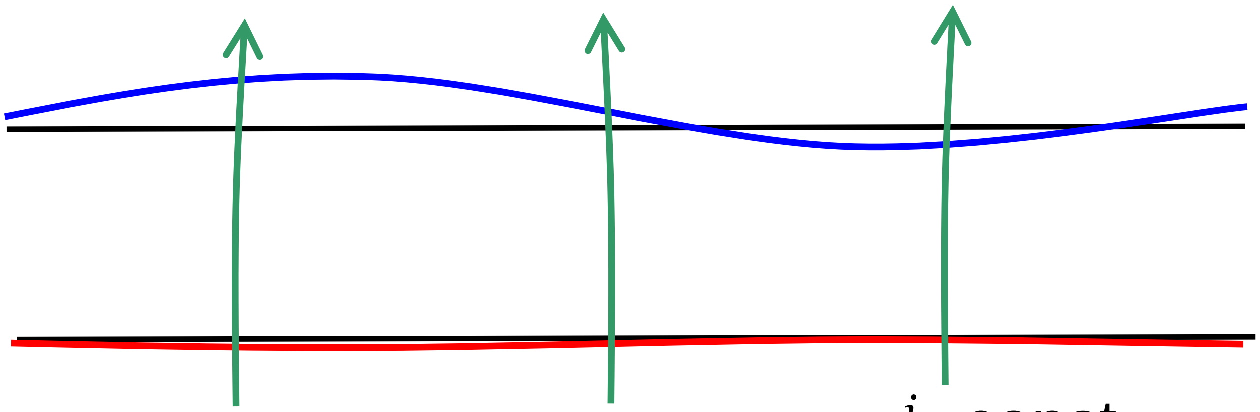
$$\delta N\left(t;t_{\rm fin}\right)\equiv\int_t^{t_{\rm fin}}\tilde{H}\,d\tau-\left(\int_t^{t_{\rm fin}}H\,d\tau\right)_{\rm background}$$

$$=\int_t^{t_{\rm fin}}\partial_t\left[\mathcal{R}+\frac{1}{3}{}^{(3)}\!\Delta\,H_T\right]dt=\mathcal{R}\left(t_{\rm fin}\right)-\mathcal{R}\left(t\right)+O\left(\varepsilon^2\right)$$



δN=0 if both $\Sigma(t)$ and $\Sigma(t_{\rm fin})$ are chosen to be 'flat' ($\mathcal{R}$=0).

Choose $\Sigma(t)$ = flat ($\mathcal{R}$=0) and $\Sigma(t_{\rm fin})$ = comoving:



$$\Rightarrow \delta N(t;t_{\rm fin}) = \mathcal{R}(t_{\rm fin}) - \mathcal{R}(t) = \mathcal{R}_{\rm C}(t_{\rm fin})$$

⇧

curvature perturbation on comoving slice (suffix ‘C’ for comoving)

By definition, $\delta N(t;\, t_{\rm fin})$ is t-independent.

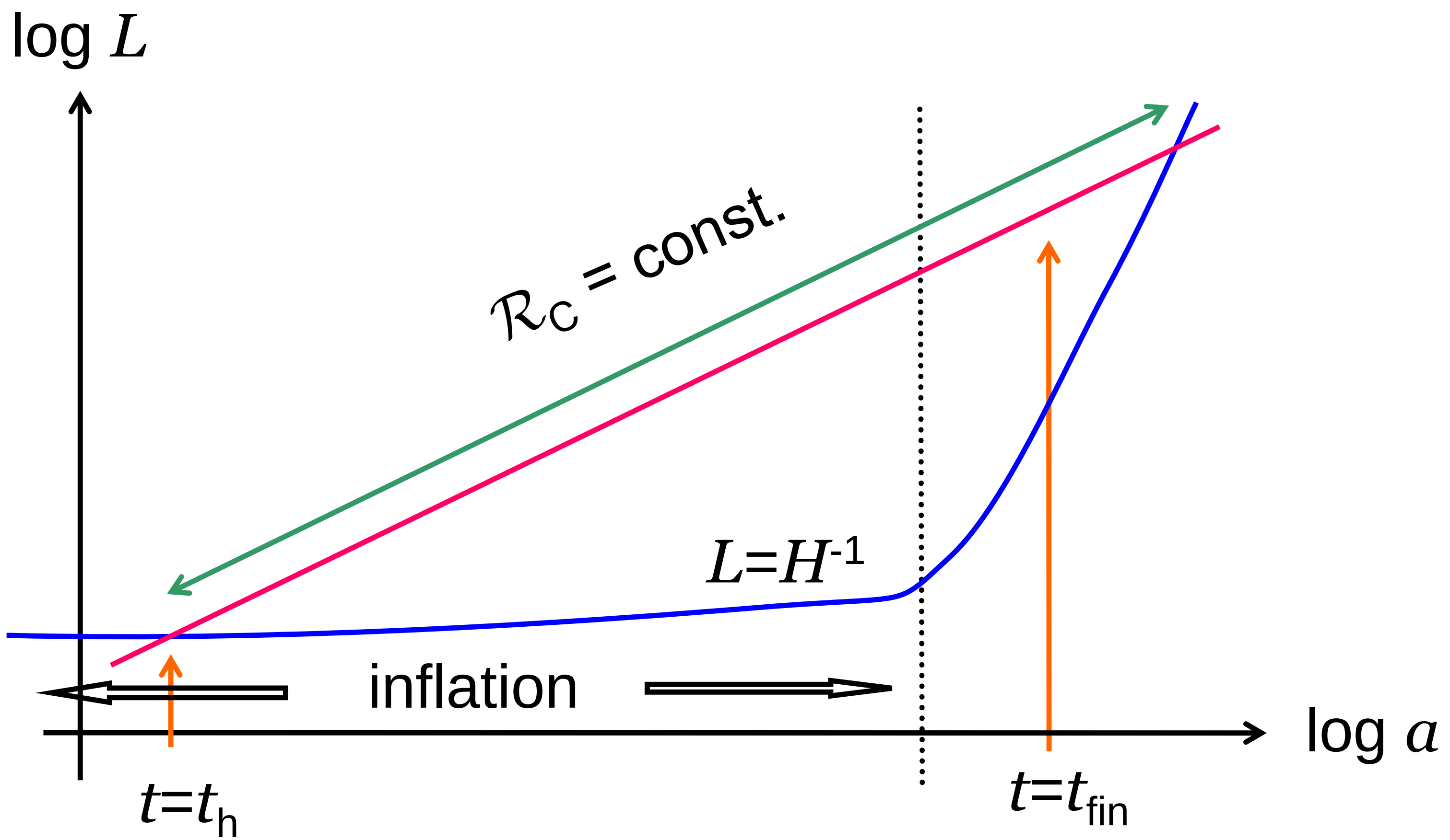
The gauge-invariant variable ‘ζ’ used in the literature is equal to $\mathcal{R}_{\rm C}$ on superhorizon scales (sometimes $\zeta = -\mathcal{R}_{\rm C}$)

