Quantum Mirror Map for Del Pezzo Geometries

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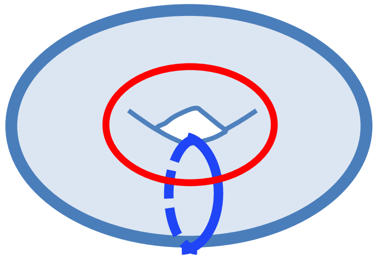
Yuji Sugimoto
(Univ. of Science and Technology of China)

with Sanefumi Moriyama
Tomohiro Furukawa
(Osaka City Univ.)

Susy. Gauge Theory



encoded in algebraic curve

e.g.) D5 del Pezzo $0=\left[\left(Q^{1/2}+Q^{-1/2}\right)\left(P^{1/2}+P^{-1/2}\right)\right]^2-z/\alpha$

A-period

Mirror Map $\Pi_A(z)\sim c_1 z+c_2 z+\ldots$

redefining the variables

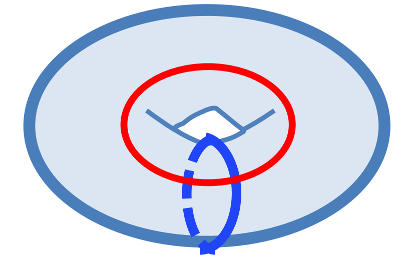
not well studied

B-period

Free energy relating to BPS indices

well studied

Susy. Gauge Theory



encoded in algebraic curve

e.g.) D5 del Pezzo $0 = \left[\left(Q^{1/2}+Q^{-1/2}\right)\left(P^{1/2}+P^{-1/2}\right)\right]^2 - z/\alpha$

A-period

We need this to reach important results.

Mirror Map $\Pi_A(z) \sim c_1 z + c_2 z + \ldots$

redefining the variables

e.g. Free energy of ABJM

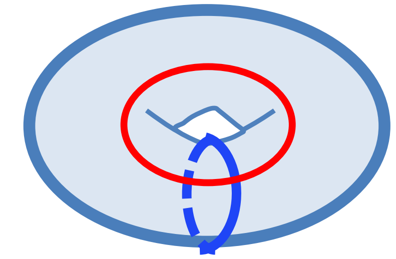
F = (pert.) + (non-pert.) for CS level

poles poles

A-period

No pole

Susy. Gauge Theory



- Some kinds of ABJM has group theoretical structure coming from symmetry of curve.

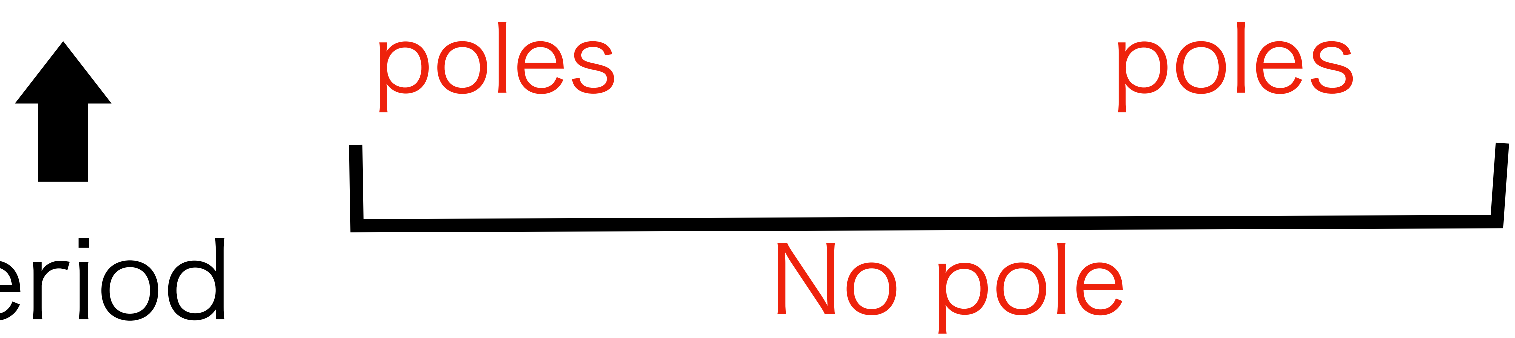
  (Kubo-Moriyama-Nosaka(2018))

- Free energy corresponds to B-period.

