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Holographic Entanglement of Purification and CFT Dual

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It from Qubit
Simons Collaboration

YITP
YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Based on

[1] Koji Umemoto and TT (Nature Phys. 14 (2018) 6, 573)

[2] Pawel Caputa, Masamichi Miyaji, Koji Umemoto and TT (1812.05268)

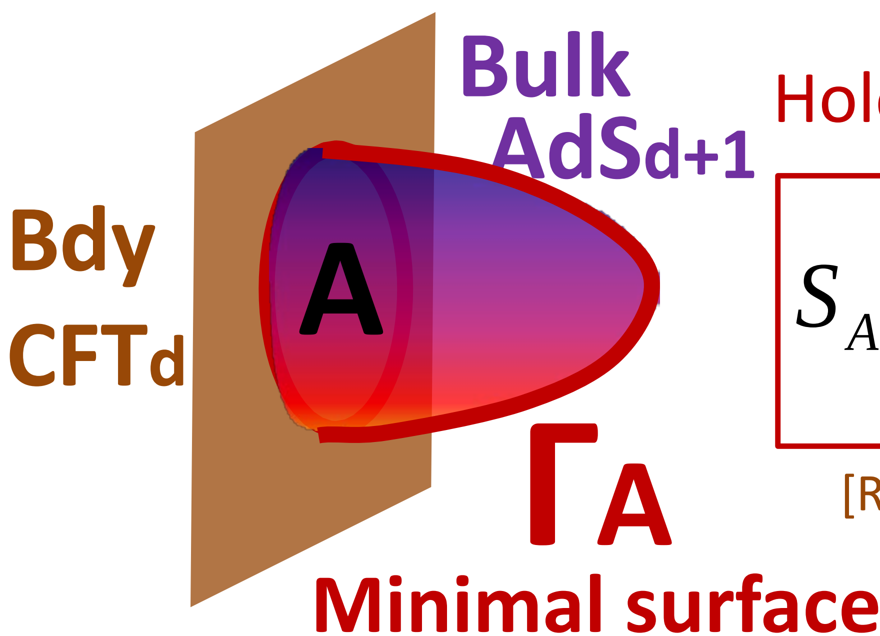
① Introduction

AdS/CFT [Maldacena 1997, Gubser-Klebanov-Polyakov, Witten 1998,…..]

➡ **``Geometrization'' of Dynamics in CFTs**

One manifestation of this feature is

Quantum entanglement of CFTs ≈ Bulk Geomety



Holographic Entanglement Entropy

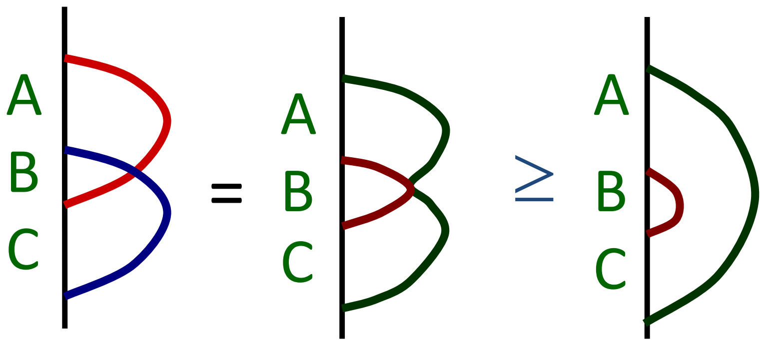
$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = \frac{\mathrm{Area}(\Gamma_A)}{4G_N}$$

[Ryu-TT 2006, Hubeny-Rangamani-TT 2007]

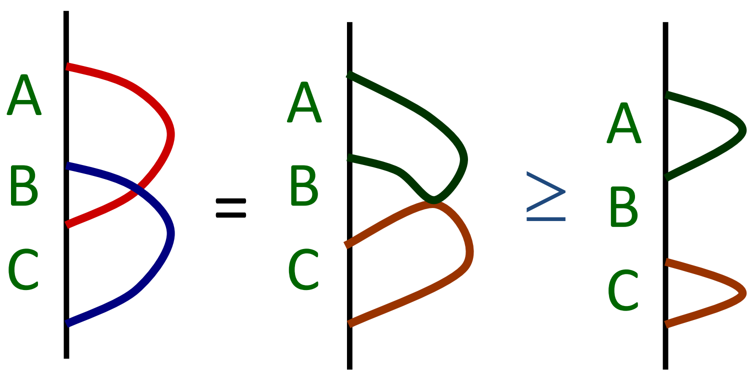
[Derivation: Casini-Huerta-Myers 2009, Lewkowycz-Maldacena 2013]

Holographic Proof of Strong Subadditivity

[Headrick-TT 07]



$$\Rightarrow S_{AB} + S_{BC} \geq S_{ABC} + S_{B}$$



$$\Rightarrow S_{AB} + S_{BC} \geq S_{A} + S_{C}$$

$$(\text{Note}: \quad AB \equiv A \cup B)$$

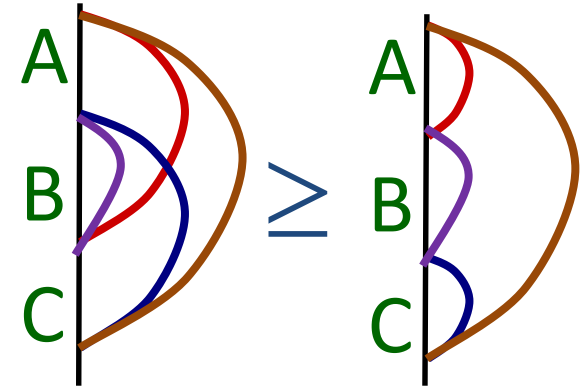
Algebraic relations in Quantum Information Theory
⇔ Geometric properties in Gravity

Monogamy of Mutual Information [Hayden-Headrick-Maloney 11]

The holographic mutual information

$$I(A:B) = S_A + S_B - S_{AB}$$

has a special property called ***monogamy***.



$$I(A:BC) \geq I(A:B) + I(A:C)$$

$$\Leftrightarrow I_3(A,B,C) \equiv S_A + S_B + S_C + S_{ABC} - S_{AB} - S_{BC} - S_{AC} \leq 0$$

Comments:

- This property is special to holographic CFTs.

  [cf. For massive free fermion QFT: $I_3 > 0$ Casini-Fosco-Huerta 05]

In this talk we will propose a *generalization of holographic entanglement entropy for mixed states*.

⇒ **Holographic Entanglement of Purification**

First we will explain this conjecture and then we will give quantitative evidences for this conjecture.

An ambitious goal

Generalize AdS/CFT such that a dual theory lives at a finite cut off surface !

