The Gravity Dual of OPE Block - Part I

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(Based on To Appear with Lung-Chuan Chen, Nozomu Kobayashi and Tatsuma Nishioka.)

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- The so-called "OPE block" in its simplest form is a bi-local operator arising from the OPE between two local primary operators: [Ferrara et al, Mack et al, 1970s]:

$$\mathcal{O}_{\Delta_1}(P_1)\,\mathcal{O}_{\Delta_2}(P_2) = \sum_{\{\Delta,J\}} \mathcal{B}_{\Delta,J}(P_1,P_2).$$

which is entirely kinematical and fixed by conformal symmetries.

- In Euclidean $\mathbb{R}^d$, the explicit form of $\mathcal{B}_{\Delta,J}(P_1,P_2)$ can be extracted from contraction with the shadow projector:

$$|\mathcal{O}_{\Delta,J}| = \frac{\alpha_{\Delta,J}\alpha_{\bar{\Delta},J}}{J!(h-1)_J}\int D^d_{\mathrm{E}}P\,|\,\mathcal{O}_{\bar{\Delta},J}(P,D_Z)\,\rangle\,\langle\,\mathcal{O}_{\Delta,J}(P,Z)\,|\,, \qquad \bar{\Delta} = d-\Delta\,, \quad h = \frac{d}{2}$$

where D_Z is the embedding space spin Todorov operator.

- The shadow transformation is given through the following:

$$\mathcal{O}_{\bar{\Delta},J}(P,Z) \equiv \frac{1}{J!(h-1)_J}\int D^d_{\mathrm{E}}P'\,\langle\,\mathcal{O}_{\bar{\Delta},J}(P,Z)\,\mathcal{O}_{\bar{\Delta},J}(P',D_{Z'})\,\rangle_{\mathrm{E}}\,\mathcal{O}_{\Delta}(P',Z')$$

The generic form of OPE block can be expressed as:

$$\mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_1,P_2) = \frac{\alpha_{\Delta,J}\alpha_{\bar{\Delta},J}}{J!(h-1)_J}\int D^d_{\mathrm{E}}P_0 \,\langle\, \mathcal{O}_{\Delta_1}(P_1)\,\mathcal{O}_{\Delta_2}(P_2)\,\mathcal{O}_{\bar{\Delta},J}(P_0,D_{Z_0})\,\rangle_{\mathrm{n}}\,\mathcal{O}_{\Delta,J}(P_0,Z_0)$$

where the normalized three point function is

$$\langle\, \mathcal{O}_1(P_1)\,\mathcal{O}_2(P_2)\,\mathcal{O}_{\bar{\Delta},J}(P_0,Z_0)\,\rangle_{\mathrm{E}} = \frac{[-2P_1\cdot C_0\cdot P_2]^J}{P_{12}^{\frac{\Delta^+_{12}-\bar{\Delta}+J}{2}}\,P_{10}^{\frac{\bar{\Delta}+\Delta^-_{12}+J}{2}}\,P_{20}^{\frac{\bar{\Delta}-\Delta^-_{12}+J}{2}}}\,, \qquad \Delta^{\pm}_{ij} = \Delta_i \pm \Delta_j$$

If we now consider the Feynman parametrization:

$$\frac{1}{P_{10}^{\frac{\bar{\Delta}+\Delta^-_{12}+J}{2}}\,P_{20}^{\frac{\bar{\Delta}-\Delta^-_{12}+J}{2}}} = 2\mathrm{B}\left(\frac{\bar{\Delta}+\Delta^-_{12}+J}{2},\frac{\bar{\Delta}-\Delta^-_{12}+J}{2}\right)\frac{1}{(P_{12})^{\frac{\bar{\Delta}+J}{2}}}\int_{-\infty}^{\infty}\mathrm{d}\lambda\, e^{\lambda\Delta^-_{12}}\frac{1}{(-2X(\lambda)\cdot P_0)^{\bar{\Delta}+J}}$$

where $\mathrm{B}(x,y)$ is the beta function.

- It is interesting to note that the combined coordinate:

$$\gamma_{12} \ : \ X^A(\lambda) \equiv \frac{e^{\lambda} P_1^A + e^{-\lambda} P_2^A}{P_{12}^{\frac{1}{2}}} \ , \quad X(\lambda)^2 = -1$$

naturally belongs to AdS_{d+1} space and is interpreted as geodesic.