Example: single-field slow-roll inflation

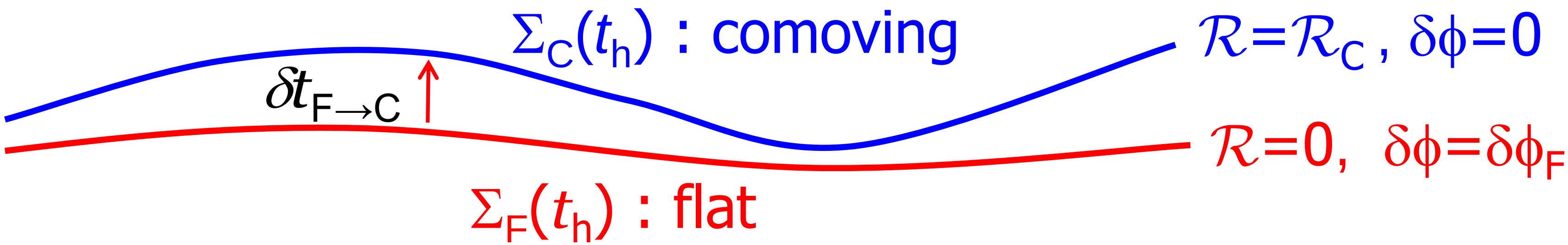
- single-field inflation, no extra degree of freedom

$\mathcal{R}_c$ becomes constant soon after horizon-crossing (t=t_h):

$$\delta N(t_h; t_{fin}) = R_c(t_{fin}) = R_c(t_h)$$



Also because ☼$_{c}$ is conserved, $\delta N = H(t_{h})\,\delta t_{F\to C}$, where $\delta t_{F\to C}$ is the time difference between the comoving and flat slices at $t=t_{h}$.



$$\phi_{F}\left(t_{h}+\delta t_{F\to C}, x^{i}\right)=\phi_{C}\left(t_{h}\right) \quad \Rightarrow \quad \delta\phi_{F}+\dot{\phi}\left(t_{h}\right)\delta t_{F\to C}=0$$

$$\Rightarrow \quad \mathcal{R}_{C}\left(t_{fin}\right)=\delta N\left(t_{h};t_{fin}\right)=H\,\delta t_{F\to C}=-H\frac{dt}{d\phi}\delta\phi_{F}\left(t_{h}\right)$$

$$=\frac{dN}{d\phi}\delta\phi_{F}\left(t_{h}\right)$$ ··· δN formula

Starobinsky ’85

$$dN=-Hdt$$

Only the knowledge of the background evolution is necessary to calculate $\mathcal{R}_{C}(t_{fin})$.

Extension to a multi-component scalar

(for slow-roll, no isocurvature perturbation)

MS & Stewart '96, MS & Tanaka '98

$$\mathcal{R}_{\mathrm{C}}(t_{\mathrm{fin}}) = \delta N = \sum_a \frac{\partial N}{\partial \phi^a} \delta\phi_{\mathrm{F}}^a(t_{\mathrm{h}}) \equiv \nabla_a N \cdot \delta\phi_{\mathrm{F}}^a(t_{\mathrm{h}})$$

$$\nabla_a N \equiv \frac{\partial N}{\partial \phi^a}$$

N.B. $\mathcal{R}_{\mathrm{C}}$ is no longer conserved:

$$\mathcal{R}_{\mathrm{C}}(t) = -H \frac{\dot{\phi} \cdot \delta\phi_{\mathrm{F}}}{\|\dot{\phi}\|^2}$$ ··· time-varying even on superhorizon

- spectrum (for mutually independent $\delta\phi_F{}^a$)

$$\frac{4\pi k^3}{(2\pi)^3} P_S(k) = \|\nabla N\|^2 \|\delta\phi_{\mathrm{F}}\|^2 = \|\nabla N\|^2 \frac{H^2(t_{\mathrm{h}})}{(2\pi)^2} \left(\geq \frac{H^4}{(2\pi)^2 \|\dot{\phi}\|^2} \right)$$

$$H^2 = \left| \dot{\phi}^a \nabla_a N \right|^2 \leq \|\dot{\phi}\|^2 \|\nabla N\|^2 \;\Rightarrow\; \|\nabla N\|^2 \geq \frac{H^2}{\|\dot{\phi}\|^2}$$

tensor-to-scalar ratio

MS & Stewart '96

- scalar spectrum: $$P_S(k)\frac{4\pi k^3}{(2\pi)^3}=\frac{H^2}{(2\pi)^2}\left\|\nabla N\right\|^2 \propto k^{n_s-1}$$

$\kappa^2 = 8\pi G$

- tensor spectrum: $$P_T(k)\frac{4\pi k^3}{(2\pi)^3}=8\kappa^2\frac{H^2}{(2\pi)^2} \propto k^{n_T}$$

$$\varepsilon_s \equiv -\frac{\dot{H}}{H^2}$$

slow-roll parameter

- tensor spectral index: $$-n_T = 2\varepsilon_s \equiv -\frac{2\dot{H}}{H^2}=\kappa^2\frac{\left\|\dot{\phi}\right\|^2}{H^2}$$

$$H=-\frac{dN}{dt}=-\dot{\phi}^a\nabla_a N$$

$$=\kappa^2\frac{\left\|\dot{\phi}\right\|^2}{\left\|\dot{\phi}\cdot\nabla N\right\|^2}\geq\kappa^2\frac{1}{\left\|\nabla N\right\|^2}=\frac{P_T}{8P_S}$$

$\Rightarrow$ $$\boxed{\frac{P_T}{P_S}\leq 8\left|n_T\right| = 16\varepsilon_s}$$

Einstein gravity

⋯ valid for any slow-roll models

('=' for a single inflaton model)

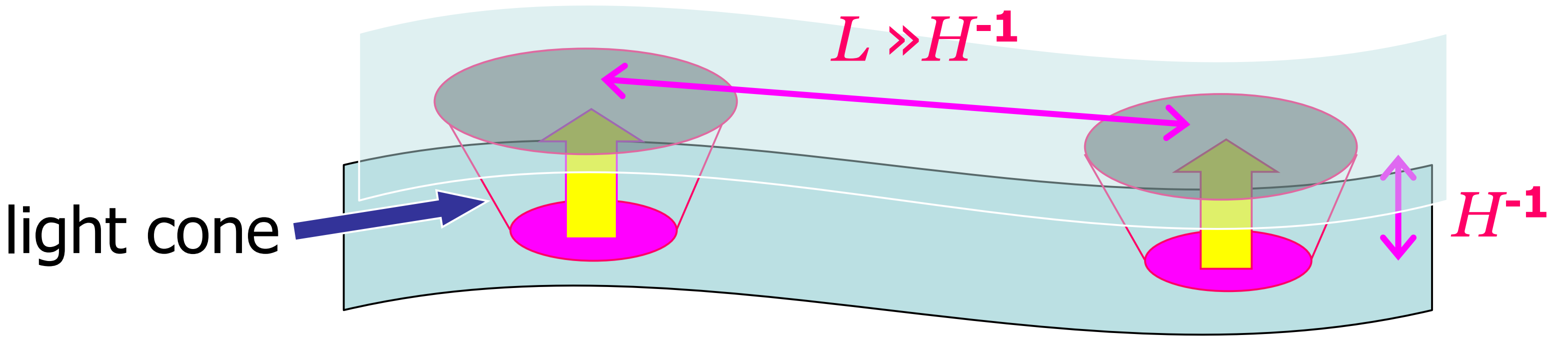
4. Non-linear extension

- On superhorizon scales, gradient expansion is valid:

$$\left|\frac{\partial}{\partial x^i} Q\right| \ll \left|\frac{\partial}{\partial t} Q\right| \sim HQ; \; H \sim \sqrt{G\rho}$$

Belinski et al. '70, Tomita '72, Salopek & Bond '90, …

This is a consequence of causality:



- At lowest order, no signal propagates in spatial directions.

