NCTS Annual Theory Meeting 2018

ALGEBRAIC ENGINEERING OF 4D N = 2 SUPER YANG-MILLS THEORIES

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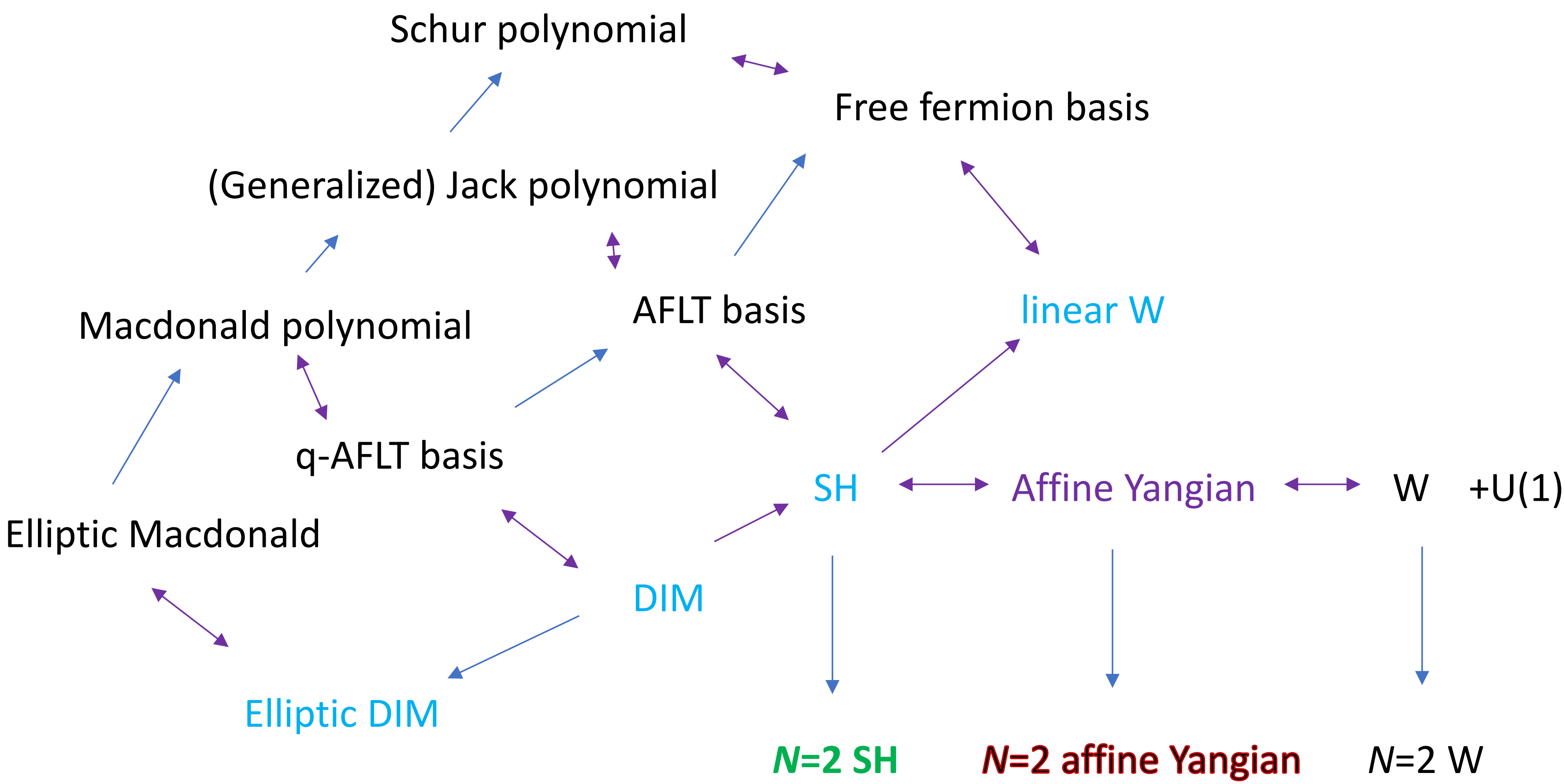
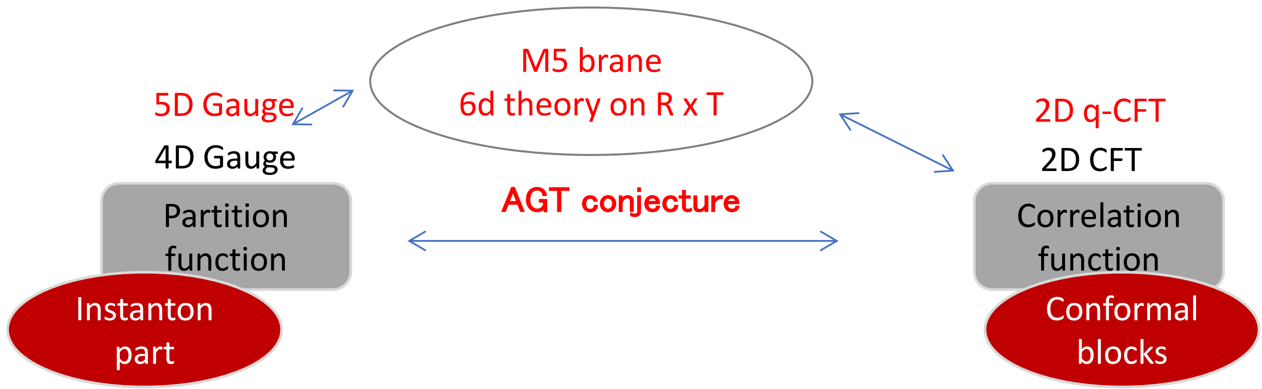
Based on arXiv:**1809.08861**, **J.-E. Bourgine** and **KZ**



