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Interacting Vortices in U(1) Gauge Theory

A Brief Review of Nielsen-Olesen Vortices

Nielsen-Olesen vortex
a point-like object in (1+2)D

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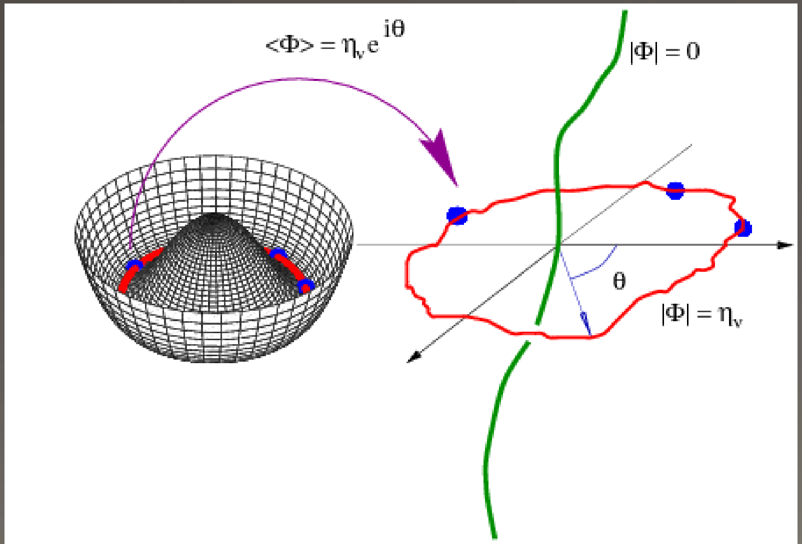
cosmic string
a straight string along z-axis



Revisit Abelian Higgs model as the simplest field theoretic model

a complex scalar field ϕ

a U(1) gauge field A_μ

action $S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \overline{D_\mu \phi} D^\mu \phi - V(\phi, \bar{\phi}) \right]$

scalar potential $V(\phi, \bar{\phi}) = \frac{\lambda}{8} (\bar{\phi}\phi - v^2)^2$

Energy for static Nielsen-Olesen vortices carrying magnetic flux

$$\frac{E}{\int_{-\infty}^{\infty} dz} = \int dxdy\left[\frac{1}{2}B^2 + \frac{1}{2}|D_i\phi|^2 + \frac{\lambda}{8}(\bar{\phi}\phi - v^2)^2\right]$$
$$= \int dxdy\left[\frac{1}{2}B^2 + \frac{1}{2}(\partial_i|\phi|)^2 + \frac{|\phi|^2}{2}(\partial_i\Omega - eA_i)^2 + \frac{\lambda}{8}(|\phi|^2 - v^2)^2\right]$$

complex scalar field = amplitude + phase $\phi = |\phi|e^{i\Omega}$

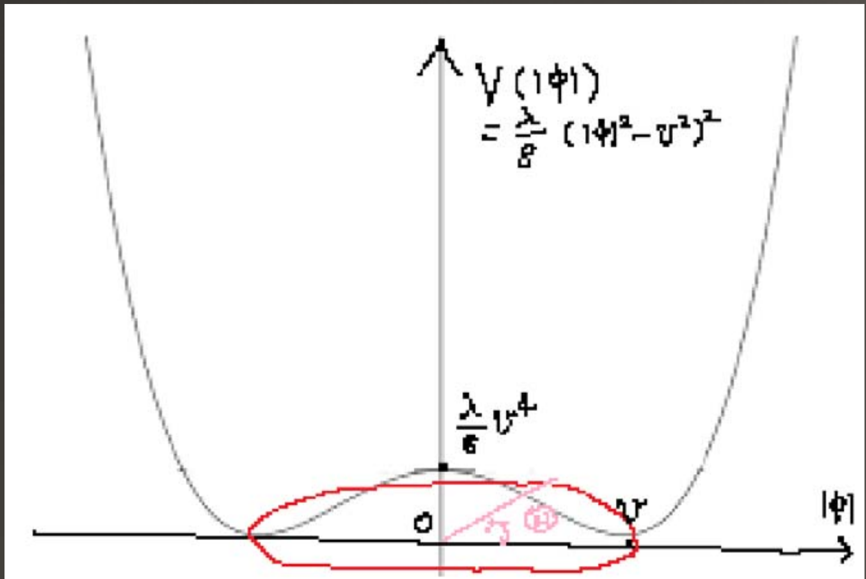
Vacuum configuration

$$|\phi| = v$$

$$0 = B = \epsilon^{ij}\partial_i A^j = \frac{1}{e}\epsilon^{ij}\partial_i\partial_j\Omega$$