Field equations reduce to ODE's

metric on superhorizon scales

- gradient expansion:

$$\partial_i \to \varepsilon \partial_i \,, \quad \varepsilon = \text{expansion parameter}$$

- metric:

$$ds^2 = -\mathcal{N}^2 dt^2 + e^{2\alpha} \tilde{\gamma}_{ij} \left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

$$\det \tilde{\gamma}_{ij} = 1, \quad \beta^{\mathrm{i}} = O(\varepsilon)$$

the only non-trivial assumption

contains GW (~ tensor) modes

$$\exp\left[\alpha(t,x^i)\right] = \underline{a(t)} \exp\left[\psi(t,x^i)\right]$$ curvature perturbation

fiducial `background'   e.g., choose $\psi(t_*,0) = 0$

- Local Friedmann equation

$$\tilde{H}^2(t,x^i) = \frac{8\pi G}{3}\rho(t,x^i) + O(\varepsilon^2)$$

x^i : comoving (Lagrangian) coordinates.

$$\frac{d}{d\tau}\rho + 3\tilde{H}\left(\rho + p\right) = 0$$

$d\tau = \mathcal{N}dt$: proper time along fluid flow

- exactly the same as the background equations.
- uniform ρ slice = uniform Hubble slice = comoving slice

  as in the case of linear theory

- no modifications/backreaction due to super-Hubble perturbations.

- energy momentum tensor:

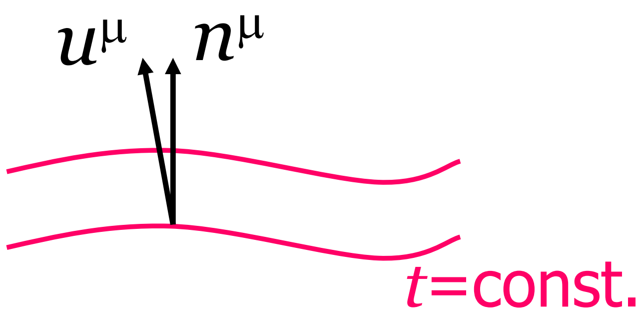
$$T^{\mu\nu} = \rho u^\mu u^\nu + p\left(g^{\mu\nu} + u^\mu u^\nu\right); \quad u_\mu \nabla_\nu T^{\mu\nu} = 0$$

$$\Rightarrow \frac{d}{d\tau}\rho + \nabla_\mu u^\mu \left(\rho + p\right) = 0\,; \;\; \nabla_\mu u^\mu = 3\frac{\partial_t \alpha}{\mathrm{N}} + O\left(\varepsilon^2\right)$$

$$v^i \equiv \frac{u^i}{u^0} = O\left(\varepsilon\right) \Longleftarrow \text{assumption} \Longrightarrow u^\mu - n^\mu = O(\varepsilon)$$

(absence of vorticity mode)

- local Hubble parameter:



$$\tilde{H} \equiv \frac{1}{3}\nabla_\mu u^\mu = \frac{1}{3}\nabla_\mu n^\mu + O\left(\varepsilon^2\right)$$

$$n_\mu dx^\mu = -\mathcal{N} dt \quad \cdots \text{normal to } t = \text{const.}$$

At leading order, local Hubble parameter is independent of the time slicing, as in linear theory

5. Nonlinear δN formula

Lyth, Malik & MS '04

Langlois & Vernizzi '05

- energy conservation:

(applicable to each independent matter component)

$$\frac{\partial_t \rho}{3(\rho+p)} + O(\varepsilon^2) = -\partial_t \alpha = -\left(\frac{\dot{a}}{a} + \partial_t \psi\right) = -\tilde{H}\mathcal{N}$$

- e-folding number:

$$N(t_1, t_2; x^i) \equiv \int_{t_1}^{t_2} \tilde{H}\mathcal{N} dt = -\frac{1}{3}\int_{t_1}^{t_2} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt$$

where x^i =const. is a comoving worldline.

$$\Rightarrow \quad \boxed{\psi(t_2, x^i) - \psi(t_1, x^i) = \Delta N(t_1, t_2; x^i)}$$

where $\Delta N(t_1, t_2; x^i) \equiv N(t_1, t_2; x^i) - N_0(t_1, t_2)$

$$= -\frac{1}{3}\int_{t_1}^{t_2} \frac{\partial_t \rho}{\rho + P(\rho)} dt - N_0(t_1, t_2)$$

To summarize:

$$\psi\left(t_2,x^i\right)-\psi\left(t_1,x^i\right)=\Delta N\left(t_1,t_2;x^i\right)$$

$$=-\frac{1}{3}\int_{t_1}^{t_2}\frac{\partial_t\rho}{\rho+P(\rho)}dt-N_0(t_1,t_2)$$

$\Uparrow$ geometry

$\Uparrow$ matter

This definition applies to any choice of time-slicing

relates the evolution of matter to geometry.

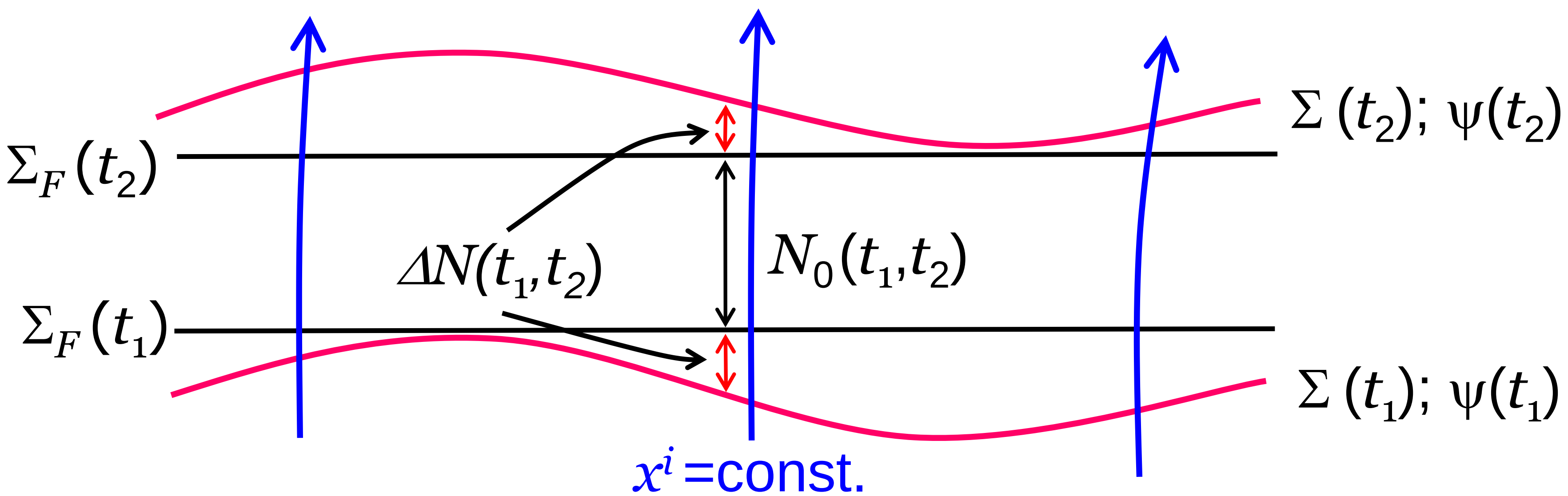
Here we use ΔN for general choice of slices.
δN is reserved for 'δN formula'.