Schur polynomial
Free fermion basis
(Generalized) Jack polynomial
AFLT basis
linear W
Macdonald polynomial
q-AFLT basis
SH
Affine Yangian
W +U(1)
Elliptic Macdonald
DIM
Elliptic DIM
N=2 SH
N=2 affine Yangian
N=2 W



Some BPS quantities of N = 1 5D quiver gauge theories, like instanton partition functions or qq-characters, can be constructed as algebraic objects of the Ding-Iohara-Miki (DIM) algebra.

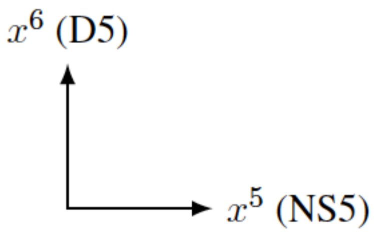
This construction is applied here to N = 2 super Yang-Mills theories in four dimensions using a degenerate version of the DIM algebra.

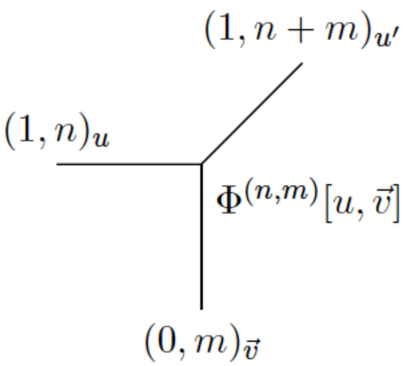
We build up the equivalent of horizontal and vertical representations, the first one being defined using vertex operators acting on a free boson's Fock space, while the second one is essentially equivalent to the action of Vasserot-Shiffmann's Spherical Hecke central algebra.

1. Degenerate DIM algebra

1.1 Horizontal representation

1.2 Vertical representation





2. Reconstructing the gauge theories' BPS quantities

2.1 Intertwiners

2.2 Instanton partition functions and qq-characters

2.3 Degenerate limit of Kimura-Pestun's quiver W-algebra

DIM algebra

$$[\bar{\psi}^{\pm}(Z), \bar{\psi}^{\pm}(W)] = 0, \quad \bar{\psi}^{+}(Z)\bar{\psi}^{-}(W) = \frac{\bar{g}(q_3^{c/2} Z/W)}{\bar{g}(q_3^{-c/2} Z/W)} \bar{\psi}^{-}(W)\bar{\psi}^{+}(Z), \quad \bar{x}^{\pm}(Z)\bar{x}^{\pm}(W) = \bar{g}(Z/W)^{\pm 1}\bar{x}^{\pm}(W)\bar{x}^{\pm}(Z),$$

$$\bar{\psi}^{+}(Z)\bar{x}^{\pm}(W) = \bar{g}(q_3^{\pm c/4} Z/W)^{\pm 1}\bar{x}^{\pm}(W)\bar{\psi}^{+}(Z), \quad \bar{\psi}^{-}(Z)\bar{x}^{\pm}(W) = \bar{g}(q_3^{\mp c/4} Z/W)^{\pm 1}\bar{x}^{\pm}(W)\bar{\psi}^{-}(Z),$$

