The gravity dual of OPE block II

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Introduction

- Renewed interest on Lorentzian CFT:
  - Analytic bootstrap [Fitzpatrick–Kaplan–Poland–Simmons-Duffin 12, Komargodski-Zhiboedov 12, $\cdots$]
  - Lorentzian inversion formula [Caron-Huot 17, Simmons-Duffin–Stanford–Witten 17]
  - Causality constraint and ANEC [Hartman-Kundu-Tajdini 15]
- We revisit the OPE structure in Lorentzian CFT and derive its holographic representation

Outline

- Review of Lorentzian CFT
- Derivation of Lorentzian OPE block in momentum space
- The gravity dual of Lorentzian OPE block
  - Spacelike: HKLL field on AdS
  - Timelike: HKLL-type field on hyperboloid
- Summary and future problem

Lorentzian CFT

- Reconstruction of Lorentzian CFT from Euclidean CFT (Lüscher-Mack theorem) [Osterwalder-Schrader 73, 75, Lüscher-Mack 75]

- Lorentzian CFT can be defined by
  - a set of primary operators $\mathcal{O}_i(x)$ (unitary irrep of $\mathrm{SO}(2,d)$):

$$i = (\Delta, J, \lambda) \in \mathrm{SO}(1,1) \times \mathrm{SO}(1,1) \times \mathrm{SO}(d-2) \ ,$$

    (Δ, J): continuous

  - the OPE coefficients C_{ijk}:

$$\mathcal{O}_i(x)\,\mathcal{O}_j(0) = \sum_k C_{ijk} \frac{1}{|x|^{\Delta_i+\Delta_j-\Delta_k}} \left[\mathcal{O}_k(0) + (\text{descendants})\right]$$

Weyl reflections in CFT

- Eigenvalue of quadratic Casimir for $i = (\Delta, J, 0)$:

$$C_2 = \Delta(\Delta - d) + J(J + d - 2)$$

- Additional "symmetry" in Lorentzian CFT [Kravchuk–Simmons-Duffin 18]:

$$\mathrm{S}_J \;:\; (\Delta, J) \;\to\; (\Delta, 2 - d - J) \qquad \text{Spin shadow transform}$$

$$\mathrm{L} \;:\; (\Delta, J) \;\to\; (1 - J, 1 - \Delta) \qquad \text{Light transform}$$

- They form the restricted Weyl group of order 8 of $\mathrm{SO}(2, d)$ with the relations $\mathrm{L}^2 = \mathrm{S}_J^2 = (\mathrm{L}\mathrm{S}_J)^2 = 1$ including

$$\mathrm{S}_\Delta \equiv \mathrm{L}\,\mathrm{S}_J\,\mathrm{L} : (\Delta, J) \;\to\; (d - \Delta, J) \qquad \text{Shadow transform}$$

OPE in CFT

- OPE is convergent on the CFT vacuum [Mack 77]

$$\mathcal{O}_i(x_1)\,\mathcal{O}_j(x_2) = \sum_k \mathcal{B}_{ijk}(x_1, x_2)$$

- $\mathcal{B}_{ijk}(x_1, x_2)$ is called the OPE block and appeared recently in different contexts, i.e. entanglement entropy
  [Czech-Lamprou-McCandlish-Mosk-Sully 16, de Boer-Haehl-Heller-Myers 16]

- Conformal invariance is strong enough to fix the form of $\mathcal{B}_{ijk}(x_1, x_2)$

Shadow projector in Lorentzian CFT?

- Naive use of the shadow projector [Ferrara-Gatto-Grillo-Parisi 72]

$$\mathbf{1} \underset{?}{=} \sum_k \int \mathrm{d}^d y \, | \, \tilde{\mathcal{O}}_{\tilde{k}}(y) \, \rangle \langle \, \mathcal{O}_k(y) \, |$$

  gives an integral representation of the OPE block:

$$\mathcal{B}_{ijk}(x_1, x_2) \underset{?}{=} \int \mathrm{d}^d y \, \langle \, \mathcal{O}_i(x_1) \, \mathcal{O}_j(x_2) \, \tilde{\mathcal{O}}_{\tilde{k}}(y) \, \rangle \, \mathcal{O}_k(y)$$

  $\tilde{\mathcal{O}}_{\bar{k}}$: shadow field with $\bar{k} = (d - \Delta_k, J)$

- Ambiguity:
  - Is the projector complete even in Lorentzian?
  - What type of three-point function is used? (Time-ordered or Wightman?)