No need for `background' universe

$\Sigma_F(t)$: hypersurface on which $\psi = 0 \leftrightarrow e^{\alpha} = a(t)$; 'flat' slice

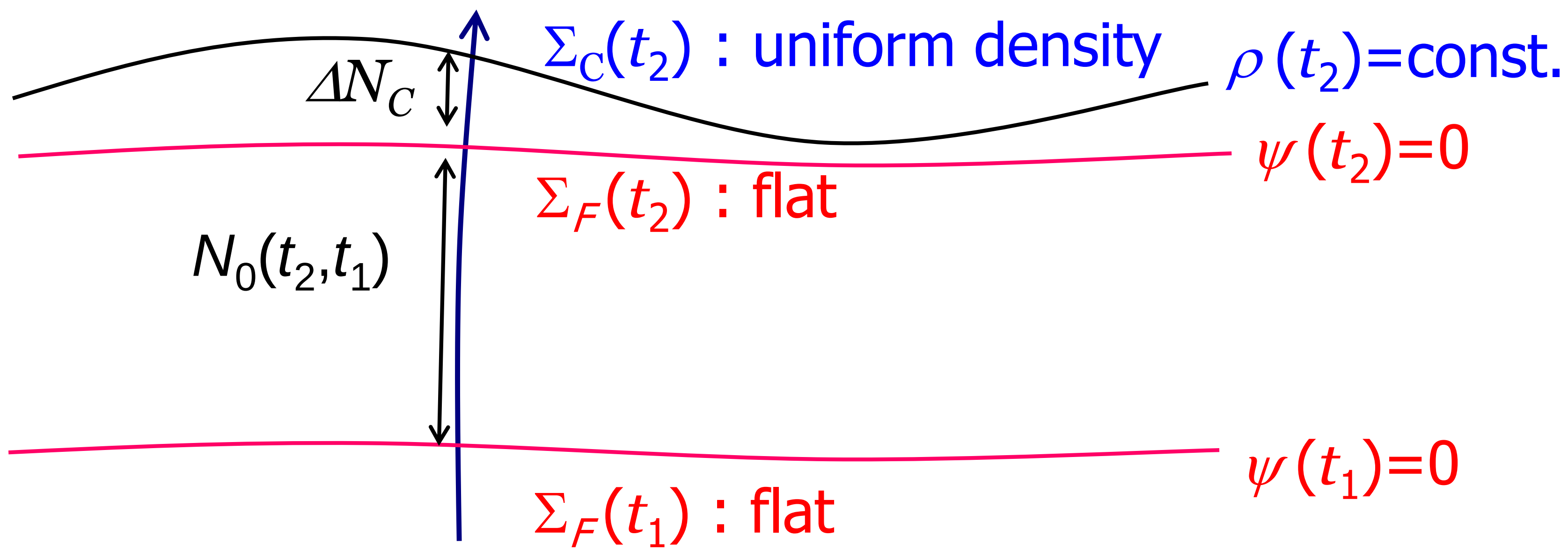
geometry is closest to homogeneous & isotropic universe

$$N(t_1,t_2;x^i) = N_0(t_1,t_2) \text{ between } \Sigma_F(t_1) \text{ and } \Sigma_F(t_2)$$



NL δN - formula

Let us take slicing such that $\Sigma(t)$ is 'flat' at $t = t_1$ [$\Sigma_F(t_1)$]
and uniform density/comoving/uniform H at $t = t_2$ [$\Sigma_C(t_2)$]:
('flat' slice: $\Sigma(t)$ on which $\psi = 0 \leftrightarrow e^{\alpha} = a(t)$)



$$\Delta N(t_1, t_2; x^i) = \Delta N_C(t_1, t_2; x^i)$$

Then

$$\boxed{\psi\left(t_1,x^i\right)=0,\ \ \psi\left(t_2,x^i\right)=\mathcal{R}_{\mathrm{C}}\left(t_2,x^i\right)=\Delta N_C\left(t_1,t_2;x^i\right)}$$

suffix C for comoving/uniform ρ/uniform H

where ΔN_C is the e-folding number from $\Sigma_F(t_2)$ to $\Sigma_C(t_2)$:

$$\delta N\left(t_1,t_2;x^i\right)\equiv\Delta N_C=-\frac{1}{3}\int_{\Sigma_F(t_1)}^{\Sigma_C(t_2)}\left.\frac{\partial_t\rho}{\rho+P}\right|_{x^i}dt+\frac{1}{3}\int_{\Sigma_F(t_1)}^{\Sigma_F(t_2)}\frac{\partial_t\rho}{\rho+P}dt$$

$$=-\frac{1}{3}\int_{\Sigma_F(t_2)}^{\Sigma_C(t_2)}\left.\frac{\partial_t\rho}{\rho+P}\right|_{x^i}dt \longleftarrow \text{indep of } t_1$$

$\Sigma_C(t)$: matter is almost homogeneous & isotropic

($\Leftrightarrow$ $\Sigma_F(t)$: geometry is closest to Friedmann universe)

6. conservation of NL curvature perturbation

For adiabatic case ($p=p(\rho)$,or single-field slow-roll case),

$$N\left(t_1,t_2;x^i\right)=-\frac{1}{3}\int_{t_1}^{t_2}\frac{\partial_t\rho}{\rho+P(\rho)}dt$$

$$=-\frac{1}{3}\int_{\rho(t_1,x^i)}^{\rho(t_2,x^i)}\frac{d\rho}{\rho+P(\rho)}=\psi(t_2,x^i)-\psi(t_1,x^i)+\ln\left[\frac{a(t_2)}{a(t_1)}\right]$$

$$\Rightarrow\ -\frac{1}{3}\int_{\rho(t_1,x^i)}^{\rho(t_2,x^i)}\frac{d\rho}{\rho+P(\rho)}$$

$$=-\frac{1}{3}\int_{\rho(t_2)}^{\rho(t_2,x^i)}\frac{d\rho}{\rho+P(\rho)}+\frac{1}{3}\int_{\rho(t_1)}^{\rho(t_1,x^i)}\frac{d\rho}{\rho+P(\rho)}-\frac{1}{3}\int_{\rho(t_1)}^{\rho(t_2)}\frac{d\rho}{\rho+P(\rho)}$$

$$=\psi(t_2,x^i)-\psi(t_1,x^i)+\ln\left[\frac{a(t_2)}{a(t_1)}\right]$$

$$\Rightarrow \quad \psi(t_1,x^i)+\frac{1}{3}\int_{\rho(t_1)}^{\rho(t_1,x^i)}\frac{d\rho}{\rho+P(\rho)}=\psi(t_2,x^i)+\frac{1}{3}\int_{\rho(t_2)}^{\rho(t_2,x^i)}\frac{d\rho}{\rho+P(\rho)}$$

$$\Rightarrow \quad \boxed{\mathcal{R}_{NL}(x^i)\equiv\psi(t,x^i)+\frac{1}{3}\int_{\rho(t)}^{\rho(t,x^i)}\frac{d\rho}{\rho+P(\rho)}} \quad \cdots\text{slice-independent}$$

non-linear generalization of
conserved ‘gauge’-invariant quantity ζ or $\mathcal{R}_c$