$$[\bar{x}^{+}(Z), \bar{x}^{-}(W)] = \kappa \left(\bar{\delta}(q_3^{-c/2} Z/W)\bar{\psi}^{+}(q_3^{-c/4} Z) - \bar{\delta}(q_3^{c/2} Z/W)\bar{\psi}^{-}(q_3^{c/4} Z) \right).$$

$$q_1 q_2 q_3 = 1$$

$$\kappa = \frac{(1-q_1)(1-q_2)}{(1-q_1 q_2)}, \quad \bar{g}(Z) = \prod_{\alpha=1,2,3} \frac{1-q_\alpha Z}{1-q_\alpha^{-1} Z}.$$

Degenerate DIM algebra and representations

$$[\psi^{\pm}(z),\psi^{\pm}(w)]=0,\quad \psi^{\pm}(z)\psi^{\mp}(w)=\frac{g(z-w-c\varepsilon_+/2)}{g(z-w+c\varepsilon_+/2)}\psi^{\mp}(w)\psi^{\pm}(z),$$

$$\psi^{+}(z)x^{\pm}(w)=g(z-w\mp c\varepsilon_+/4)^{\pm1}x^{\pm}(w)\psi^{+}(z),\quad \psi^{-}(z)x^{\pm}(w)=g(z-w\pm c\varepsilon_+/4)^{\pm1}x^{\pm}(w)\psi^{-}(z),$$

$$x^{\pm}(z)x^{\pm}(w)=g(z-w)^{\pm1}x^{\pm}(w)x^{\pm}(z),$$

$$[x^{+}(z),x^{-}(w)]=-\frac{\varepsilon_1\varepsilon_2}{\varepsilon_+}\left[\delta(z-w+c\varepsilon_+/2)\psi^{+}(z+c\varepsilon_+/4)-\delta(z-w-c\varepsilon_+/2)\psi^{-}(z-c\varepsilon_+/4)\right].$$

$$\varepsilon_+=-\varepsilon_3=\varepsilon_1+\varepsilon_2$$

$$g(z)=\prod_{\alpha=1,2,3}\frac{z+\varepsilon_\alpha}{z-\varepsilon_\alpha},$$

The coproduct can be seen as a degenerate limit of the Drinfeld coproduct for the DIM algebra,

$$\begin{aligned}
\Delta(x^+(z)) &= x^+(z)\otimes 1+\psi^-(z-\varepsilon_+c_{(1)}/4)\otimes x^+(z-\varepsilon_+c_{(1)}/2),\\
\Delta(x^-(z)) &= 1\otimes x^-(z)+x^-(z-\varepsilon_+c_{(2)}/2)\otimes \psi^+(z-\varepsilon_+c_{(2)}/4),\\
\Delta(\psi^\pm(z)) &= \psi^\pm(z\mp\varepsilon_+c_{(2)}/4)\otimes \psi^\pm(z\pm\varepsilon_+c_{(1)}/4),
\end{aligned}$$

$$\Delta(c)=c_{(1)}+c_{(2)} \text{ with } c_{(1)}=c\otimes 1 \text{ and } c_{(2)}=1\otimes c.$$

Horizontal representation

The horizontal representation is built using vertex operators acting on the Fock space of a free boson.

$$[\alpha_n, \alpha_m] = n\delta_{n+m} \qquad\qquad [P, Q] = 1$$

$$\varphi_+(z) = P\log z - \sum_{n>0} \frac{z^{-n}}{n}\alpha_n, \quad \varphi_-(z) = Q + \sum_{n>0} \frac{z^n}{n}\alpha_{-n}$$

$$e^{\varphi_+(z)}e^{\varphi_-(w)} = (z-w) : e^{\varphi_+(z)}e^{\varphi_-(w)} :$$

$$V_-(z) = e^{\varphi_-(z-\varepsilon_2/2)-\varphi_-(z+\varepsilon_2/2)} \qquad V_+(z) = e^{\varphi_+(z+\varepsilon_1/2)-\varphi_+(z-\varepsilon_1/2)}$$

$$V_+(z)V_-(w) = S(z-w-\varepsilon_+/2)^{-1} : V_+(z)V_-(w) :,$$

$$S(z) = \frac{(z+\varepsilon_1)(z+\varepsilon_2)}{z(z+\varepsilon_+)}$$

$$\eta^\pm(z) = V_-(z \pm \varepsilon_+/4)^{\pm 1} V_+(z \mp \varepsilon_+/4)^{\pm 1}, \quad \xi^\pm(z) = V_\pm(z \mp \varepsilon_+/2) V_\pm(z \pm \varepsilon_+/2)^{-1},$$