- We can now express the OPE block as the following integral:

$$\mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_1,P_2) = 2\frac{\alpha_{\Delta,J}\alpha_{\bar{\Delta},J}}{J!(h-1)_J}\mathrm{B}\left(\frac{\bar{\Delta}+\Delta_{12}^{-}+J}{2},\frac{\bar{\Delta}-\Delta_{12}^{-}+J}{2}\right)\int D_{\mathrm{E}}^{d}P_0$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{1}{(-2P_1\cdot X(\lambda))^{\Delta_1}(-2P_2\cdot X(\lambda))^{\Delta_2}} \frac{\left[V_{0,12}\right]^J|_{Z_0\to D_{Z_0}}}{(-2P_0\cdot X(\lambda))^{\bar{\Delta}+J}}\mathcal{O}_{\Delta,J}(P_0,Z_0)$$

where the unique tensor structure is:

$$V_{0,12} \equiv \frac{P_1\cdot C_0\cdot P_2}{P_1\cdot P_2} = \frac{-2P_1\cdot C_0\cdot P_2}{P_{12}}$$

- For the scalar case, we can express the scalar OPE block into:

$$\mathcal{B}^{(\mathrm{E})}_{\Delta}(P_1, P_2) = 2\alpha_{\bar{\Delta}} \mathrm{B}\left(\frac{\bar{\Delta}+\Delta^{-}_{12}}{2}, \frac{\bar{\Delta}-\Delta^{-}_{12}}{2}\right) \frac{1}{P_{12}^{\frac{\Delta^{+}_{12}}{2}}} \int_{-\infty}^{\infty} \mathrm{d}\lambda \, e^{\lambda \Delta^{-}_{12}} \, \Phi^{(\mathrm{E})}_{\Delta}(X(\lambda))$$

  where we have introduced so-called HKLL scalar field $\Phi^{(\mathrm{E})}_{\Delta}(X)$ [Hamilton et. al. 2006] such that:

$$\Phi^{(\mathrm{E})}_{\Delta}(X(\lambda)) = \alpha_{\Delta} \int D^{d}_{\mathrm{E}} P_0 \frac{1}{(-2P_0 \cdot X(\lambda))^{\bar{\Delta}}} \mathcal{O}_{\Delta}(P_0) \ , \quad \alpha_{\Delta} = \frac{\Gamma(\Delta)}{\pi^{h} \Gamma(h-\Delta)} \ ,$$

  which was first derived from solving the AdS scalar equation of motion for arbitrary bulk point X.
- The natural interpretation of OPE block is that we integrating $\Phi^{(\mathrm{E})}_{\Delta}(X)$ along the geodesic γ_{12} with measure $e^{\Delta_{12}\lambda}$. [Czech et. al. 2016], [de Boer et. al. 2016], [da Cunha et. al. 2016]

- For $J \neq 0$, notice that we have the identity along γ_{12}:

$$V_{0,12} = -\frac{\mathrm{d}X(\lambda)}{\mathrm{d}\lambda} \cdot C_0 \cdot X(\lambda)$$

and rewrite the spinning OPE block as

$$\begin{aligned}\mathcal{B}^{(\mathrm{E})}_{\Delta,J}(P_1,P_2) &= \frac{2\alpha_{\bar{\Delta},J}}{J!(h-\frac{1}{2})_J}\mathrm{B}\left(\frac{\bar{\Delta}+\Delta^-_{12}+J}{2},\frac{\bar{\Delta}-\Delta^-_{12}+J}{2}\right)\int_{-\infty}^{\infty}\mathrm{d}\lambda\, e^{\lambda\Delta^-_{12}}\left[\frac{\mathrm{d}X(\lambda)}{\mathrm{d}\lambda}\cdot K\right]^J \Phi^{(\mathrm{E})}_{\Delta,J}(X(\lambda),W)\\ &= 2\alpha_{\bar{\Delta},J}\mathrm{B}\left(\frac{\bar{\Delta}+\Delta^-_{12}+J}{2},\frac{\bar{\Delta}-\Delta^-_{12}+J}{2}\right)\frac{1}{P_{12}^{\frac{\Delta^+_{12}}{2}}}\int_{-\infty}^{\infty}\mathrm{d}\lambda\, e^{\lambda\Delta^-_{12}}\,\Phi^{(\mathrm{E})}_{\Delta,J}\left(X(\lambda),\frac{\mathrm{d}X(\lambda)}{\mathrm{d}\lambda}\right)\end{aligned}$$