(ψ and ρ can be evaluated on any time slice)

ex.: single-field slow-roll inflation

$$d\rho\approx V'd\phi,\quad \rho+P=\dot{\phi}^2\approx\frac{V'^2}{3V} \quad \Rightarrow \quad \frac{1}{3}\int_{\rho}^{\rho+\delta\rho}\frac{d\rho}{\rho+P}=\int_{\phi}^{\phi+\delta\phi}\frac{V}{V'}d\phi=\delta N$$

$$\Rightarrow \quad \mathcal{R}_{NL}=\delta N\big|_{\psi=0} \quad (t=t_{\mathrm{h}})$$

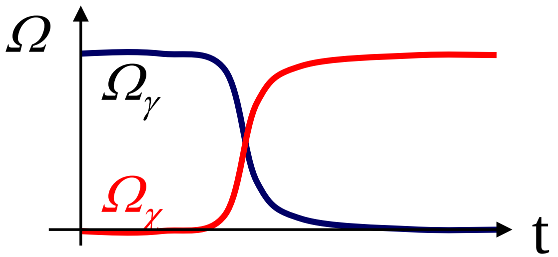
Example 2: Curvaton model

2-field model: inflaton (ϕ) + curvaton (χ)

$$V = V(\phi) + \frac{1}{2} m_\chi^2 \chi^2 \qquad m_\chi^2 \ll H^2 \approx \frac{8\pi G V}{3}$$

- during inflation ϕ dominates.
- after inflation, χ begins to dominate if it does not decay.

$\rho_\phi = \rho_\gamma \propto a^{-4}$ and $\rho_\chi \propto a^{-3}$, hence $\Omega_\chi / \Omega_\gamma \propto a$



- final curvature pert amplitude depends on when χ decays.

- Before curvaton decay

$$\mathcal{R}_\chi = \psi + \frac{1}{3}\ln\left(\frac{\rho_\chi(t,x^i)}{\bar{\rho}_\chi(t)}\right) \qquad \mathcal{R}_\gamma = \psi + \frac{1}{4}\ln\left(\frac{\rho_\gamma(t,x^i)}{\bar{\rho}_\gamma(t)}\right)$$

$$\Rightarrow \quad \rho_\chi(t,x^i) + \rho_\gamma(t,x^i) = \bar{\rho}_\chi e^{-3(\mathcal{R}_\chi - \psi)} + \bar{\rho}_\gamma e^{-4(\mathcal{R}_\gamma - \psi)}$$

- On homogeneous total density slices, $\psi = \zeta$

$$\rho_\chi(t,x^i) + \rho_\gamma(t,x^i) = \bar{\rho}_\chi e^{-3(\mathrm{R}_\chi - \zeta)} + \bar{\rho}_\gamma e^{-4(\mathrm{R}_\gamma - \zeta)} = \bar{\rho}_\chi + \bar{\rho}_\gamma$$

nonlinear version of $\zeta = \mathcal{R}_C = \sum_A \frac{(\rho_A + P_A)\mathcal{R}_A}{\rho + P}$

- With sudden decay approx, final curvature pert amp ζ is determined by

$$\boxed{\left(1-\Omega_\chi\right)\ e^{4(\mathcal{R}_\gamma - \zeta)} + \Omega_\chi \ \ e^{3(\mathcal{R}_\chi - \zeta)} = 1}$$

MS, Valiviita & Wands (2006)

Ω_χ : density fraction of χ at the moment of its decay

7. NL δN for 'slowroll' inflation

MS & Tanaka '98, Lyth & Rodriguez '05

- In slow-roll inflation, all decaying mode solutions of the (multi-component) inflaton field ϕ die out.

- If ϕ is slow rolling (or already at an attractor stage) when the scale of our interest leaves the horizon, N is only a function of ϕ (independent of $d\phi/dt$), no matter how complicated the subsequent evolution is.

- Nonlinear δN for multi-component inflation :

$$\delta N = N\left(\phi^A + \delta\phi^A\right) - N\left(\phi^A\right)$$

$$= \sum_n \frac{1}{n!} \frac{\partial^n N}{\partial\phi^{A_1} \partial\phi^{A_2} \cdots \partial\phi^{A_n}} \delta\phi^{A_1} \delta\phi^{A_2} \cdots \delta\phi^{A_n}$$

where $\delta\phi = \delta\phi_F$ (on flat slice) at horizon-crossing.

($\delta\phi_F$ may contain non-gaussianity from subhorizon interactions)

eg, DBI inflation

example: multi-brid inflation

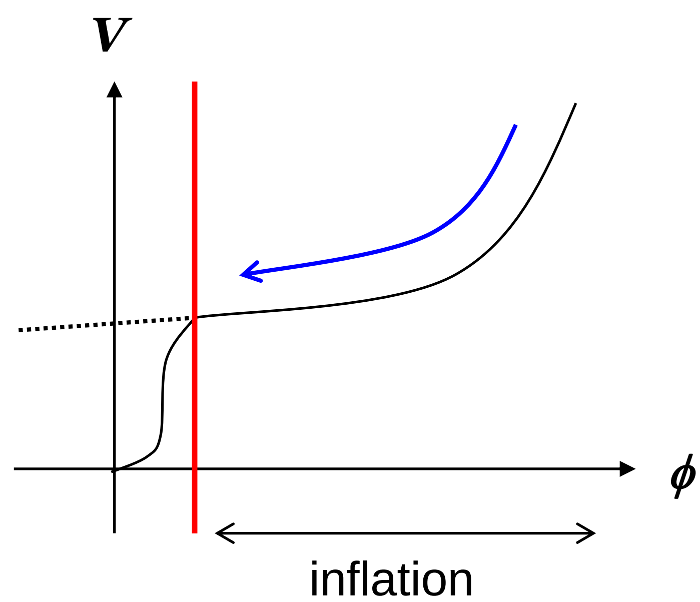
MS '08

$$(\phi_1, \phi_2, \cdots \phi_n) \;+\; \chi$$

inflaton — waterfall field

$$L_\phi = -\frac{1}{2}\sum_{A=1,2} g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^A - V(\phi)$$

$$V = V_0 \exp\left[\sum_A u_A(\phi_A)\right]$$



N as a time variable: $dN = -Hdt$

- slow-roll eom:

$$\frac{d\phi_A}{dN} = -\frac{1}{H}\frac{d\phi_A}{dt} = \frac{1}{3H^2}\frac{\partial V}{\partial\phi_A} = \frac{1}{V}\frac{\partial V}{\partial\phi_A} = u_A'(\phi_A)$$

$$3H^2 = \kappa^2 V$$

$$\kappa^2 = 8\pi G = M_{Pl}^{-2} = 1$$

- transformation of field variables:

$$\frac{d\phi_A}{dN} = \frac{1}{3V}\frac{\partial V}{\partial \phi_A} = u'_A(\phi_A) \quad \Rightarrow \quad \frac{1}{u'_A(\phi_A)}\frac{d\phi_A}{dN} = 1$$

set $$\frac{dq_A}{q_A} \equiv \frac{d\phi_A}{u'_A(\phi_A)} \quad \Rightarrow \quad \frac{d\ln q_A}{dN} = 1$$