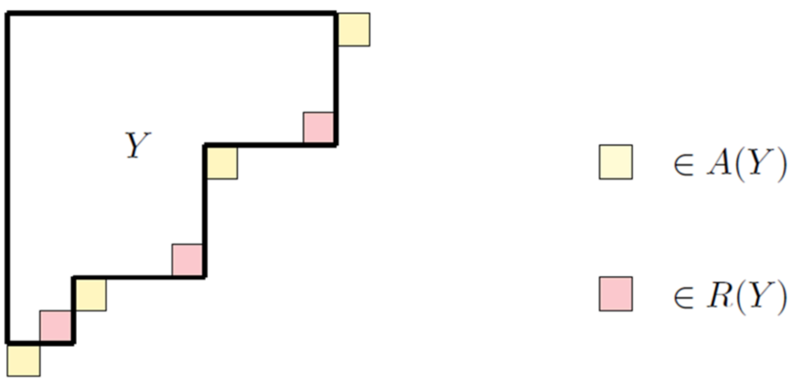
$$\rho_u^{(H)}(x^\pm(z)) = u^{\pm 1}\eta^\pm(z), \quad \rho_u^{(H)}(\psi^\pm(z)) = \xi^\pm(z), \quad \rho_u^{(H)}(c) = 1.$$

Vertical representations and SHc algebra

$$\rho^{(m)}_{\vec{a}}(x^+(z))\,|\vec{a},\vec{\lambda}\rangle\!\rangle = \sum_{x\in A(\vec{\lambda})} \delta(z-\phi_x)\,\operatorname*{Res}_{z=\phi_x} \mathcal{Y}_{\vec{\lambda}}(z)^{-1}\,|\vec{a},\vec{\lambda}+x\rangle\!\rangle,$$

$$\rho^{(m)}_{\vec{a}}(x^-(z))\,|\vec{a},\vec{\lambda}\rangle\!\rangle = \sum_{x\in R(\vec{\lambda})} \delta(z-\phi_x)\,\operatorname*{Res}_{z=\phi_x} \mathcal{Y}_{\vec{\lambda}}(z+\varepsilon_+)\,|\vec{a},\vec{\lambda}-x\rangle\!\rangle,$$

$$\rho^{(m)}_{\vec{a}}(\psi^\pm(z))\,|\vec{a},\vec{\lambda}\rangle\!\rangle = \left[\Psi_{\vec{\lambda}}(z)\right]_\pm |\vec{a},\vec{\lambda}\rangle\!\rangle.$$



$$x = (l,i,j)$$

$$\phi_x = a_l+(i-1)\varepsilon_1+(j-1)\varepsilon_2$$

$$\Psi_{\vec{\lambda}}(z) = \frac{\mathcal{Y}_{\vec{\lambda}}(z+\varepsilon_+)}{\mathcal{Y}_{\vec{\lambda}}(z)}, \quad \mathcal{Y}_{\vec{\lambda}}(z) = \frac{\prod_{x\in A(\vec{\lambda})} z-\phi_x}{\prod_{x\in R(\vec{\lambda})} z-\varepsilon_+-\phi_x}.$$

The connection between the vertical representation and the action of SHc algebra

$$\rho_{\vec{a}}^{(m)}(x^{\pm}(z)) = [X^{\pm}(z)]_{+} - [X^{\pm}(z)]_{-} \qquad \psi^{\pm}(z) = [\Psi(z)]_{\pm}$$

$$X^{+}(z)\,|\vec{a},\vec{\lambda}\rangle\!\rangle = \sum_{x\in A(\vec{\lambda})} \frac{1}{z-\phi_x} \operatorname*{Res}_{z=\phi_x} \mathcal{Y}_{\vec{\lambda}}(z)^{-1}\,|\vec{a},\vec{\lambda}+x\rangle\!\rangle,$$

$$X^{-}(z)\,|\vec{a},\vec{\lambda}\rangle\!\rangle = \sum_{x\in R(\vec{\lambda})} \frac{1}{z-\phi_x} \operatorname*{Res}_{z=\phi_x} \mathcal{Y}_{\vec{\lambda}}(z+\varepsilon_{+})\,|\vec{a},\vec{\lambda}-x\rangle\!\rangle,$$

$$\Psi(z)\,|\vec{a},\vec{\lambda}\rangle\!\rangle = \Psi_{\vec{\lambda}}(z)\,|\vec{a},\vec{\lambda}\rangle\!\rangle$$