+Tensor Network picture ⇒ **Surface/State correspondence**

[Miyaji-TT 15]

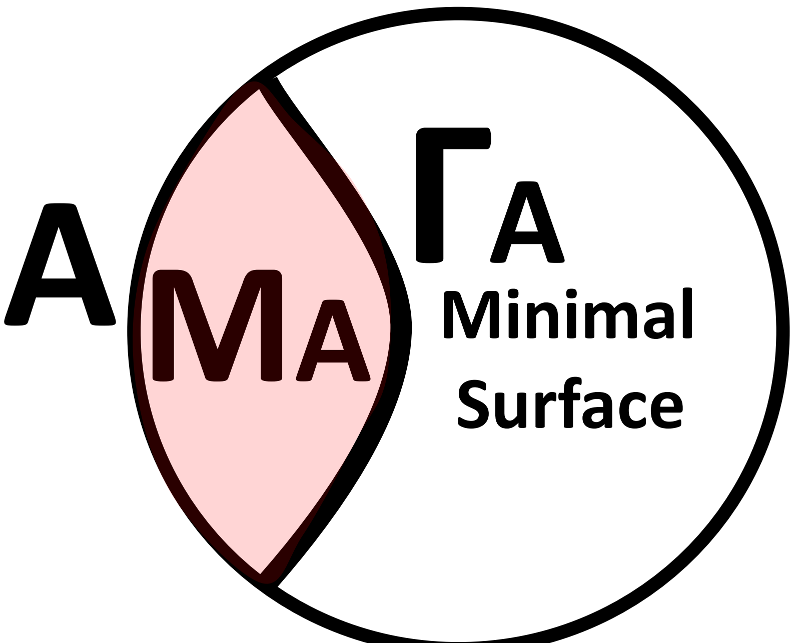
Contents

② Entanglement Wedges

Which bulk region is dual to a given region A in CFT ?

⇒ Entanglement Wedge M_A (note: we took a time slice)

M_A = A region surrounded by A and Γ_A (on time slice)



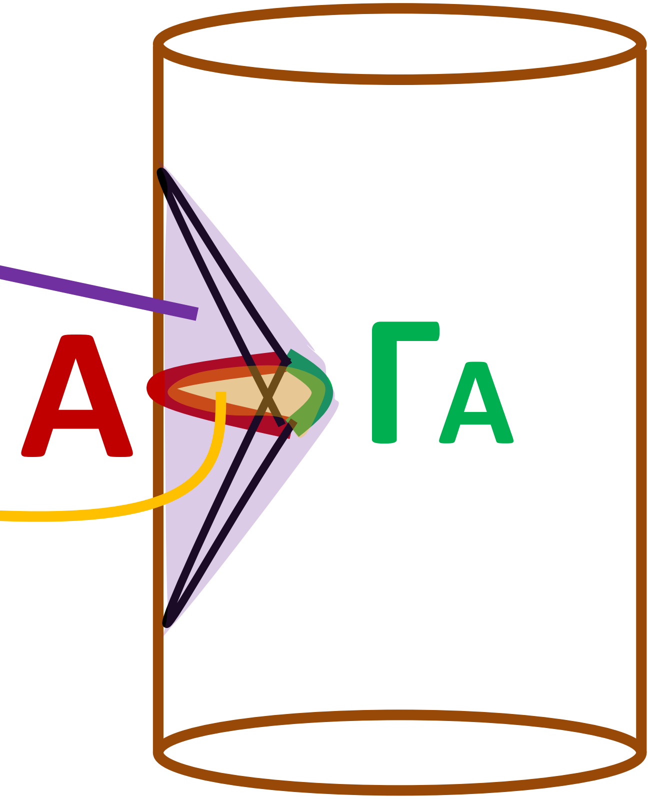
Time slice of AdS

$$\rho_A \text{ in CFT}$$
$$\Leftrightarrow \rho_{MA} \text{ in AdS gravity}$$

[Hamilton-Kabat-Lifschytz-Lowe 2006, Czech-Karczmarek-Nogueira-Raamsdonk 2012, Wall 2012, Headrick-Hubeny-Lawrence-Rangamani 2014, Jafferis-Lewkowycz-Maldacena-Suh 2015, Dong-Harlow-Wall 2016, . . .]

Covariant Definition of EW

Entanglement Wedge
=Domain of dependence
of M_A

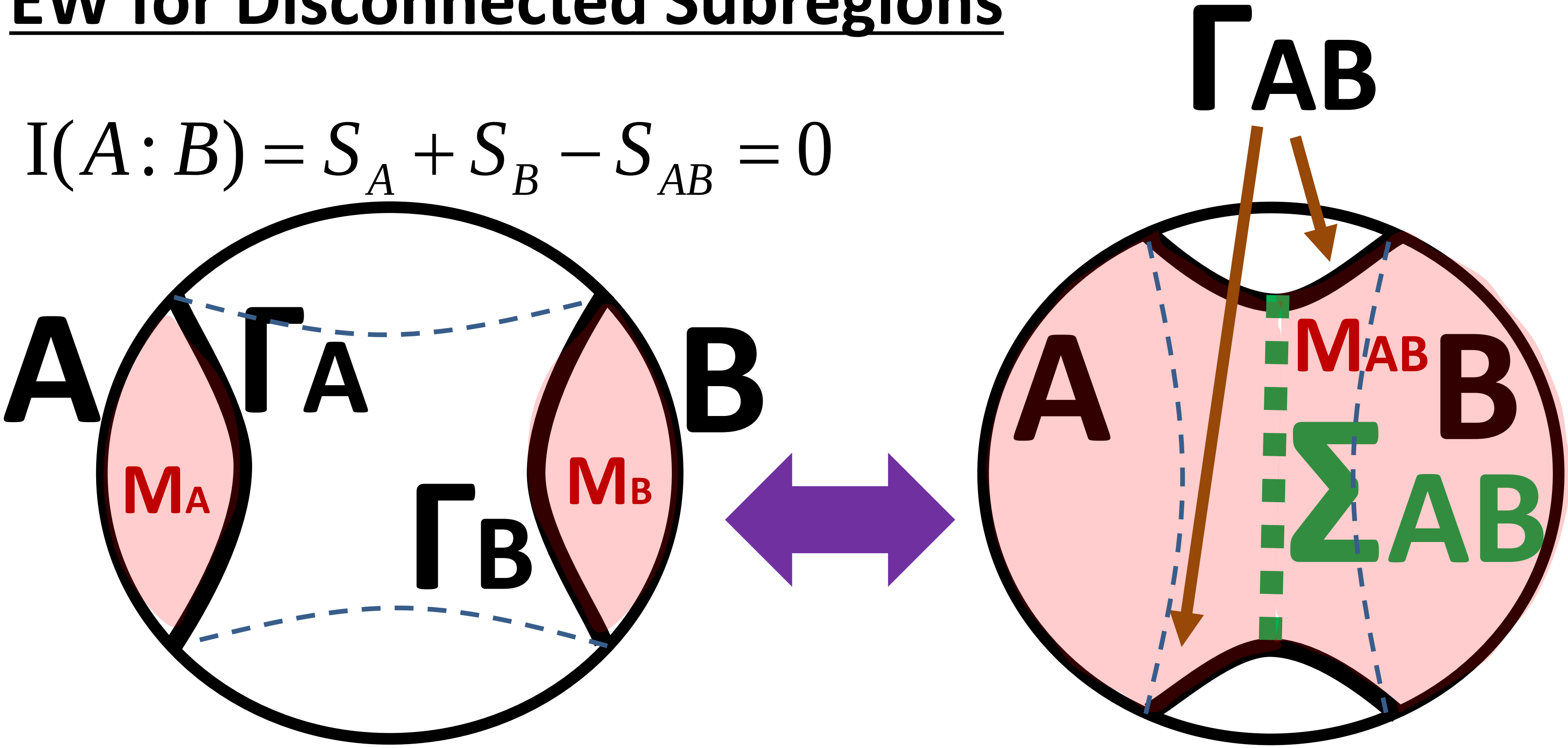


Note: Our arguments below assume a static asymptotically AdS spacetime, which has a canonical time slice.