  Then, what about A-period ?

e.g. Free energy of ABJM

$$F = (\text{pert.}) + (\text{non-pert.}) \quad \text{for CS level}$$



Generalized ABJM theory [Honda, Moriyama, 2014]

$(U(N)_k \times U(N)_{-k} \times U(N)_k \times U(N)_{-k})$

encoded in D_5 del Pezzo with quantization

$$\widehat{H} = \left[\left(\widehat{Q}^{1/2} + \widehat{Q}^{-1/2}\right)\left(\widehat{P}^{1/2} + \widehat{P}^{-1/2}\right)\right]^2 - z/\alpha$$

$$\widehat{Q}\widehat{P} = q\widehat{P}\widehat{Q}$$

$$\Pi_A(z) \longrightarrow \Pi_A(z, \hbar)$$

$$\left(q = \mathrm{e}^{\mathrm{i}\hbar}\right)$$

Quantum Mirror Map

Generalized ABJM theory [Honda, Moriyama, 2014]
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encoded in D_5 del Pezzo with quantization

$$\widehat{H} = \left[\left(\widehat{Q}^{1/2} + \widehat{Q}^{-1/2}\right)\left(\widehat{P}^{1/2} + \widehat{P}^{-1/2}\right)\right]^2 - z/\alpha$$

$$\widehat{Q}\widehat{P} = q\widehat{P}\widehat{Q}$$

$[SU(2)]^3$ sym.

- $[SU(2)]^3$ comes from breaking of D_5 sym.
- The sym. breaking is observed in B-period

(Kubo-Moriyama-Nosaka(2018))

↓

We consider quantum mirror map of D_5 del Pezzo

Result

A-period is similar to B-period

- expressed by Weyl characters
- coef. of characters = Integer
- has same reps. as those in B-period

Plan

A-period is similar to B-period

① • expressed by Weyl characters

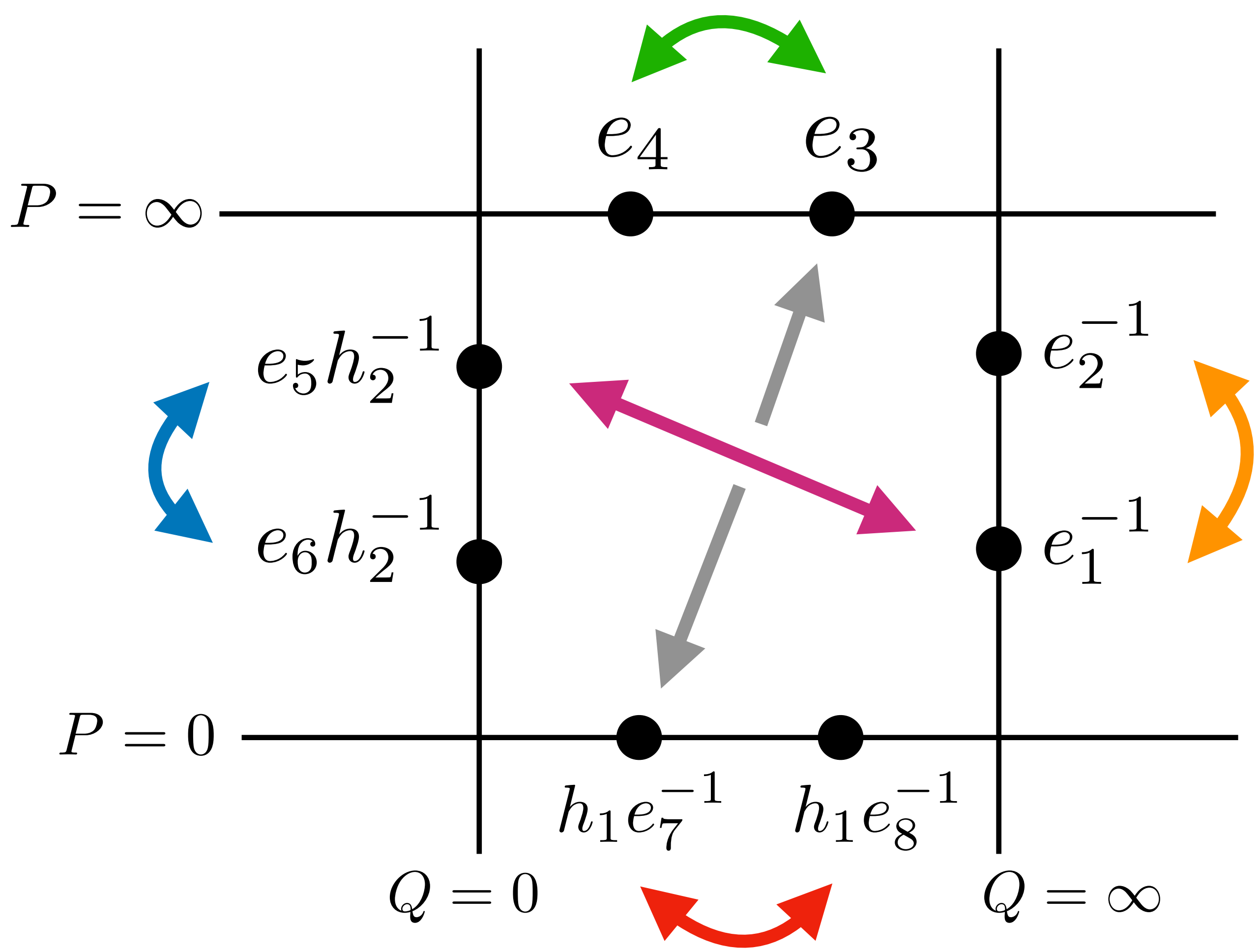
② [
- coef. of characters = Integer
- has same reps. as those in B-period
]