$X^{\pm}(z)$, $\Psi(z)$ and $\partial_z \Phi(z)$

identify them with the SHc currents

$\sqrt{-\varepsilon_1 \varepsilon_2} D_{\pm 1}(z)$, $E(z)$ and $D_0(z)$

Reconstructing the gauge theories' BPS quantities

Intertwiners

$$\rho_{u'}^{(H)}(e)\Phi^{(m)}[u,\vec{a}] = \Phi^{(m)}[u,\vec{a}] \left(\rho_{\vec{a}}^{(m)} \otimes \rho_{u}^{(H)} \Delta(e)\right),$$
$$\left(\rho_{\vec{a}}^{(m)} \otimes \rho_{u}^{(H)} \Delta'(e)\right) \Phi^{*(m)}[u,\vec{a}] = \Phi^{*(m)}[u,\vec{a}]\rho_{u'}^{(H)}(e),$$

the solution can be expanded on the vertical basis, their components are operators acting on the horizontal Fock space:

$$\Phi^{(m)}[u,\vec{a}] = \sum_{\vec{\lambda}} \mathcal{Z}_{\text{vect.}}(\vec{a},\vec{\lambda})\, \Phi_{\vec{\lambda}}^{(m)}[u,\vec{a}]\, \langle\langle \vec{a},\vec{\lambda}|\,, \quad \Phi_{\vec{\lambda}}^{(m)}[u,\vec{a}] = t_{\vec{\lambda}}^{(m)}[u,\vec{a}] \ :\prod_{l=1}^{m} \Phi_{\emptyset}(a_l) \prod_{x\in\vec{\lambda}} \eta^{+}(\phi_x) :,$$

$$\Phi^{*(m)}[u,\vec{a}] = \sum_{\vec{\lambda}} \mathcal{Z}_{\text{vect.}}(\vec{a},\vec{\lambda})\, \Phi_{\vec{\lambda}}^{*(m)}[u,\vec{a}]\ |\vec{a},\vec{\lambda}\rangle\rangle, \quad \Phi_{\vec{\lambda}}^{*(m)}[u,\vec{a}] = t_{\vec{\lambda}}^{*(m)}[u,\vec{a}] \ :\prod_{l=1}^{m} \Phi_{\emptyset}^{*}(a_l) \prod_{x\in\vec{\lambda}} \eta^{-}(\phi_x) :$$

Instanton partition functions and qq-characters

For the purpose of illustration, we present here the main results concerning (A1 quiver) pure U(m) gauge theories.

$$\mathcal{T}_{U(m)} = \Phi^{(m)}[u,\vec{a}] \cdot \Phi^{*(m)}[u^*,\vec{a}] = \sum_{\vec{\lambda}} \mathcal{Z}_{\text{vect.}}(\vec{a},\vec{\lambda})\, \Phi^{(m)}_{\vec{\lambda}}[u,\vec{a}] \otimes \Phi^{*(m)}_{\vec{\lambda}}[u^*,\vec{a}].$$

$$\mathcal{Z}_{\text{inst.}}[U(m)] = \left(\langle\emptyset| \otimes \langle\emptyset|\right) \mathcal{T}_{U(m)} \left(|\emptyset\rangle \otimes |\emptyset\rangle\right) = \sum_{\vec{\lambda}} \mathfrak{q}^{|\vec{\lambda}|} \mathcal{Z}_{\text{vect.}}(\vec{a},\vec{\lambda}),$$