However, it is straightforward to extend our arguments to general non-static setups in a covariant way.

EW for Disconnected Subregions

$$\mathrm{I}(A:B) = S_A + S_B - S_{AB} = 0$$



$M_{AB} = M_A \cup M_B$, $\Gamma_{AB} = \Gamma_A \cup \Gamma_B$

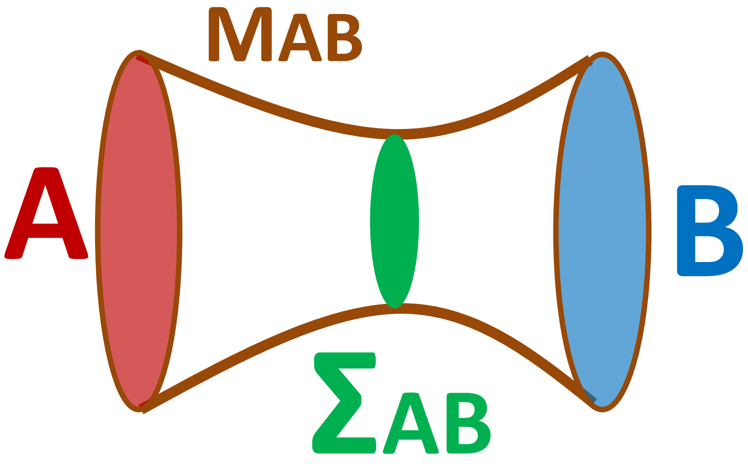
$$\mathrm{I}(A:B) > 0$$

Σ_{AB} = Minimal Surface which divides M_{AB} into A side and B side

Entanglement Wedge Cross Section

We define a quantity called ***EW cross section*** by

$$E_W(\rho_{AB}) = \frac{\mathrm{Area}(\Sigma_{AB})}{4G_N}$$



Our Conjecture [Umemoto-TT 17, Nguyen-Devakul-Halbasch-Zaletel-Swingle 17]

$$E_W(\rho_{AB}) = E_P(\rho_{AB})$$

Entanglement of Purification

[Terhal-Horodecki-Leung-Divincenzo quant-ph/0202044]

Note: When ρ_{AB} is a pure state,
we simply have $E_W(\rho_{AB}) = E_P(\rho_{AB}) = S_A = S_B$.

③ Entanglement of Purification (EoP)

(3-1) Purification

A given density matrix for H_C : $\rho_C = \sum_i \lambda_i |i\rangle_C \langle i|.$

We can always describe this state as a pure state by extending the Hilbert space:

$$H_C \to H_C \otimes H_D \qquad |\Psi\rangle_{CD} = \sum_i \sqrt{\lambda_i}\, |i\rangle_C |i\rangle_D$$

such that $\rho_C = \mathrm{Tr}_\mathrm{D}\left[|\Psi\rangle\langle\Psi|\right].$

Note: there are infinite many ways to do this.

(3-2) Definition of EoP

Consider all purifications $|\Psi\rangle_{A\tilde{A}B\tilde{B}}$ of ρ_{AB} in the extended Hilbert space: $H_A \otimes H_B \to H_A \otimes H_B \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$.

Then, **Entanglement of Purification (EoP)** is defined by

$$\mathrm{E_P}(\rho_{AB}) = \underset{\text{All purifications } |\Psi\rangle \text{ of } \rho_{AB}}{\mathrm{Min}} \; S_{A\tilde{A}}\left(|\Psi\rangle_{A\tilde{A}B\tilde{B}}\right)$$

$$\rho_{AB} = \mathrm{Tr}_{\tilde{A}\tilde{B}}\left[|\Psi\rangle\langle\Psi|\right].$$

Entanglement Entropy

Note: $\mathrm{E_p}(\rho_{AB}) \geq 0$ and $\mathrm{E_p}(\rho_{AB}) = 0 \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B$.

(3-3) Properties of EoP [Bagchi-Pat, 1502.01272]

[1] In general, $\mathrm{E_p}(\rho_{AB}) \leq \min\{S_A, S_B\}$.

If ρ_{AB} is pure, then $\mathrm{E_p}(\rho_{AB}) = S_A = S_B$.

[2] $\mathrm{E_p}(\rho_{AB}) \geq \mathrm{I}(A:B)/2$.

[3] $\mathrm{E_p}(\rho_{A(BC)}) \geq \mathrm{I}(A:B)/2 + \mathrm{I}(A:C)/2$.

[4] If ρ_{ABC} is pure, then

$\mathrm{E_p}(\rho_{A(BC)}) \leq \mathrm{E_p}(\rho_{AB}) + \mathrm{E_p}(\rho_{AC})$. [Polygamy]

[5] $\mathrm{E_p}(\rho_{A(BC)}) \geq \mathrm{E_p}(\rho_{AB})$.

Comment: These all follows from strong subadditivity.

④ Holographic Entanglement of Purification

To test our conjecture $E_W(\rho_{AB}) = E_P(\rho_{AB})$,

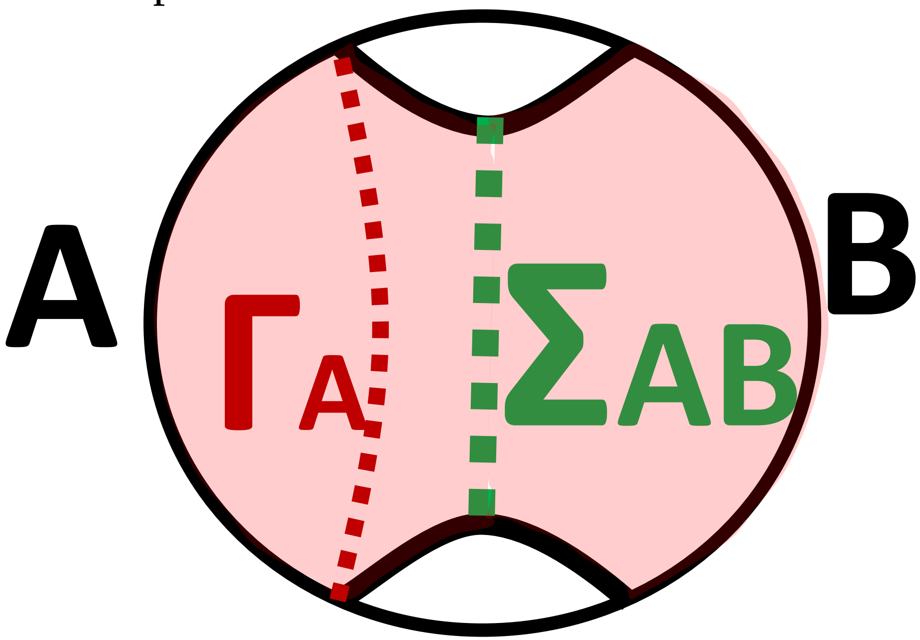
let us confirm the properties of $E_W(\rho_{AB})$.