- Here we have introduced the spinning generalization of HKLL field:

$$\begin{aligned}\Phi^{(\mathrm{E})}_{\Delta,J}(X,W) &\equiv \frac{\alpha_{\Delta,J}}{2^J J!(h-1)_J}\int D^d_{\mathrm{E}}P_0\, K^{(\mathrm{n})}_{\bar{\Delta},J}(X,P_0;W,D_{Z_0})\,\mathcal{O}_{\Delta,J}(P_0,Z_0)\\ &= \frac{\alpha_{\Delta,J}}{2^J}\int D^d_{\mathrm{E}}P_0\,\frac{1}{(-2P_0\cdot X)^{\bar{\Delta}}}\,\mathcal{O}_{\Delta,J}(P_0,W\cdot\mathcal{J}(P_0,X))\end{aligned}$$

[HYC + Chen, Kobayashi, Nishioka, To Appear]

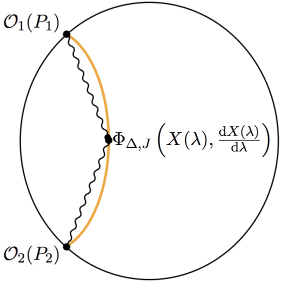
- It is interesting to note that the boundary polarization vector:

$$(W \cdot \mathcal{J}(P_0, X))_A = W_A - \frac{(W \cdot P_0)}{(X \cdot P_0)} X_A$$

is necessary for preserving the bulk symmetry: $W^A \to W^A + \alpha X^A$.

- We therefore have the Euclidean Spinning OPE block as integrating the pull-back of $\Phi^{(\mathrm{E})}_{\Delta,J}(X, W)$ along γ_{12}:

$$\mathcal{B}_{\Delta,J}(P_1, P_2) = \int_{-\infty}^{\infty} \mathrm{d}\lambda$$



Another way to derive $\Phi^{(\mathrm{E})}_{\Delta,J}(X,W)$ is to consider differential operator:

$$D_P^A = Z^A \left(Z \cdot \frac{\partial}{\partial Z} \right) - C^{AB} \frac{\partial}{\partial P^B} \,, \qquad C^{AB} = Z^A P^B - P^A Z^B$$

on both sides of scalar conformal integral

$$\int D^d_{\mathrm{E}} P_0 \frac{\alpha_{\bar{\Delta}}}{(-2P_0 \cdot X)^{\bar{\Delta}}} \frac{1}{(-2\tilde{P} \cdot P_0)^{\Delta}} = \frac{(-X^2)^{h-\bar{\Delta}}}{(-2\tilde{P} \cdot X)^{\Delta}}$$

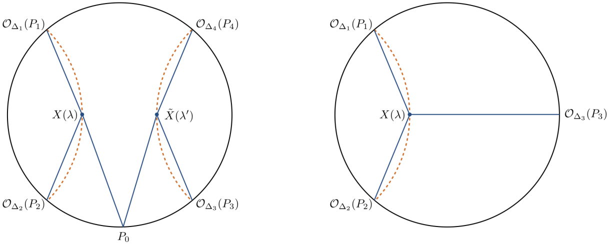
We can deduce the following relation:

$$\begin{aligned} &\langle \mathcal{O}_{\Delta,J}(\tilde{P},\tilde{Z})\, \Phi^{(\mathrm{E}) A_1 \ldots A_J}_{\Delta}(X) \rangle_{\mathrm{E}} \\ &= \int D^d_{\mathrm{E}} P_0 \frac{\alpha_{\bar{\Delta}}}{(-2X \cdot P_0)^{\bar{\Delta}}} \langle \mathcal{O}_{\Delta,J}(\tilde{P},\tilde{Z})\, \mathcal{J}^{A_1 B_1}(P_0,X) \ldots \mathcal{J}^{A_J B_J}(P_0,X)\, \mathcal{O}_{\Delta, B_1,\ldots,B_J}(P_0) \rangle_{\mathrm{E}} \end{aligned}$$

This allows us to directly verify $\Phi^{(\mathrm{E})}_{\Delta,J}(X,W)$ satisfies the bulk equation of motion.