The partition function can also be recovered as a scalar product of Gaiotto states, corresponding here to the vevs of the intertwiners:

$$|G,\vec{a}\rangle\!\rangle = (1\otimes\langle\emptyset|)\,\Phi^{*(m)}[u^*,\vec{a}]\,|\emptyset\rangle = \sum_{\vec{\lambda}} \mathcal{Z}_{\text{vect.}}(\vec{a},\vec{\lambda})\ \langle\emptyset|\,\Phi^{*(m)}_{\vec{\lambda}}[u^*,\vec{a}]\,|\emptyset\rangle\ |\vec{a},\vec{\lambda}\rangle\!\rangle,$$

$$\langle\!\langle G,\vec{a}| = \langle\emptyset|\,\Phi^{(m)}[u,\vec{a}]\,(1\otimes|\emptyset\rangle) = \sum_{\vec{\lambda}} \mathcal{Z}_{\text{vect.}}(\vec{a},\vec{\lambda})\ \langle\emptyset|\,\Phi^{(m)}_{\vec{\lambda}}[u,\vec{a}]\,|\emptyset\rangle\ \langle\!\langle\vec{a},\vec{\lambda}|\,,$$

$$\left\langle \mathcal{T}_{U(m)} \right\rangle = \langle\!\langle G,\vec{a}|G,\vec{a}\rangle\!\rangle.$$

$$\rho_{\vec{a}}^{(m)}(x^+(z))\,|G,\vec{a}\rangle\!\rangle = -u^*\left([\mathcal{Y}(z+\varepsilon_+)_+ - [\mathcal{Y}(z+\varepsilon_+)]_-\right)|G,\vec{a}\rangle\!\rangle,$$
$$\rho_{\vec{a}}^{(m)}(x^-(z))\,|G,\vec{a}\rangle\!\rangle = -(u^*)^{-1}\left([\mathcal{Y}(z)^{-1}]_+ - [\mathcal{Y}(z)^{-1}]_-\right)|G,\vec{a}\rangle\!\rangle,$$

The fundamental qq-character can be recovered

$$\chi(z) = \mathcal{Z}_{\text{inst.}}[U(m)]^{-1}\sum_{\vec{\lambda}}\mathfrak{q}^{|\vec{\lambda}|}\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{\lambda})\left(\mathcal{Y}_{\vec{\lambda}}(z+\varepsilon_+)+\frac{\mathfrak{q}}{\mathcal{Y}_{\vec{\lambda}}(z)}\right).$$

These results can be extended to linear quivers of higher rank.

$$\Phi_{\vec{\lambda}}^{(m)}[u,\vec{a}]\Phi_{\vec{\lambda}^*}^{*(m^*)}[u^*,\vec{a}^*] = (-1)^{m|\vec{\lambda}^*|}\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{\lambda};\vec{a}^*,\vec{\lambda}^*|\varepsilon_+/2)\prod_{l,l^*=1}^{m,m^*}F(a_l-a_{l^*}^*+\varepsilon_+/2)\ :\Phi_{\vec{\lambda}}^{(m)}[u,\vec{a}]\Phi_{\vec{\lambda}^*}^{*(m^*)}[u^*,\vec{a}^*]:,$$
$$\Phi_{\vec{\lambda}^*}^{*(m^*)}[u^*,\vec{a}^*]\Phi_{\vec{\lambda}}^{(m)}[u,\vec{a}] = (-1)^{m^*|\vec{\lambda}|}\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{\lambda};\vec{a}^*,\vec{\lambda}^*|\varepsilon_+/2)\prod_{l,l^*=1}^{m,m^*}F(a_{l^*}^*-a_l+\varepsilon_+/2)\ :\Phi_{\vec{\lambda}}^{(m)}[u,\vec{a}]\Phi_{\vec{\lambda}^*}^{*(m^*)}[u^*,\vec{a}^*]:.$$

Degenerate limit of Kimura-Pestun's quiver W-algebra

$$f(z-w)T(z)T(w) - f(w-z)T(w)T(z) = -\frac{\varepsilon_1\varepsilon_2}{\varepsilon_+}\left(\delta(z-w+\varepsilon_+) - \delta(z-w-\varepsilon_+)\right),$$

We claim that this algebra is the degenerate version of KP's quiver W-algebra in the case of a single node.

The q-Virasoro algebra is defined in terms of its stress-energy tensor

$$\bar{f}(W/Z)\bar{T}(Z)\bar{T}(W) - \bar{f}(Z/W)\bar{T}(W)\bar{T}(Z) = \frac{(1-q_1)(1-q_2)}{(1-q_1q_2)}\left(\bar{\delta}(Z/q_3W) - \bar{\delta}(q_3Z/W)\right)$$