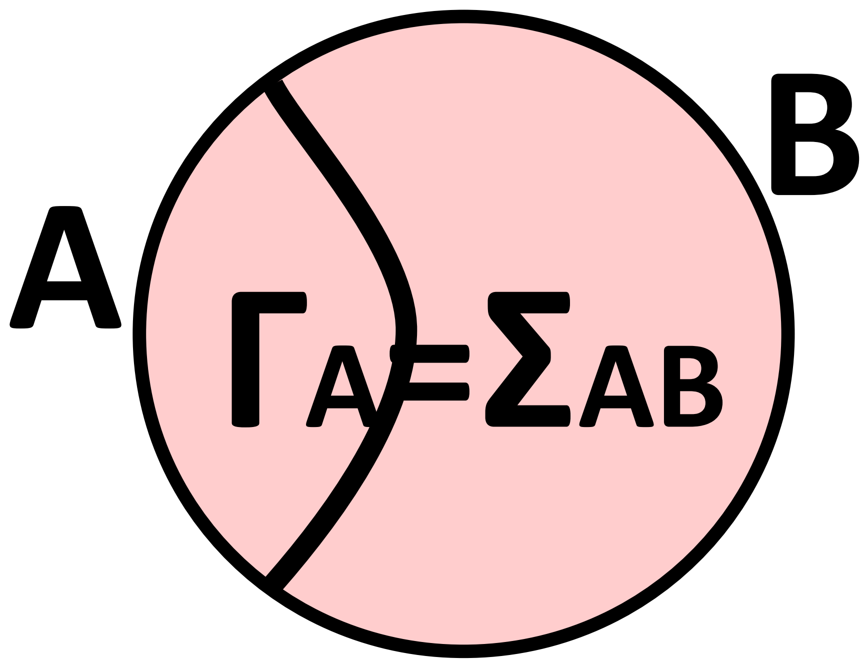
Property [1]

$$E_p(\rho_{AB}) \leq \min\{S_A, S_B\}.$$

If ρ_{AB} is pure, then

$$E_p(\rho_{AB}) = S_A = S_B.$$





Property [2]

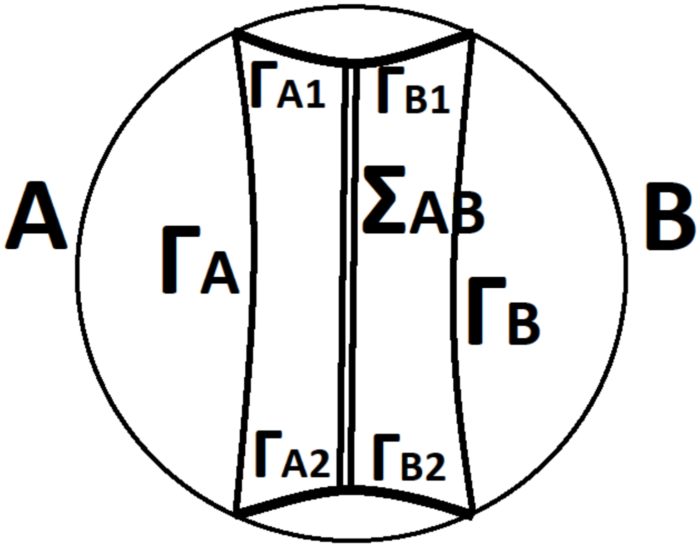
$$\mathrm{Area}(\Gamma_{\mathrm{A}}) \le \mathrm{Area}(\Gamma_{\mathrm{A1}}) + \mathrm{Area}(\Gamma_{\mathrm{A2}}) + \mathrm{Area}(\Sigma_{AB})$$

$$\mathrm{Area}(\Gamma_{\mathrm{B}}) \le \mathrm{Area}(\Gamma_{\mathrm{B1}}) + \mathrm{Area}(\Gamma_{\mathrm{B2}}) + \mathrm{Area}(\Sigma_{AB})$$

$$\mathrm{Area}(\Gamma_{A}) + \mathrm{Area}(\Gamma_{\mathrm{B}}) - \mathrm{Area}(\Gamma_{\mathrm{AB}}) \le 2 \cdot \mathrm{Area}(\Sigma_{AB})$$

$$\mathrm{E_p}(\rho_{AB}) \ge \mathrm{I}(A:B)/2 \ .$$

[Freedman-Headrick 2016]



Property [3]

Monogamy Mutual Info.

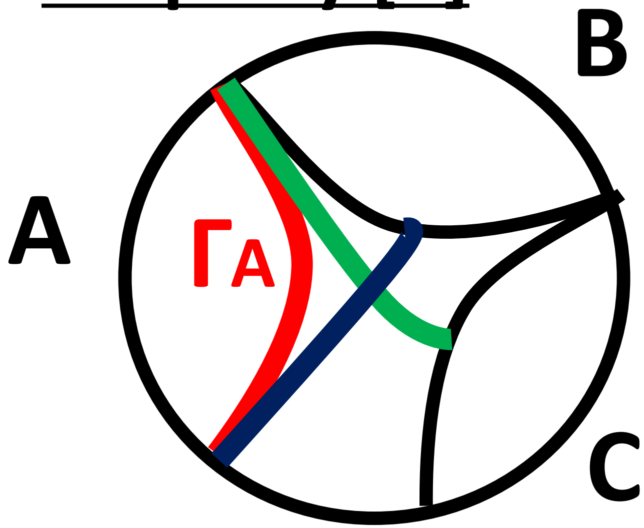
$$\mathrm{I}(A:BC) \ge I(A:B) + I(A:C)$$

holds for holographic EE.