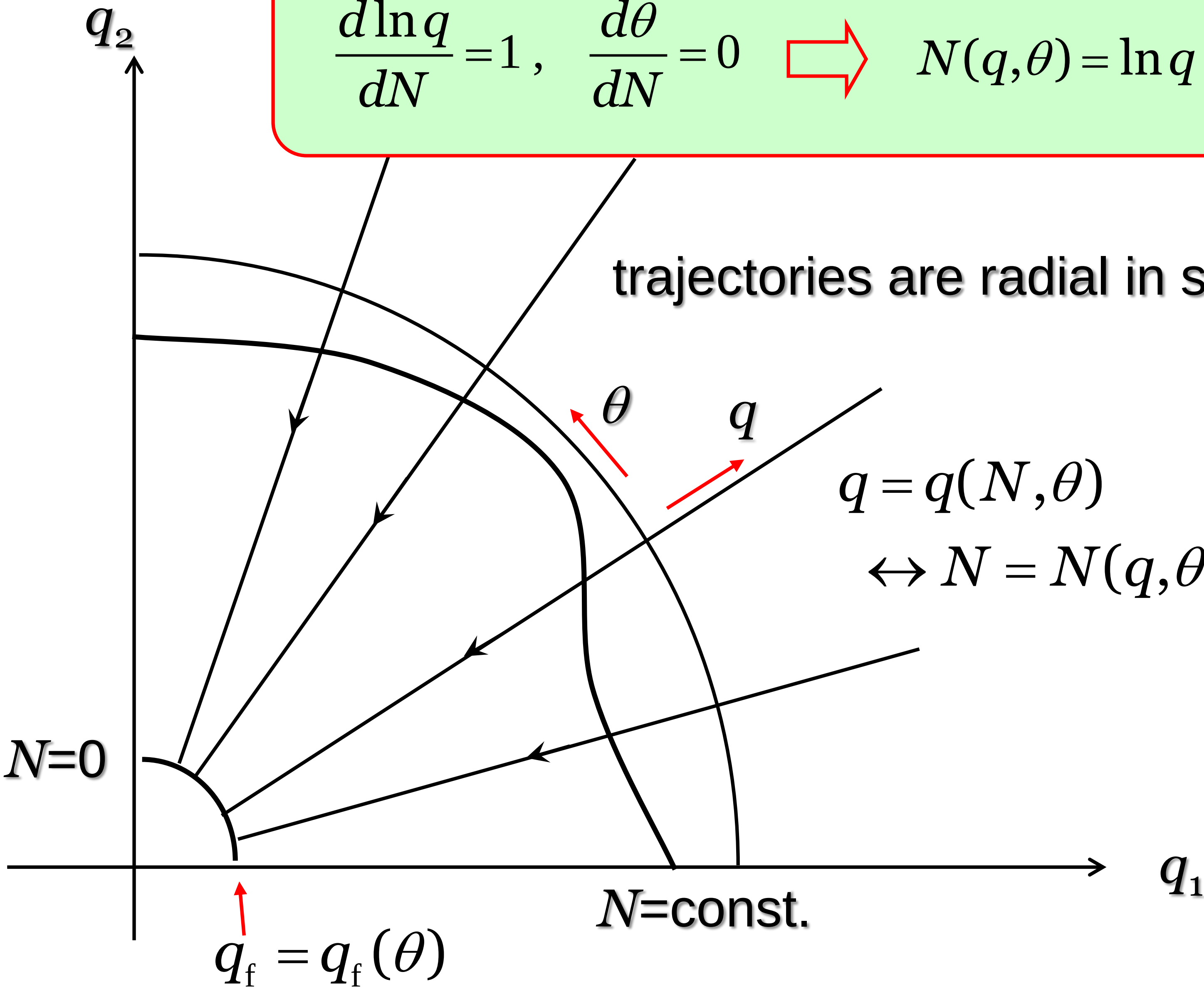
$$q_A = q n_A \;;\quad \sum_{\mathrm{A}} n_A^2 = 1 \quad \Rightarrow \quad \frac{d\ln q}{dN} = 1\,,\quad \frac{dn_A}{dN} = 0$$

angular coordinates n_A are conserved.

- For two field case,

$$q_1 = q\cos\theta, \quad q_2 = q\sin\theta, \quad \theta = \text{const.}$$

$$\frac{d\ln q}{dN}=1\,,\quad \frac{d\theta}{dN}=0 \quad \Rightarrow \quad N(q,\theta)=\ln q-\ln q_f(\theta)$$



trajectories are radial in space $(q_{1,}\,q_2)$

$$q=q(N,\theta)$$
$$\leftrightarrow N=N(q,\theta)=N(\phi_1,\phi_2)$$

For exponential pot.: $V = V_0 \exp\left[\sum_A u_A(\phi_A)\right] = V_0 \exp\left[\sum_A m_A \phi_A\right]$

$$\frac{dq_A}{q_A} = \frac{d\phi_A}{u'_A(\phi_A)} = \frac{d\phi_A}{m_A} \quad \Rightarrow \quad q_A = e^{\phi_A/m_A}, \quad q^2 = q_1{}^2 + q_2{}^2$$

$$\Rightarrow \quad N = \ln q - \ln q_{\mathrm{f}}(\theta) = \frac{1}{2}\ln\left[\frac{e^{2\phi_1/m_1} + e^{2\phi_2/m_2}}{e^{2\phi_{1,\mathrm{f}}/m_1} + e^{2\phi_{2,\mathrm{f}} m_2}}\right]$$

Assume that inflation ends at $g_1^2\phi_1^2 + g_2^2\phi_2^2 = \sigma^2$
and the universe is thermalized instantaneously.

realized by $V_0 = \frac{1}{2}\left(g_1^2\phi_1{}^2 + g_2^2\phi_2{}^2\right)\chi^2 + \frac{\lambda}{4}\left(\chi^2 - \frac{\sigma^2}{\lambda}\right)^2$

Parametrize orbits by an angle at the end of inflation

$$\phi_{1,\mathrm{f}} = \frac{\sigma}{g_1}\cos\gamma, \quad \phi_{2,\mathrm{f}} = \frac{\sigma}{g_2}\sin\gamma$$

⇨ $$\ln\left[\frac{q_1}{q_2}\right]=\frac{\phi_1}{m_1}-\frac{\phi_2}{m_2}=\frac{\sigma\cos\gamma}{g_1 m_1}-\frac{\sigma\sin\gamma}{g_2 m_2}$$

(∵ const of motion)

This determines γ in terms of ϕ_1 & ϕ_2.

⇨ $$N=N(\phi_1,\phi_2)=\frac{1}{2}\ln\left[\frac{e^{2\phi_1/m_1}+e^{2\phi_2/m_2}}{e^{2\sigma\cos\gamma/g_1 m_1}+e^{2\sigma\sin\gamma/g_2 m_2}}\right]$$

where $\gamma=\gamma(\phi_1,\phi_2)$

- δN valid to full nonlinear order is simply given by

$$\delta N=N(\phi_1+\delta\phi_1,\phi_2+\delta\phi_2)-N(\phi_1,\phi_2)$$

- To be precise, one has to add a correction term to adjust the energy density difference at the end of inflation

$$N = \frac{1}{2}\ln\left[\frac{e^{2\phi_1/m_1} + e^{2\phi_2/m_2}}{e^{2\sigma\cos\gamma/g_1 m_1} + e^{2\sigma\sin\gamma/g_2 m_2}}\right] + N_c$$

where

$$N_c = \frac{1}{4}\ln\left[\frac{V_{\mathrm{f}}}{V_0}\right] = \frac{\sigma}{4}\left(\frac{m_1}{g_1}\cos\gamma + \frac{m_2}{g_2}\sin\gamma\right)$$

$$V_{\mathrm{f}} = V_0 \exp\left[\frac{m_1\sigma}{g_1}\cos\gamma + \frac{m_2\sigma}{g_2}\sin\gamma\right]$$

(assuming instantaneous thermalization)