D5 Del Pezzo geometry Kubo-Moriyama-Nosaka(2018)

$$\begin{aligned} 0 = \; & e_3 e_4 Q^{-1} P - (e_3+e_4)P + QP \\ & - h_2^{-1} e_3 e_4 (e_5+e_6) Q^{-1} - \frac{z}{\alpha} - (e_1^{-1}+e_2^{-1})Q \\ & + h_1^2 (e_1 e_2 e_7 e_8)^{-1} Q^{-1} P^{-1} - h_1 (e_1 e_2)^{-1} (e_7^{-1}+e_8^{-1}) P^{-1} + (e_1 e_2)^{-1} Q P^{-1} \end{aligned}$$

- $h_1^2 h_2^2 = \prod_{i=1}^{8} e_i$

classical curve



invariant under exchanging of asymptotic values

⬇

D5 Weyl sym.

D_5 Del Pezzo geometry Kubo-Moriyama-Nosaka(2018)

$$\frac{\widehat{H}}{\alpha}+\frac{z}{\alpha}= e_3e_4\widehat{Q}^{-1}\widehat{P} -(e_3+e_4)\widehat{P} +\widehat{Q}\widehat{P}$$
$$-h_2^{-1}e_3e_4(e_5+e_6)\widehat{Q}^{-1} +\frac{E}{\alpha} -(e_1^{-1}+e_2^{-1})\widehat{Q}$$
$$+h_1^2(e_1e_2e_7e_8)^{-1}\widehat{Q}^{-1}\widehat{P}^{-1} -h_1(e_1e_2)^{-1}(e_7^{-1}+e_8^{-1})\widehat{P}^{-1} +(e_1e_2)^{-1}\widehat{Q}\widehat{P}^{-1}$$

quantum curve

10 parameters - (2 + 2 + 1) parameters

- 8 asymptotic values determine curve — 2
- $(\widehat{P},\widehat{Q})\sim(A\widehat{P},B\widehat{Q})$ — 2
- $h_1^2h_2^2=\prod_{i=1}^{8}e_i$ — 1