[Hayden-Headrick-Maloney 2011]

$$\mathrm{E_p}(\rho_{A(BC)}) \ge \mathrm{I}(A:B)/2 + \mathrm{I}(A:C)/2 .$$

Property [4]



$$\mathrm{E_p}(\rho_{A(BC)}) \le \mathrm{E_p}(\rho_{AB}) + \mathrm{E_p}(\rho_{AC})$$

Property [5]

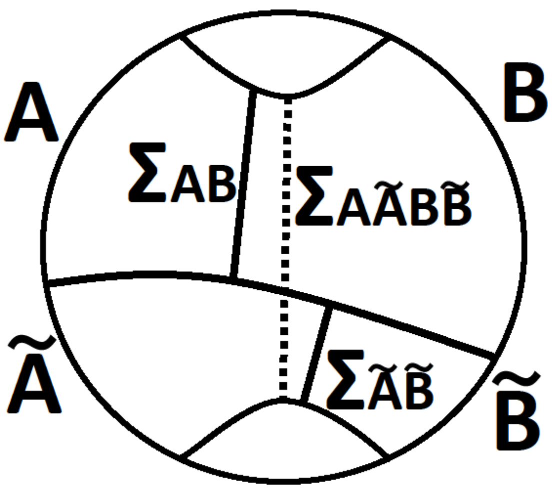
EW nesting property: $\mathrm{C} \supset \mathrm{D} \;\Rightarrow\; \mathrm{M_C} \supset \mathrm{M_D}$

Therefore, $\mathrm{ABC} \supset \mathrm{AB} \;\Rightarrow\; M_{\mathrm{A(BC)}} \supset M_{\mathrm{AB}}$

➡ $\mathrm{E_p}(\rho_{A(BC)}) \geq \mathrm{E_p}(\rho_{AB})$.

Strong Super-additivity (Special to Hol. CFTs)

$$\mathrm{M}_{A\tilde{A}B\tilde{B}} \supset \mathrm{M}_{AB} \cup \mathrm{M}_{\tilde{A}\tilde{B}}, \quad \mathrm{M}_{AB} \cap \mathrm{M}_{\tilde{A}\tilde{B}} = \phi$$



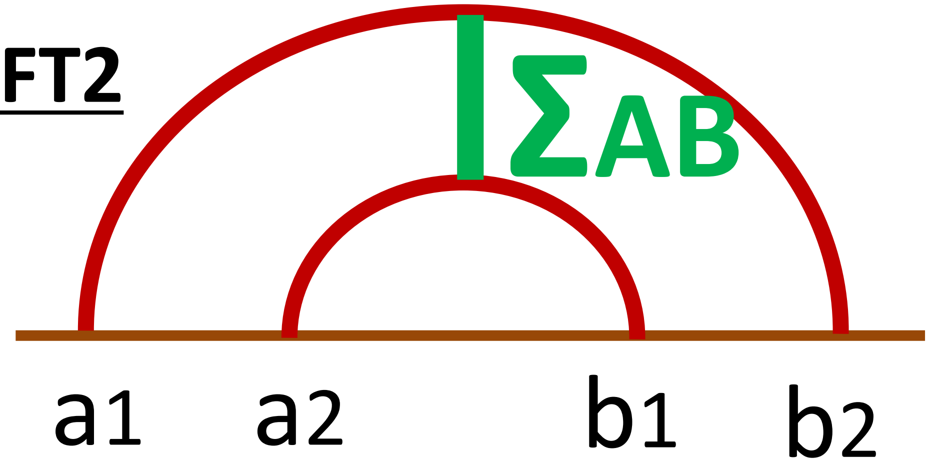
⬇

$$\mathrm{Area}(\Sigma_{A\tilde{A}B\tilde{B}}) \geq \mathrm{Area}(\Sigma_{AB}) + \mathrm{Area}(\Sigma_{\tilde{A}\tilde{B}})$$

⬇

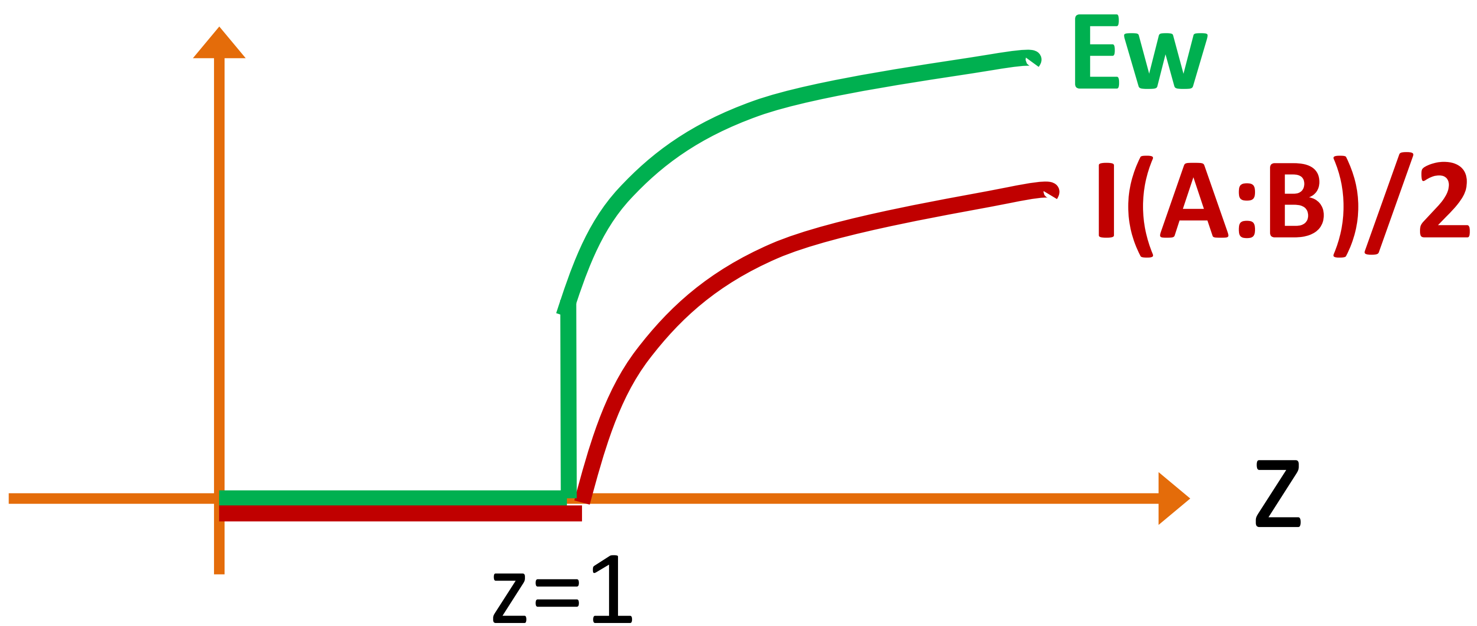
$$\mathrm{E_p}(\rho_{A\tilde{A}(B\tilde{B})}) \geq \mathrm{E_p}(\rho_{AB}) + \mathrm{E_p}(\rho_{\tilde{A}\tilde{B}}).$$