Connection with Geodesic Witten Diagram

For $J \in \mathbb{Z}_{\geq 0}$ in $\mathbb{R}^d$, the holographic dual is "Geodesic Witten Diagram" (GWD) [Hijano et. al.]:



Here we have diagrammatically introduced the "split representation", and their building block: three point GWD [Chen, Kyono, Kuo.]

This picture also generalizes holographically the earlier proposal for "two point" function of OPE block yields conformal block:

$$\begin{aligned}
&\langle\, \mathcal{B}_{\Delta,J}(P_1,P_2)\, \mathcal{B}_{\Delta,J}(P_3,P_4)\,\rangle \\
&= (2\alpha_{\bar{\Delta},J})^2 \mathrm{B}\left(\frac{\bar{\Delta}+\Delta_{12}^{-}+J}{2}, \frac{\bar{\Delta}-\Delta_{12}^{-}+J}{2}\right) \mathrm{B}\left(\frac{\bar{\Delta}+\Delta_{34}^{-}+J}{2}, \frac{\bar{\Delta}-\Delta_{34}^{-}+J}{2}\right) \\
&\times \int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\lambda' \frac{\left\langle \Phi^{(\mathrm{E})}_{\Delta,J}\left(X(\lambda), \frac{\mathrm{d}X(\lambda)}{\mathrm{d}\lambda}\right)\, \Phi^{(\mathrm{E})}_{\Delta,J}\left(\tilde{X}(\lambda'), \frac{\mathrm{d}\tilde{X}(\lambda')}{\mathrm{d}\lambda'}\right)\right\rangle}{(-2P_1\cdot X(\lambda))^{\Delta_1}(-2P_2\cdot X(\lambda))^{\Delta_2}(-2P_3\cdot \tilde{X}(\lambda'))^{\Delta_3}(-2P_4\cdot \tilde{X}(\lambda'))^{\Delta_4}}
\end{aligned}$$

where we can directly identify the pull back of bulk to bulk propagator for spin-J tensor field.[Hijano et. al.]

Generalization to Lorentzian CFT/AdS?

- It is interesting to ask if we can extend the construction of Euclidean Holographic OPE block to Lorentzian case? [See Tatsuma's talk]

$$\mathcal{B}^{(\mathrm{L})}_{\Delta,J}(P_1,P_2) \sim \int_{?} D^d_{\mathrm{L}} P_0 \,\langle\, \mathcal{O}_{\Delta_1}(P_1)\,\mathcal{O}_{\Delta_2}(P_2)\,\mathcal{O}_{\tilde{\Delta},J}(P_0,D_{Z_0})\,\rangle_{\mathrm{n}}\,\mathcal{O}_{\Delta,J}(P_0,Z_0)\ ??$$

  Basically three issues:

  1. Integration Region?
  2. Which 3 pt function?
  3. Which bulk field and geodesic?

- This belongs to part of the systematic program of relating boundary primary fields to bulk ones (i.e. Lorentzian HKLL?), their correlation functions to the corresponding Witten diagrams in Lorentzian setting.

Non-local Primary Operators in Lorentzian CFT

- Now consider CFT in $\mathbb{M}^{1,d-1}$, a primary operator $\mathcal{O}^{\lambda}_{\Delta,J}(x)$ is labeled by (Δ, J, λ) of $SO_{\Delta}(1,1) \times SO_{J}(1,1) \times SO_{\lambda}(d-2) \subset SO(2,d)$.
- The restrict Weyl group is now D_8, whose elements can be generated by two transformations: [Kravchuk, Simmons-Duffin]

$$\mathbf{S}_J[(\Delta, J, \lambda)] = (\Delta, 2-d-J, \lambda^{\mathrm{R}}), \quad \mathbf{L}[(\Delta, J, \lambda)] = (1-J, 1-\Delta, \lambda)$$

which satisfy $(\mathbf{S}_J)^2 = (\mathbf{L})^2 = (\mathbf{L}\mathbf{S}_J)^4 = 1$.