However, this correction is negligible

if $\quad m_1, m_2 \ll M_{Pl} = 1$

- δN to 2nd order in $\delta\phi$:

$$\delta N = \frac{\delta\phi_1 g_1 \cos\gamma + \delta\phi_2 g_2 \sin\gamma}{m_1 g_1 \cos\gamma + m_2 g_2 \sin\gamma} + \frac{g_1^2 g_2^2}{2\sigma} \frac{(m_2\delta\phi_1 - m_1\delta\phi_2)^2}{(m_1 g_1 \cos\gamma + m_2 g_2 \sin\gamma)^3}$$

- comoving curvature perturbation spectrum

$$\mathrm{P}_S(k) = \frac{g_1^2 \cos^2\gamma + g_2^2 \sin^2\gamma}{(m_1 g_1 \cos\gamma + m_2 g_2 \sin\gamma)^2} \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=Ha}$$

spectral index: $n_s = 1 - (m_1^2 + m_2^2)$

tensor/scalar: $r = \dfrac{\mathrm{P}_T(k)}{\mathrm{P}_S(k)} = 8 \dfrac{(m_1 g_1 \cos\gamma + m_2 g_2 \sin\gamma)^2}{g_1^2 \cos^2\gamma + g_2^2 \sin^2\gamma}$

- cf: single-field case $\phi = mN + \phi_{\mathrm{f}} \Leftrightarrow N = \dfrac{\phi - \phi_{\mathrm{f}}}{m}$

No non-Gaussianity if $\delta\phi$ is Gaussian

$$n_s = 1 - m^2, \quad r = 8m^2, \quad f_{NL}^{\mathrm{local}} = 0$$

Let

$$\delta_L N \equiv \frac{\delta\phi_1 g_1 \cos\gamma + \delta\phi_2 g_2 \sin\gamma}{m_1 g_1 \cos\gamma + m_2 g_2 \sin\gamma}, \quad S \equiv \frac{\delta\phi_1 g_2 \sin\gamma - \delta\phi_2 g_1 \cos\gamma}{m_2 g_1 \cos\gamma - m_1 g_2 \sin\gamma}$$

"true" entropy perturbation

$$\left(\quad \langle \delta_L N \cdot S \rangle = 0 \ \text{ for } \ \langle \delta\phi^A \delta\phi^B \rangle = \left(\frac{H}{2\pi}\right)^2 \delta^{AB} \quad \right)$$

$$\Rightarrow \quad \delta N = \delta_L N + \frac{3}{5} f_{NL}^{\text{local}} \left(\delta_L N + S\right)^2$$

linear entropy perturbation contributes at 2nd order

$$f_{NL}^{\text{local}} = \frac{5 g_1^2 g_2^2}{6\sigma (g_1^2 \cos^2\gamma + g_2^2 \sin^2\gamma)^2} \frac{(m_2 g_1 \cos\gamma - m_1 g_2 \sin\gamma)^2}{m_1 g_1 \cos\gamma + m_2 g_2 \sin\gamma}$$

$$\Rightarrow \quad f_{NL}^{\text{local}} = O\left(gm/\sigma\right) \ \text{ for } \ m_1, m_2 \sim O(m), \ g_1, g_2 \sim O(g).$$

$$\boxed{\text{practically any non-Gaussianity is possible}}$$

$$(N.B., \ f_{NL}^{\text{local}} > 0)$$

• example of parameters

$$1 = M_{Pl} = (8\pi G)^{-1/2} = 2.43\times 10^{18}\,\mathrm{GeV}$$

model parameters: $m_1^2 \sim 0.005,\;\; m_2^2 \sim 0.035$

assume $m_1 \cos\gamma \gg m_2 \sin\gamma$

$$g_1{}^2 = g_2{}^2 \equiv g^2$$

outputs:

$$\left.\begin{aligned} n_s &= 1-(m_1^2+m_2^2) \sim 0.96 \\ r &\approx 8m_1^2 \sim 0.04 \end{aligned}\right\}$$

independent of waterfall field

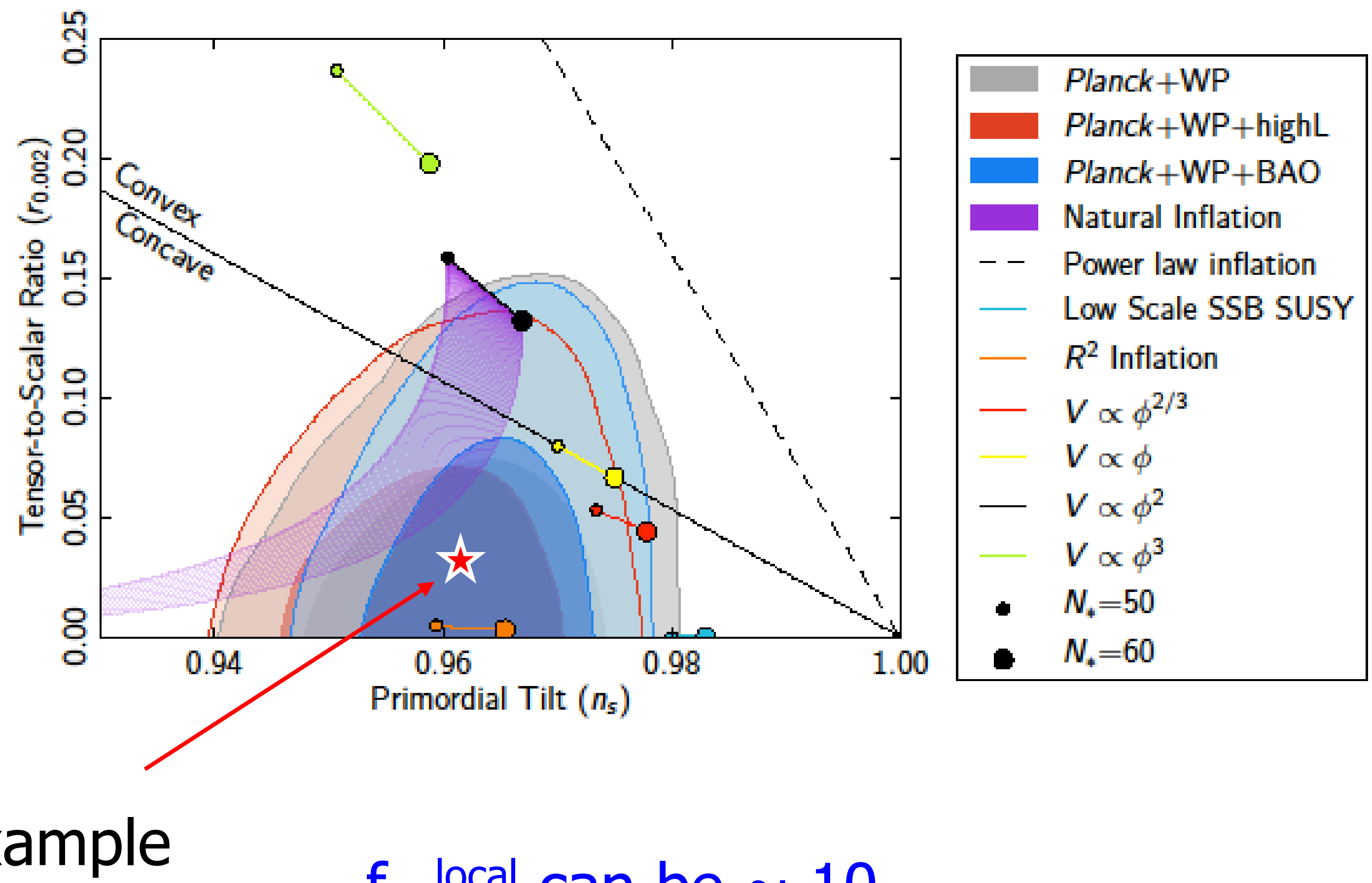
$$3H^2 = \sigma^4/4\lambda \sim 1.5\times 10^{-9} \quad \left(\Leftrightarrow \mathrm{P}_{\mathrm{R}}(k) \sim 2.5\times 10^{-5}\right)$$

$$\Rightarrow \quad \sigma^2 \sim \lambda^{1/2}\times 10^{-4}$$

$$\Rightarrow \quad \boxed{f_{NL}^{\mathrm{local}} \approx \frac{5gm_2^2}{6m_1\sigma} \sim 40\frac{g}{\lambda^{1/4}}}$$