Example 1: Pure AdS3/CFT2



$$z \equiv \frac{(a_2 - a_1)(b_2 - b_1)}{(b_1 - a_2)(b_2 - a_1)}$$

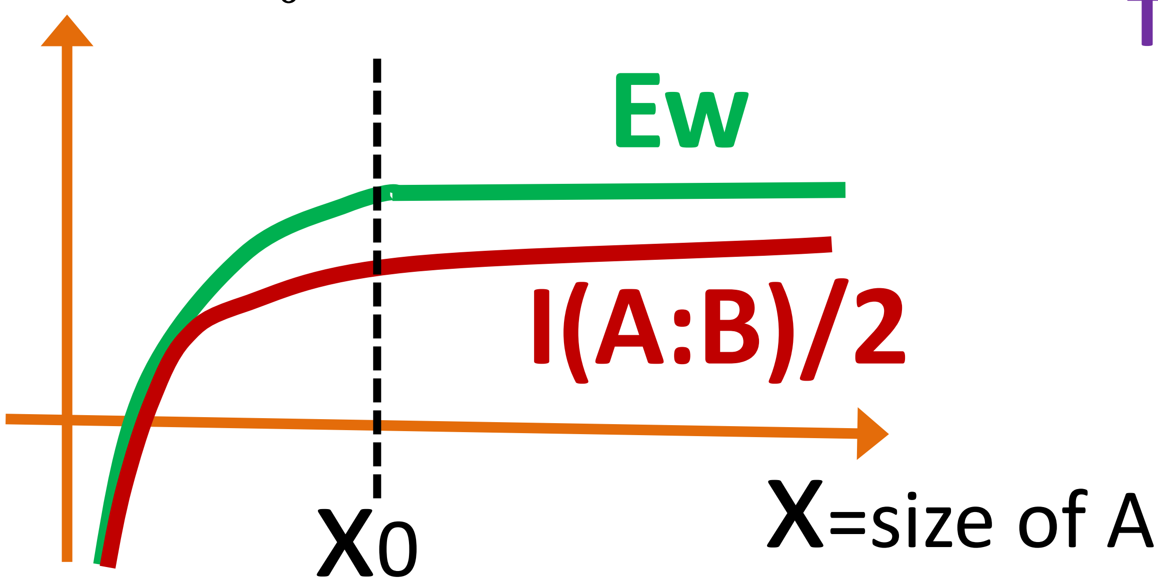
$$E_W = \frac{c}{6}\log\left(1 + 2z + \sqrt{z(z+1)}\right), \qquad I(A:B) = \frac{c}{3}\log z.$$

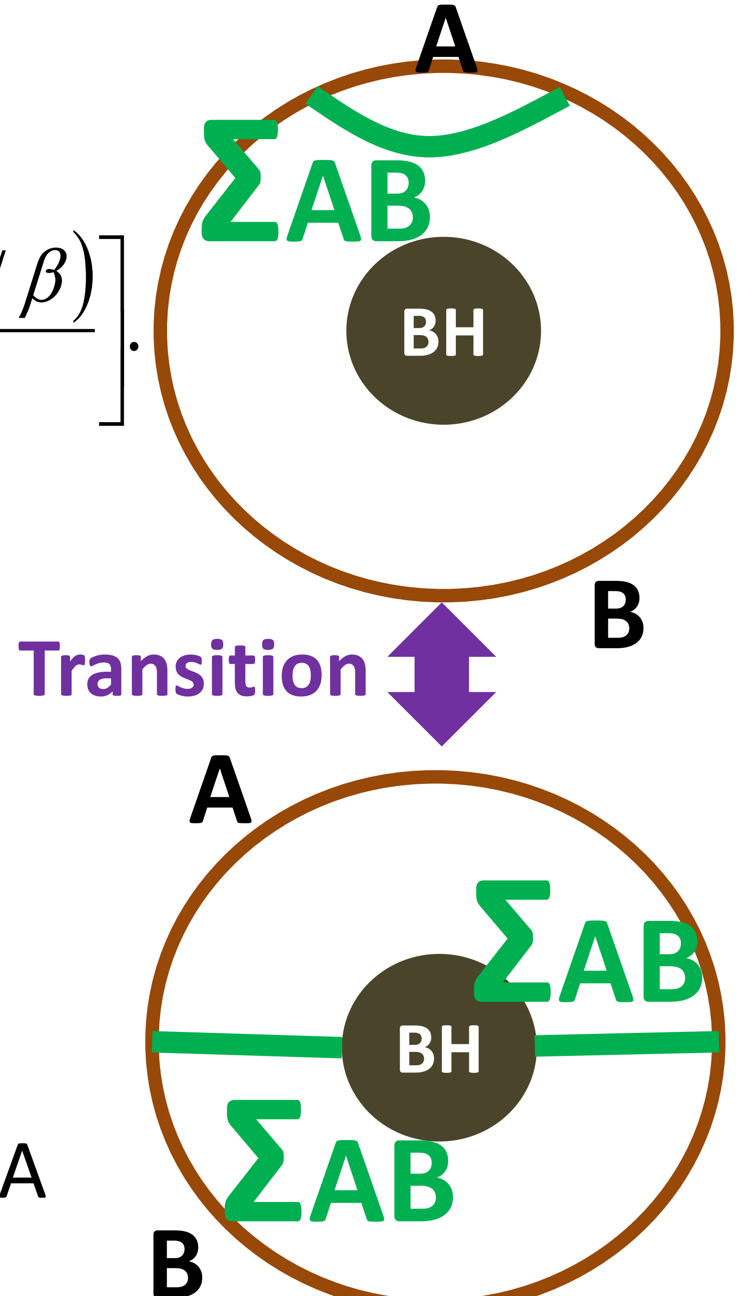


Example 2: BTZ/Thermal CFT

$$E_W = \frac{c}{3} \cdot \mathrm{Min}\left[\log \frac{\beta}{\pi\varepsilon}, \quad \log \frac{\beta \sinh(\pi l / \beta)}{\pi\varepsilon} \right].$$

$$x_0 = \beta \log(1 + \sqrt{2})$$





[Nice Matching with Finite Temp. MPS: Nguyen-Devakul-Halbasch-Zaletel-Swingle 2017]

It is not a priori clear which class of quantum states in CFTs we should perform the minimization of $S_{A\tilde{A}}$ to get the Hol EoP.