$\Omega = \Lambda(x,y)$ and $A^i = \frac{1}{e}\partial_i\Lambda$ satisfying $\epsilon^{ij}\partial_i\partial_j\Lambda = 0$

Q. topology of vacua



superselection $\langle v, \Theta_1 | v, \Theta_2 \rangle \propto \delta(\Theta_1 - \Theta_2)$

vacuum manifold $\mathcal{M}_v = \mathrm{S}^1$

boundary of xy-plane $\partial(\mathbb{R}^2) = \mathrm{S}^1_\infty$

complex scalar field $\dfrac{\phi}{|\phi|} : \mathrm{S}^1_\infty \longrightarrow \mathrm{S}^1$

nontrivial topology = first homotopy

$\Pi_1(\mathrm{S}^1) = \mathbb{Z}$

Q. topological soliton = Nielsen-Olesen vortex

n vortices superimposed at the origin $\boxed{\Omega = n\theta + \Lambda(x)}$

single valuedness of field $\boxed{e^{in\theta} \xrightarrow{\text{rotate } s \text{ times}} e^{in(\theta+2\pi s)} = e^{in\theta}e^{2\pi isn}}$ for any integer s

n vortices separated in xy-plane $\boxed{\Omega(x,y) = \sum_{p=1}^{n} \tan^{-1}\frac{y-y_p}{x-x_p} + \Lambda(x)}$ $\rightarrow$ $\boxed{|\phi|(\boldsymbol{x}_p) = 0}$

quantized magnetic flux

$$
\begin{aligned}
\Phi_B &= \int dxdyB = \int dxdy\frac{1}{2}\epsilon_{ij}F_{ij} = \int dxdy\,\epsilon_{ij}\partial_i A^j \\
&= \oint_{S^1_\infty} d\ell_i A^i \\
&= \oint_{S^1_\infty} d\ell_i \frac{1}{e}\partial_i\Omega = \frac{1}{e}\int dxdy\,\epsilon_{ij}\partial_i\partial_j\Omega \\
&= \frac{1}{e}\int dxdy\left[\sum_{p=1}^{n}\epsilon_{ij}\partial_i\partial_j\tan^{-1}\frac{y-y_p}{x-x_p} + \epsilon_{ij}\partial_i\partial_j\Lambda(x)\right] \\
&= \frac{1}{e}\sum_{p=1}^{n}\int dxdy\epsilon_{ij}\partial_i\partial_j\tan^{-1}\frac{y-y_p}{x-x_p} \\
&= \frac{2\pi}{e}\sum_{p=1}^{n}\int dxdy\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}_p) = \frac{2\pi}{e}n.
\end{aligned}
$$

= first Chern number

= winding number

n = vorticity

Focus the questions to

1. tractability
2. intriguing limit, theoretically and experimentally

$$\lambda = e^2$$

Q. Why is this coupling so special?

A1. Masses of U(1) gauge boson and Higgs become equal.

$$m_{\mathrm{g}} = m_{\mathrm{H}} = ev$$

A2. Bogomolny bound

Reshuffle the energy terms to show a lower bound (Bogomolny bound)

$$
\begin{aligned}
\frac{E}{\int dz} &= \int d^2x \left[\frac{1}{2} \left| (D_1 \pm i D_2)\phi \right|^2 + \frac{1}{2} \left[B \mp \frac{e}{2}(v^2 - |\phi|^2) \right]^2 \pm \frac{1}{2e} \partial_i \left(\epsilon^{ij} j_j \right) \pm \frac{ev^2}{2} B \right] \\
&= \int d^2x \left[\frac{1}{2} \left| (D_1 \pm i D_2)\phi \right|^2 + \frac{1}{2} \left[B \mp \frac{e}{2}(v^2 - |\phi|^2) \right]^2 \right] \pm \frac{ev^2}{2} \Phi_B \\
&= \int d^2x \left[\frac{1}{2} \left| (D_1 \pm i D_2)\phi \right|^2 + \frac{1}{2} \left[B \mp \frac{e}{2}(v^2 - |\phi|^2) \right]^2 \right] \pm n m_{\mathrm{BPS}} \\
&\geq |n| m_{\mathrm{BPS}}.
\end{aligned}
$$

where $m_{\mathrm{BPS}} = \pi v^2 = \dfrac{\pi v}{e} m_{\mathrm{H}} = \dfrac{\pi v}{e} m_{\mathrm{g}}$