  (cf: no ambiguity in Euclidean CFT)

Momentum space shadow formalism

- In Lorentzian CFT the projector can be constructed in momentum space [Gillioz-Lu-Luty 16, Gillioz 18]

$$\mathbf{1} = \sum_k \int \frac{\mathrm{d}^d_{\mathrm{L}} p}{(2\pi)^d} \, \Theta(p^0) \, \Theta(-p^2) \, | \, \tilde{\mathcal{O}}_{\bar{k}}(-p) \, \rangle \langle \, \mathcal{O}_k(p) \, |$$

- $| \, \mathcal{O}_k(p) \, \rangle$ spans orthonormal basis with the positive definite inner product:

$$\langle \, \mathcal{O}_k(p_1) \, | \, \mathcal{O}_k(p_2) \, \rangle = (2\pi)^d \, \delta^{(d)}(p_1 + p_2) \, W_k(p)$$

with the Wightman 2-pt function $W_k(p)$

- In momentum space the shadow field is defined by

$$| \, \tilde{\mathcal{O}}_{\bar{k}}(p) \, \rangle \equiv W_{\bar{k}}(p) \, | \, \mathcal{O}_k(p) \, \rangle$$

Lorentzian OPE block

- Inserting the projector the OPE block becomes

$$\mathcal{B}_{ijk}(x_1, x_2) = \int_{p^0 \geq 0,\, p^2 \leq 0} \frac{\mathrm{d}^d p}{(2\pi)^d}\, W_{ijk}(x_1, x_2, -p)\, W_{\bar{k}}(-p)\, \mathcal{O}_k(p)$$

- The Wightman three-point function W_{ijk} is given by analytically continuing the Euclidean correlator:

$$x_i^d = \mathrm{i}\, x_i^0 + \epsilon_i \ , \qquad \epsilon_1 > \epsilon_2 > \cdots > 0$$

- Explicit form of the OPE block was already derived in 70s
[cf: Dobrev-Mack-Petkova-Petkova-Todorov 77]

- There are two distinct cases in Lorentzian OPE:
  - Spacelike: $x_{12}^2 \equiv (x_1 - x_2)^2 > 0$
  - Timelike: $x_{12}^2 < 0$

Spacelike OPE block

- For a pair of scalar primaries, $i = j =$ scalar, the OPE has only the spin-J contributions, $k = (\Delta, J)$

$$\mathcal{O}_{\Delta_1}(x_1)\,\mathcal{O}_{\Delta_2}(x_2) = \sum_{\Delta,J} \mathcal{B}_{\Delta,J}(x_1, x_2)$$

- First, we will start with the spacelike case

- We proceed to rewrite $\mathcal{B}_{\Delta,J}(x_1, x_2)$ as follows:
  - Rewrite the three-point function by introducing a Schwinger/Feynman parameter u
  - Perform Fourier transform to the position space

Scalar OPE block

- For a scalar primary ($J = 0$) we find (up to a constant) [da Cunha-Guica 16]

$$\mathcal{B}_\Delta(x_1, x_2) = \frac{1}{(x_{12}^2)^{\frac{\Delta_{12}^+}{2}}} \int_0^1 \mathrm{d}u \, u^{\frac{\Delta_{12}^-}{2} - 1} (1-u)^{-\frac{\Delta_{12}^-}{2} - 1} \, \Phi_\Delta^{(\mathrm{L})}\left(x^\mu(u), \eta(u)\right)$$

- The parameters defined by

$$x^\mu(u) \equiv u \, x_1^\mu + (1-u) \, x_2^\mu \,, \qquad \eta(u) \equiv \sqrt{u(1-u) \, x_{12}^2}$$

can be identified with the geodesic connecting the boundary points x_1 and x_2 in the Poincaré AdS_{d+1} coordinates

$$\mathrm{d}s^2 = \frac{\mathrm{d}\eta^2 + \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu}{\eta^2} \,, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$$