⇒ Several possibilities:

(i) All quantum states in CFTs

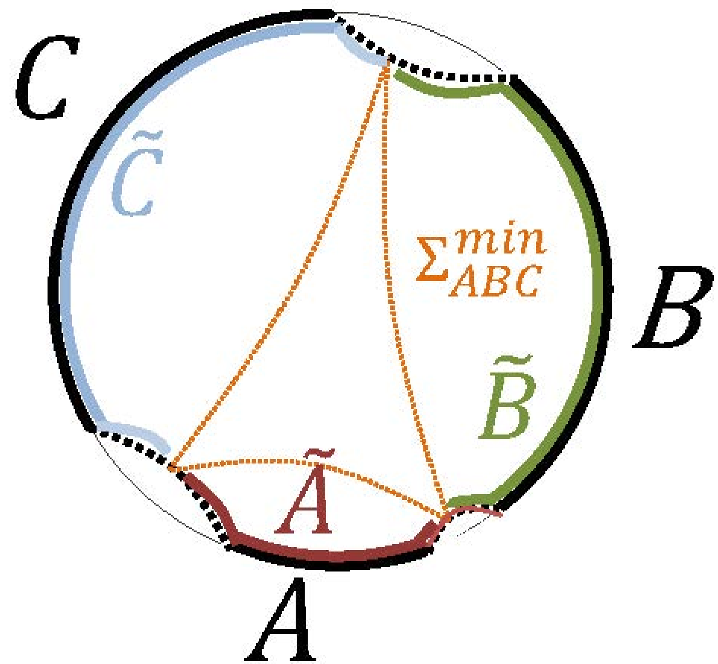
(ii) Quantum States with classical gravity duals

⇒ We should explore CFT calculations !

Multi-partite Generalization [Umemoto-Zhou 2018]

$$\Delta_P(A:B:C) := \min_{|\psi\rangle_{AA'BB'CC'}} [S_{AA'} + S_{BB'} + S_{CC'}],$$

$= A(\Sigma_{min})/4G$



⑤ HEoP from Path-integral Optimization

(5-1) Path-integral optimization [Caputa-Kundu-Miyaji -Watanabe-TT 17]

Consider 2d CFTs defined on a flat space:

$ds^2 = dx^2 + dz^2$.   z= Euclidean time (=-τ)

Path-integral Optimization

= A special Weyl transformation which

(i) preserves quantum wave functional at the time z=ε,

(ii) minimizes path-integral complexity (=Liouville action).

$$ds^2 = e^{2\phi(x,z)}(dx^2 + dz^2). \quad \text{with} \quad e^{2\phi}|_{z=\varepsilon} = 1 \ .$$

The rule of UV cut off: a area of one lattice site =ε^2.

Path-Integral Complexity I[ϕ]

$$I[\phi] = \mathrm{Log}\left[\frac{\Psi_{g=e^{2\phi}\delta_{ab}}}{\Psi_{g=\delta_{ab}}}\right] = S_L[\phi],$$

Liouville Action $S_L[\phi]$

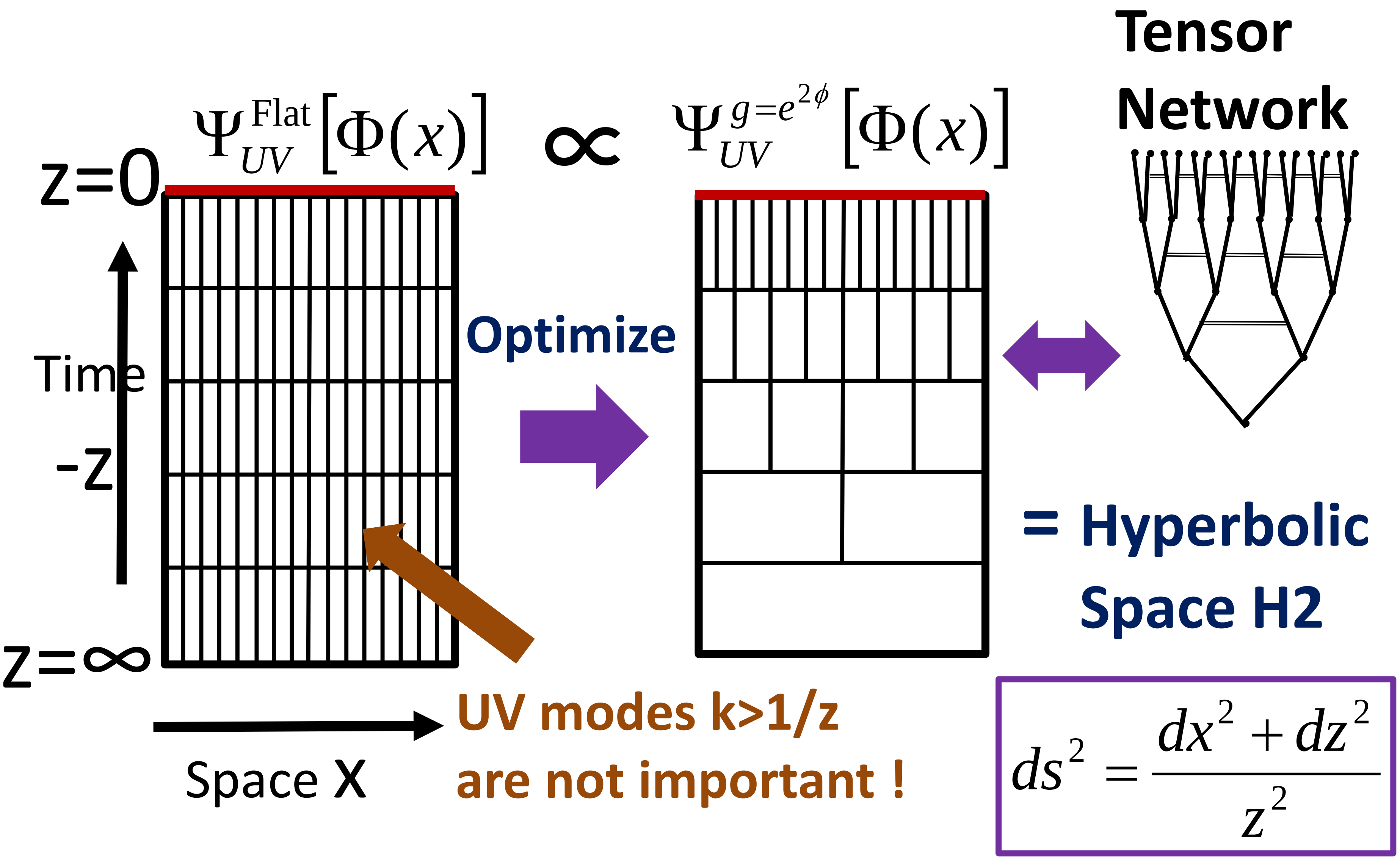
$$S_L[\phi] = \frac{c}{24\pi}\int dxdz\left[(\partial_x\phi)^2 + (\partial_z\phi)^2 + e^{2\phi}\right]$$

$$= \frac{c}{24\pi}\int dxdz\left[(\partial_x\phi)^2 + \left(\partial_z\phi + e^{\phi}\right)^2\right] + (\text{surface term})$$