BPS (Bogomolny-Prasad-Sommerfield) equations

$$
\begin{aligned}
(D_1 \pm i D_2)\phi &= 0, \\
B &= \pm \frac{e}{2}(v^2 - |\phi|^2)
\end{aligned}
\quad \rightarrow \quad e A_i = \pm \epsilon^{ij} \partial_j \ln |\phi| + \partial^i \Omega
$$

$$
\rightarrow \quad \nabla^2 \ln \frac{|\phi|^2}{\prod_{p=1}^{n} |\boldsymbol{x} - \boldsymbol{x}_p|^2} = e^2 (|\phi|^2 - v^2)
$$

Reshuffle stress tensor terms

$$
\begin{aligned}
T_{ij} &= \frac{1}{2}\left(\overline{D_i\phi}D_j\phi + \overline{D_j\phi}D_i\phi\right) + \delta_{ij}\left[\frac{1}{2}B^2 - \frac{1}{2}|D_k\phi|^2 - \frac{e^2}{8}(|\phi|^2 - v^2)^2\right] \\
&= \frac{\delta_{ij}}{2}\left[B - \frac{e}{2}(v^2 - |\phi|^2)\right]\left[B + \frac{e}{2}(v^2 - |\phi|^2)\right] \\
&\quad + \frac{1}{8}\left\{\left[(\overline{D_i\phi \mp i\epsilon_{ik}D_k\phi})(D_j\phi \pm i\epsilon_{jl}D_l\phi) + (\overline{D_j\phi \mp i\epsilon_{jk}D_k\phi})(D_i\phi \pm i\epsilon_{il}D_l\phi)\right]\right. \\
&\qquad \left. + \left[(\overline{D_i\phi \pm i\epsilon_{ik}D_k\phi})(D_j\phi \mp i\epsilon_{jl}D_l\phi) + (\overline{D_j\phi \pm i\epsilon_{jk}D_k\phi})(D_i\phi \mp i\epsilon_{il}D_l\phi)\right]\right\}
\end{aligned}
$$

$\rightarrow$ vanishes everywhere : no pressure & stress

Q. more?

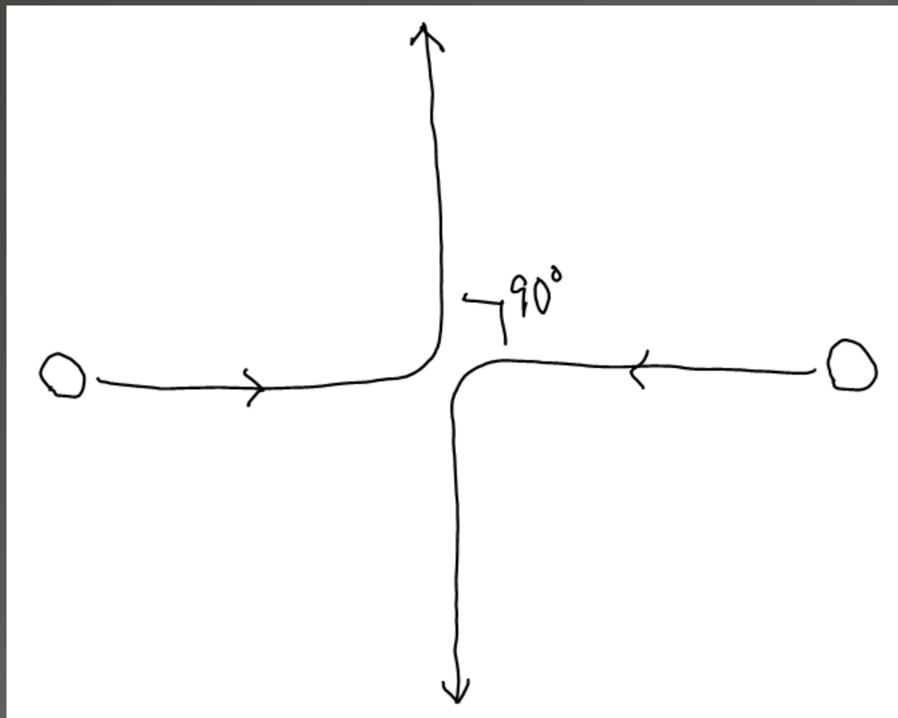
A. N=2 supersymmetry

(1+2)-dimensional U(1) gauge theory with N=2 supersymmetry

→ bosonic sector = Abelian Higgs model with $\lambda = e^2$ & specific Yukawa coupling