Holographic representation of spacelike OPE block

- $\Phi_{\Delta}^{(\mathrm{L})}$ is so called the HKLL representation of an AdS scalar field:

$$\Phi_{\Delta}^{(\mathrm{L})}(t, \boldsymbol{x}, \eta) = \int_{t'^2 + \boldsymbol{y}'^2 \le \eta^2} \mathrm{d}^d y' \left(\frac{\eta}{\eta^2 - t'^2 - \boldsymbol{y}'^2} \right)^{\bar{\Delta}} \mathcal{O}_{\Delta}(t + t', \boldsymbol{x} + \mathrm{i}\, \boldsymbol{y}')$$

- As a consistency check, the equation of motion for $\Phi_{\Delta}^{(\mathrm{L})}$ follows from the quadratic Casimir equation for $\mathcal{B}_{\Delta}(x_1, x_2)$ with the canonical identification

$$m^2 = \Delta(d - \Delta)$$

- By a change of variable $u = 1/(1 + e^{2\lambda})$ this matches the Euclidean result [see Heng-Yu's talk] up to the difference of spacetime signature as expected

Timelike OPE block

- The OPE block is analytic in x_{12}^2, so we can obtain the timelike OPE block via analytic continuation:

$$\mathcal{B}_\Delta(x_1, x_2) = \frac{1}{(x_{12}^2)^{\frac{\Delta_{12}^+}{2}}} \int_0^1 \mathrm{d}u\, u^{\frac{\Delta_{12}^-}{2}-1}(1-u)^{-\frac{\Delta_{12}^-}{2}-1}\, \Phi_\Delta^{(\mathrm{LT})}(x^\mu(u), \chi(u))$$

- Here the parameters

$$x^\mu(u) = u\, x_1^\mu + (1-u)\, x_2^\mu\ , \quad \chi(u) = \sqrt{u(1-u)\,|x_{12}^2|}$$

are on the geodesic on a hyperboloid in $\mathbb{R}^{d,2}$ with the metric:

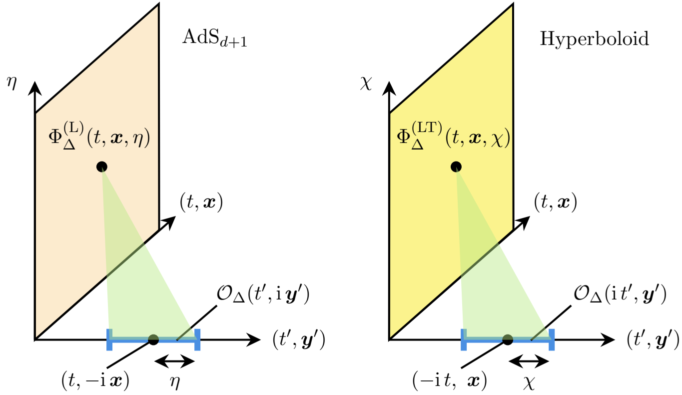
$$\mathrm{d}s^2 = \frac{-\mathrm{d}\chi^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}{\chi^2}$$

- This is not AdS, but an analytic continuation of de Sitter space!

Holographic representation of timelike OPE block

- In the timelike OPE block, we introduce an HKLL-type representation of a scalar field on the hyperboloid:

$$\Phi_{\Delta}^{(\mathrm{LT})}(t,\boldsymbol{x},\chi) \equiv \int_{t'^2+\boldsymbol{y}'^2\leq\chi^2} \mathrm{d}^d y' \left(\frac{\chi}{\chi^2-t'^2-\boldsymbol{y}'^2}\right)^{\bar{\Delta}} \mathcal{O}_{\Delta}(t+\mathrm{i}\,t',\boldsymbol{x}+\boldsymbol{y}')$$



Summary

- Derived holographic representations of Lorentzian OPE blocks
- A new holographic rep of a bulk field on the hyperboloid found in the timelike case
- Can be generalized to higher-spin fields as in Euclidean case [Heng-Yu's talk]

Future problem

- How to derive surface Witten representation of the timelike OPE block proposed by [Czech-Lamprou-McCandlish-Mosk-Sully 16, de Boer-Haehl-Heller-Myers 16]?

$$\mathcal{B}_{\Delta}(x_1, x_2) = \frac{1}{\left(x_{12}^2\right)^{\frac{\Delta_{12}^+}{2}}} \int \mathrm{d}^{d-1}\xi \sqrt{h}\ \Phi_0\left(x^\mu(\xi), \eta(\xi)\right)$$