$$\Rightarrow \text{Minimum}: e^{2\varphi} = \frac{\epsilon^2}{z^2}$$

➡ **Hyperbolic plane (H_2) = Time slice of AdS3**

A Sketch: Optimization of Path-Integral



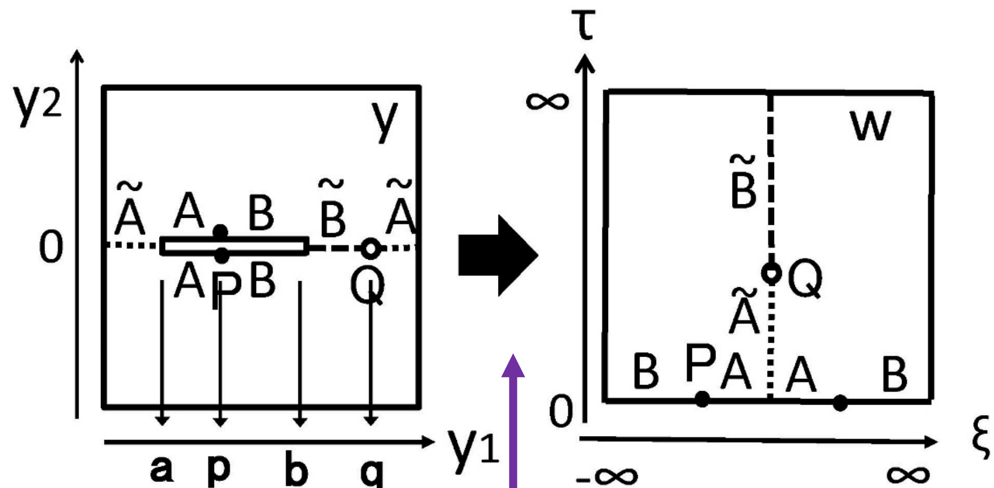
(5-2) Optimizing density matrices [Caputa-Miyaji-Umemoto-TT 18]

Consider the setup when AB= a single interval:

A=[a,p]   B=[p,b]   ⇒ Purification $\tilde{B}$=[b,q]

$\tilde{A}$=[-∞,a]U[q,∞]

ρ_{AB}=



Conformal Map

$$w = \sqrt{\frac{y-a}{b-y}}$$

The final optimized metric looks like

$$ds^2 = \frac{\epsilon^2}{\tau^2} \cdot dw d\bar{w} = \frac{\epsilon^2}{\tau^2} \cdot \frac{(b-a)^2}{4|b-y|^3|y-a|} \cdot dy d\bar{y} \equiv e^{2\tilde{\phi}} \cdot dy d\bar{y}.$$

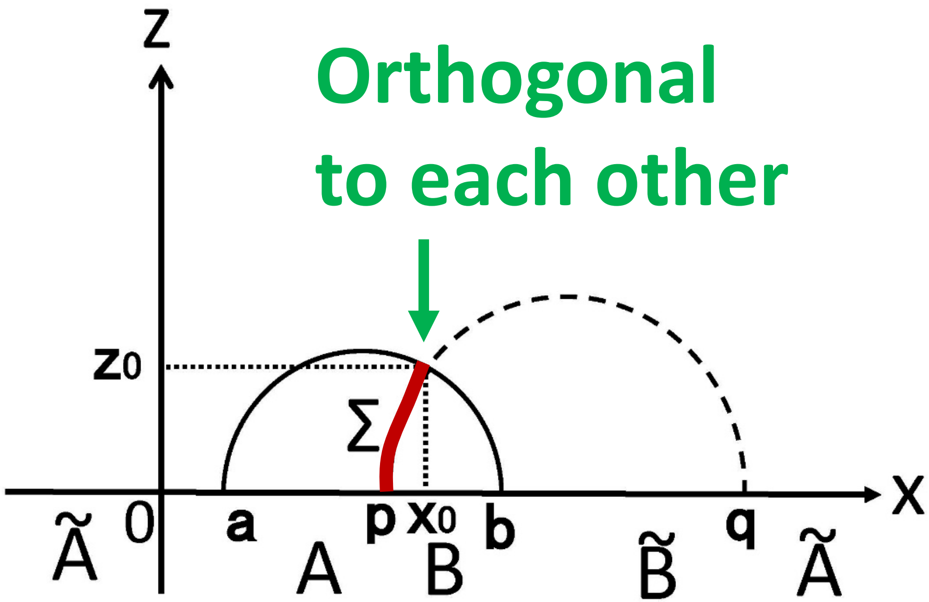
In this setup, we obtain $e^{2\tilde{\phi}_P} = 1$ and $e^{2\phi_Q} = \frac{\epsilon^2 (b-a)^2}{4(q-a)^2(q-b)^2}$. The entanglement entropy $S_{A\tilde{A}} = S_{B\tilde{B}}$ is found to be

$$S_{A\tilde{A}} = \frac{c}{3} \log\left(\frac{q-p}{\epsilon}\right) + \frac{c}{6}\tilde{\phi}_P + \frac{c}{6}\tilde{\phi}_Q$$
$$= \frac{c}{6} \log\left[\frac{(b-a)(q-p)^2}{2\epsilon(q-a)(q-b)}\right],$$

Minimize w.r.t q

$$S^{min}_{A\tilde{A}} = \frac{c}{6} \log\left[\frac{2(p-a)(b-p)}{\epsilon(b-a)}\right],$$
$$\text{at} \quad q = \frac{2ab-(a+b)p}{a+b-2p}.$$

Calculations of Hol EoP



$$ds^2 = \frac{dx^2 + dz^2}{z^2}$$

$$L(\Sigma) = \frac{q-p}{2}\int_{\epsilon}^{z_0} \frac{dz}{z\sqrt{\frac{(q-p)^2}{4} - z^2}} = \log\left[\frac{2(p-a)(p-b)}{\epsilon(b-a)}\right].$$

➡ The Hol EoP L(Σ)/4G agrees with the CFT result !

⑥ Conclusions

Our Conjecture

A gravity dual of Entanglement of Purification (EoP)
= the minimal cross section of Entanglement Wedge (Ew).

Our observation:

Ew=EE for a purified state with minimal path-integral complexity

Future problems

(1) Understand which class we really minimize the EE

(2) Derivation of Hol. EoP formula from AdS/CFT

(3) Developing Numerical Calculation of EoP

[Bhattachrayya-Jahn-Umemoto-TT, in preparation]

(4) Relations to other interpretations

[Hirai-Tamaoka-Yokota 18, Flam-Ryu 18, Tamaoka 18]

Thank you very much !

Operational Definition of EoP

The **entanglement of purification** is equal to the ``Entanglement Cost'' for the **LOq** process.

$$E_P^{\infty}(\rho_{AB}) = \text{ \# of EPR pairs needed to create } \rho_{AB} \text{ via LOq .}$$

LOq = Local Operations

+ small number of communications.

(Fact: For pure states, LOq is enough to extract EPR pairs)

However, note that EoP is not an entanglement measure, but a **correlation measure** between A and B.

$$\boxed{E_D(\rho_{AB}) \le E_{SQ}(\rho_{AB}) \le E_C(\rho_{AB}) \le E_F(\rho_{AB}) \le E_P(\rho_{AB})\,.}$$