→ 5 parameters $(e_1,e_3,e_5,\bar{h}_1,\bar{h}_2)$

$(\bar{h}_1=qh_1,\ \bar{h}_2=q^{-1}h_2)$

Quantum mirror map

Aganagic, Cheng, Dijkgraaf Krefl, Vafa(2011)

$$\Pi_A(z,\hbar) \sim \oint \frac{\log P[X]}{X}\mathrm{d}X = Ez^{-1} + \left(-\frac{E^2}{2} - A_2\right) z^{-2} + \cdots$$

$$\left[P[X] = \frac{\Psi[q^{-1}X]}{\Psi[X]},\ \widehat{Q}\Psi[X] = X\Psi[X] \qquad \left[\frac{\widehat{H}}{\alpha} + \frac{z}{\alpha}\right]\Psi[X] = 0 \right.$$

solve Schrödinger eq. order by order]

$$\frac{A_2}{\alpha^2 q^{-1}} = \frac{e_3}{e_1} + \frac{\overline{h}_1}{e_1} + \frac{\overline{h}_1 e_3}{e_1} + \frac{e_3 e_5}{\overline{h}_2^2} + \frac{e_3 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3^2 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3}{\overline{h}_2} + \frac{e_3}{\overline{h}_2 e_1} + \frac{e_3 e_5}{\overline{h}_2} + \frac{e_3 e_5}{\overline{h}_2 e_1}$$

- This is invariant under Weyl transf.
- It contains 10 terms.

Quantum mirror map

$$\Pi_A(z,\hbar) \sim \oint \frac{\log P[X]}{X}\mathrm{d}X = Ez^{-1} + \left(-\frac{E^2}{2} - A_2\right) z^{-2} + \cdots$$

$$\left[P[X] = \frac{\Psi[q^{-1}X]}{\Psi[X]},\ \widehat{Q}\Psi[X] = X\Psi[X] \qquad \left[\frac{\widehat{H}}{\alpha} + \frac{z}{\alpha}\right]\Psi[X] = 0 \right]$$

[solve Schrödinger eq. order by order]

$$\frac{A_2}{\alpha^2 q^{-1}} = \frac{e_3}{e_1} + \frac{\overline{h}_1}{e_1} + \frac{\overline{h}_1 e_3}{e_1} + \frac{e_3 e_5}{\overline{h}_2^2} + \frac{e_3 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3^2 e_5}{\overline{h}_1 \overline{h}_2^2} + \frac{e_3}{\overline{h}_2} + \frac{e_3}{\overline{h}_2 e_1} + \frac{e_3 e_5}{\overline{h}_2} + \frac{e_3 e_5}{\overline{h}_2 e_1}$$

$$\alpha = q^{1/2}\overline{h}_2^{1/2} e_1^{1/4} e_3^{-1/2} e_5^{-1/4}$$

$$A_2 = \chi_{\mathbf{10}}$$

Plan

A-period is similar to B-period

① ✓ • expressed by Weyl characters

② [
- coef. of characters = Integer
- has same reps. as those in B-period

Multi-covering & BPS

$$\Pi_A(z,\hbar) \sim \sum_{l=1}^{\infty} (-1)^{l+1} A_l z^{-l}$$

$$A_4 \ni \frac{3}{2}\chi_{\mathbf{54}} + \frac{5}{2}\chi_{\mathbf{45}} + \frac{11}{2}\chi_{\mathbf{1}}$$

fractional

➡ Does this have multi-covering structure ?

- A-period of A_1 geometry has multi-covering structure.
  (ABJM theory)
- By this structure, coefficients are integers.

[Hatsuda-Marino-Moriyama-Okuyama(2013)]

multi-covering structure

(coef. of n-th order) = $\sum_{j\leq n}$ (coef. of j-th order)

Multi-covering & BPS

$$\Pi_A(z,\hbar) =: \log z_{\text{eff}} - \log z$$

$$\Rightarrow \log z \sim -A_1 z_{\text{eff}}^{-1} + \left(A_2 - A_1^2\right) z_{\text{eff}}^{-2} - \left(A_3 - 3A_1A_2 + \frac{3}{2}A_1^3\right) z_{\text{eff}}^{-3} + \left(A_4 - 2A_2^2 - 4A_1A_3 - 8A_1^2A_2 - \frac{8}{3}A_1^4\right) z_{\text{eff}}^{-4} + \ldots$$