Just for fun…

Planck 2013 constraint on r & n_s



f_{NL}^{local} can be ~ 10

8. Conformal frame (in)dependence - why bother? -

In cosmology, we encounter various frames of the metric which are conformally equivalent.

Einstein frame, Jordan frame, string frame, ...

They are mathematically equivalent, so one can work in any frame as long as mathematical manipulations are concerned.

But it is often said that there exists a unique physical frame on which we should consider actual ‘physics.’

How does physics depend/not depend on choice of conformal frames?

Two typical frames in scalar-tensor theory

$(\boldsymbol{\phi} + \boldsymbol{g})$

- Jordan(-Brans-Dicke) frame

  "gravitational" part : $F(\phi)R+L(\phi)$

  matter part: $L(\psi, A, \ldots)$ ~ minimal coupling with g

  (matter assumed to be universally coupled with g
  ··· for baryons, experimentally consistent)

- Einstein frame

  "gravitational" part : $R+L(\phi)$ ~ minimal coupling between g and ϕ

  matter part: $G(\phi)L(\psi, A, \ldots)$ ψ : fermion, A : vector, ...

  (if non-universal coupling:

  $$\Rightarrow \sum_A G_A(\phi)L_A(Q_A); \;\; Q_A = \psi, A, \cdots.$$
)

conformal transformation

- metric and scalar curvature

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

$$R \to \tilde{R} = \Omega^{-2}\left[R - (D-1)\left(2\frac{\Box\Omega}{\Omega} - (D-4)g^{\mu\nu}\frac{\partial_\mu\Omega\partial_\nu\Omega}{\Omega^2} \right)\right]$$

- matter fields (for $D = 4$)

$$\phi \to \tilde{\phi} = \Omega^{-(D-2)/2}\phi \quad \left(= \Omega^{-2}\phi \right)$$ scalar

$$A_\mu \to \tilde{A}_\mu = \Omega^{-(D-4)/2}A_\mu \quad \left(= A_\mu \right)$$ vector

$$\psi \to \tilde{\psi} = \Omega^{-(D-1)/2}\psi \quad \left(= \Omega^{-3/2}\psi \right)$$ fermion

cosmological perturbations

Makino & MS '91, Komatsu &Futamase '99,...

- tensor-type perturbation

$$ds^2 = -dt^2 + a^2(t)\left(\delta_{ij} + h_{ij}\right)dx^i dx^j$$

$$= a^2(\eta)\left[-d\eta^2 + \left(\delta_{ij} + h_{ij}\right)dx^i dx^j\right]$$

$$\Downarrow$$

$$\boxed{\partial_j h^{ij} = h^j{}_j = 0}$$

$$d\tilde{s}^2 = \Omega^2 ds^2$$

$$= \Omega^2(x^\mu)a^2(\eta)\left[-d\eta^2 + \left(\delta_{ij} + h_{ij}\right)dx^i dx^j\right]$$

Definition of h_{ij} is apparently Ω-**in**dependent.

- vector-type perturbation

$$ds^2 = a^2\left[-d\eta^2 + 2B_j dx^j d\eta + \left(\delta_{ij} + \partial_i H_j + \partial_j H_i\right) dx^i dx^j\right]$$

$$\Downarrow$$

$$\boxed{\partial_j B^j = \partial_j H^j = 0}$$

$$d\tilde{s}^2 = \Omega^2 ds^2$$

$$= \Omega^2 a^2\left[-d\eta^2 + 2B_j dx^j d\eta + \left(\delta_{ij} + \partial_i H_j + \partial_j H_i\right) dx^i dx^j\right]$$

Definitions of B_j and H_j are aslo Ω-**in**dependent.

(spatial) tensor & vector are conformal frame-independent

This means in particular $P_T(k)$ formula ($\sim H^2$) from inflation is most easily computed in the Einstein frame.

- scalar-type perturbation

$$ds^2 = a^2(\eta)\Big[-(1+2A)d\eta^2 + 2\partial_j B dx^j d\eta$$
$$+\Big((1+2\mathcal{R})\delta_{ij} + 2\partial_i\partial_j E\Big)dx^i dx^j\Big]$$

$$\Downarrow$$

$$d\tilde{s}^2 = \Omega^2 ds^2$$
$$= \Omega^2 a^2\Big[-(1+2A)d\eta^2 + 2\partial_j B dx^j d\eta$$
$$+\Big((1+2\mathcal{R})\delta_{ij} + 2\partial_i\partial_j E\Big)dx^i dx^j\Big]$$

Definitions of B and E are Ω-independent.

But A and $\mathcal{R}$ are Ω-dependent!

$$\Omega(t,x^i) = \Omega_0(t)\big[1+\omega(t,x^i)\big]$$

$$\Rightarrow \quad A \to A+\omega\,, \quad \mathcal{R} \to \mathcal{R}+\omega$$

Nevertheless, if $\Omega = \Omega(\phi)$

- The important, curvature perturbation $\mathcal{R}_c$, conserved on superhorizon scales, is defined on comoving hypersurfaces.

$$\mathcal{R}_c \equiv \mathcal{R} - \frac{H}{\dot{\phi}}\delta\phi = \mathcal{R} - \frac{1}{a}\frac{da}{d\phi}\delta\phi$$

uniform ϕ $(\delta\phi = 0)$

frame-independent

- For scalar-tensor theory with

$$L = \frac{1}{2} f(\phi)R + K(X,\phi)\,, \quad X \equiv -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

we have $\Omega = \Omega(\phi)$

$\mathcal{R}_c = \mathcal{R}_{\delta\phi=0}$ is Ω-independent!

$\mathcal{R}_c$ is conformal-frame independent in the adiabatic limit

⇔if not in the adiabatic limit, the notion of adiabatic perturbation depends on choice of conformal frames

generalization to NL perturbation

Gong, Hwang, Park, Song & MS '11

White, Minamitsuji & MS '13

....

- Generalization is straightforward for perturbations on superhorizon scales

δN formalism:

$\mathcal{R}_c(t_f) = \delta N$ between the final comoving surface $(t=t_f)$ and an initial flat surface

although the number of e-folds N depends on conformal frames, δN is frame-independent in the adiabatic limit

9. Summary

- There exists a NL generalization of comoving curvature perturbation $\mathcal{R}_{\mathrm{C}}$ which is conserved for an adiabatic perturbation on superhorizon scales.
- There exists a NL generalization of δN formula, which may be useful in evaluating non-Gaussianity from inflation.
- (NL)δN formula is independent of conformal frames if evaluated in the adiabatic limit.