- Using the combinations of $\mathbf{S}_L$ and $\mathbf{L}$, the eight elements of D_8 are:

w		order	Δ'	J'	λ'
1		1	Δ	J	λ
S_Δ	$= \mathrm{LS}_J\mathrm{L}$	2	$d-\Delta$	J	λ^R
S_J		2	Δ	$2-d-J$	λ^R
S	$= (\mathrm{S}_J\mathrm{L})^2$	2	$d-\Delta$	$2-d-J$	λ
L		2	$1-J$	$1-\Delta$	λ
F	$= \mathrm{S}_J\mathrm{LS}_J$	2	$J+d-1$	$\Delta-d+1$	λ
R	$= \mathrm{S}_J\mathrm{L}$	4	$1-J$	$\Delta-d+1$	λ^R
$\overline{\mathrm{R}}$	$= \mathrm{LS}_J$	4	$J+d-1$	$1-\Delta$	λ^R

The non-local primary operators corresponding to $\mathbf{S}_J$ (Spin-Shadow) and $\mathbf{L}$ (Light Ray) transformations can be constructed through integrals:

$$\begin{aligned}
\text{Shadow} &: \tilde{\mathcal{O}}_{d-\Delta,J}(x,z) = i\int d^dx' \frac{1}{((x-y)^2)^{d-\Delta}}\mathcal{O}_{\Delta,J}(x',\mathcal{I}(x-x')z), \\
\text{Spin Shadow} &: \tilde{\mathcal{O}}_{\Delta,2-d-J}(x,z) = \int D^{d-2}z'(-2z\cdot z')^{2-d-J}\mathcal{O}_{\Delta,J}(x,z'), \\
\text{Light Ray} &: \tilde{\mathcal{O}}_{1-J,1-\Delta}(x,z) = \int_{-\infty}^{+\infty} d\alpha(-\alpha')^{-\Delta-J}\mathcal{O}_{\Delta,J}(x-\frac{z}{\alpha},z).
\end{aligned}$$

here we have introduced the embedding function for continuous J:

$$\mathcal{O}_{\Delta,J}(\alpha x,\beta z) = \alpha^{-\Delta}\beta^{J}\mathcal{O}_{\Delta,J}(x,z), \quad \text{c. f.} \quad \mathcal{O}_{\Delta,J}(x,z) = z^{\mu_1}\ldots z^{\mu_J}\mathcal{O}_{\Delta,\mu_1\ldots\mu_J}(x)$$

while the integration measure over z^μ is:

$$D^{d-2}z \equiv \frac{d^dz\theta(z^0)\delta(z^2)}{\mathrm{vol}\mathbb{R}^+}$$

Both shadow and spin-shadow transformations can be treated equally.

For example, if we now include the spin-shadow transformation to define "full shadow" projector ($2\varepsilon = d-2$):

$$|\mathcal{O}_{\Delta,J}| = \beta_{\Delta,J}\beta_{\bar{\Delta},\bar{J}} \int D^{2h}P' \int D^{2\varepsilon}z' |\, \mathcal{O}_{\bar{\Delta},\bar{J}}(P',z')\,\rangle\,\langle\,\mathcal{O}_{\Delta,J}(P',z')\,|$$

For space-like $P_{12} > 0$, we are led to the following proposal for the spinning Lorentzian OPE block:

$$\mathcal{B}^{(\mathrm{L})}_{\Delta,J}(P_1,P_2) = 2\beta_{\bar{\Delta},\bar{J}}\mathrm{B}\left(\frac{\bar{\Delta}+\Delta^{-}_{12}+\bar{J}}{2},\frac{\bar{\Delta}-\Delta^{-}_{12}+\bar{J}}{2}\right)\frac{1}{P_{12}^{\frac{\Delta^{+}_{12}}{2}}}\int_{-\infty}^{\infty}\mathrm{d}\lambda\, e^{\lambda\Delta^{-}_{12}}\,\Phi^{(\mathrm{L})}_{\Delta,J}\left(X(\lambda),\frac{\mathrm{d}x(\lambda)}{\mathrm{d}\lambda}\right)$$

where the HKLL field is now:

$$\Phi^{(\mathrm{L})}_{\Delta,J}(X,w) = \beta_{\Delta,J}\int D^{2\varepsilon}z_0\int_{(-2P_0\cdot X)\geq 0} D^{d}_{\mathrm{L}}P_0\,\frac{\left[w\cdot\mathcal{J}(x-x_0)\cdot z_0\right]^{\bar{J}}}{(-2P_0\cdot X)^{\bar{\Delta}}}\mathcal{O}_{\Delta,J}(P_0,z_0)$$

with $\mathcal{J}_{\mu\nu}(x-y) = \delta_{\mu\nu} - 2\frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2+\eta^2}$.