consider inverse function

$$\left[\begin{array}{l} A_1 = 0 \\ A_2 = \chi_{\mathbf{10}} \\ A_3 = (q^{1/2} + q^{-1/2})\chi_{\mathbf{16}} \\ A_4 = \left(q^2 + q^{-2}\right)\chi_{\mathbf{1}} + \left(q^{3/2} + q^{-3/2}\right)(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + \underline{\frac{3\chi_{\mathbf{54}} + 5\chi_{\mathbf{45}} + 11\chi_{\mathbf{1}}}{2}} \end{array}\right.$$

still fractional

Multi-covering & BPS

$$\Pi_A(z,\hbar) \;=:\; \log z_{\rm eff} - \log z$$

$$\Rightarrow\quad \log z \sim -\color{red}{\underline{\color{black}{A_1}}}\color{black}{\,z_{\rm eff}^{-1} + \left(\color{blue}{\underline{\color{black}{A_2 - A_1^2}}}\color{black}{\right) z_{\rm eff}^{-2} - \left(\color{green}{\underline{\color{black}{A_3 - 3A_1A_2 + \frac{3}{2}A_1^3}}}\color{black}{\right) z_{\rm eff}^{-3}}}}$$

$$+ \left(\color{orange}{\underline{\color{black}{A_4 - 2A_2^2 - 4A_1A_3 - 8A_1^2A_2 - \frac{8}{3}A_1^4}}}\color{black}{\right) z_{\rm eff}^{-4} + \ldots}$$

$$\color{red}{\rule{2cm}{2pt}}\color{black}{\;= \epsilon_1(q, e_i, \bar{h}_i)}$$

$$\color{blue}{\rule{2cm}{2pt}}\color{black}{\;= -\epsilon_2(q, e_i, \bar{h}_i) + \frac{\epsilon_1(q^2, e_i^2, \bar{h}_i^2)}{2}}$$

$$\color{green}{\rule{2cm}{2pt}}\color{black}{\;= \epsilon_3(q, e_i, \bar{h}_i) + \frac{\epsilon_1(q^3, e_i^3, \bar{h}_i^3)}{3}}$$

$$\color{orange}{\rule{2cm}{2pt}}\color{black}{\;= -\epsilon_4(q, e_i, \bar{h}_i) + \frac{\epsilon_2(q^2, e_i^2, \bar{h}_i^2)}{2} + \frac{\epsilon_1(q^4, e_i^4, \bar{h}_i^4)}{4}}$$

multi-covering structure

$$\epsilon_1 = 0, \quad -\epsilon_2 = \chi_{\mathbf{10}},$$

$$\epsilon_3 = \left(q^{1/2} + q^{-1/2}\right)\chi_{\mathbf{16}}\ , -\epsilon_4 = \left(q^2 + q^{-2}\right)\chi_{\mathbf{1}} + \left(q + q^{-1}\right)\left(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}\right) + 4\chi_{\mathbf{1}}$$

integer !

Multi-covering & BPS

[Moriyama-Nosaka-Yano(2017)]

d	(j_L, j_R)	BPS	$(-1)^{d-1}\sum_{\|\boldsymbol{d}\|=1}(N^{\boldsymbol{d}}_{j_L,j_R})_{d^+-d^-}$	representations
1	$(0;0)$	16	$8_{+1}+8_{-1}$	**16**
2	$(0,\frac{1}{2})$	10	$1_{+2}+8_0+1_{-2}$	**10**
3	$(0,1)$	16	$8_{+1}+8_{-1}$	**16**
4	$(0,\frac{1}{2})$	1	1_0	**1**
	$(0,\frac{3}{2})$	45	$8_{+2}+29_0+8_{-2}$	**45**
	$(\frac{1}{2},2)$	1	1_0	**1**

A-period

B-period

$$\begin{cases} \epsilon_1 = 0, \\ -\epsilon_2 = \chi_{\mathbf{10}}, \\ \epsilon_3 = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\chi_{\mathbf{16}}, \\ -\epsilon_4 = (q^2+q^{-2})\chi_{\mathbf{1}} + (q+q^{-1})(\chi_{\mathbf{45}} + 3\chi_{\mathbf{1}}) + 4\chi_{\mathbf{1}}, \end{cases}$$

same reps. as B-period for each order

Physical meaning ?? ➡ Future work

Summary

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A-period is similar to B-period

- expressed by Weyl characters
- coef. of characters = Integer
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Physical meaning ?? ➡ Future work