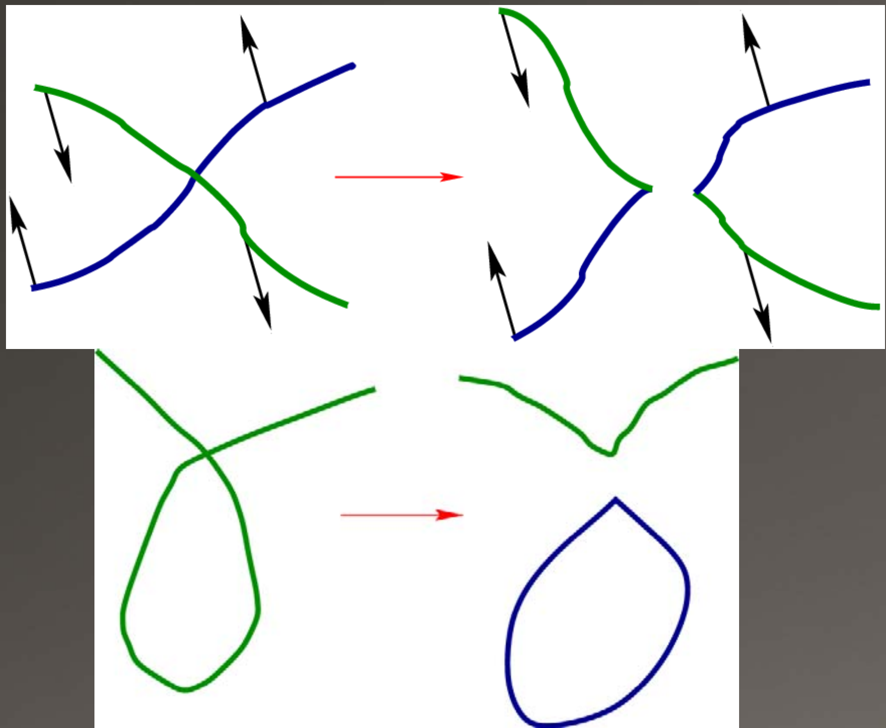
Q. Physical implication of $\lambda = e^2$

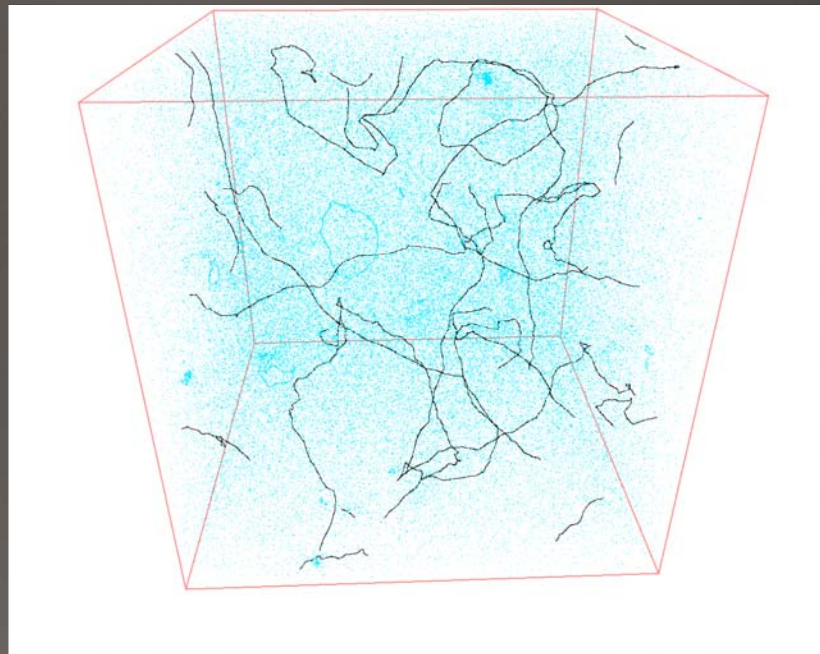
1. 90 degree scattering of slowly moving two BPS vortices in head-on collision



Q. Physical implication of $\lambda = e^2$

2. Interconnection by two encountered cosmic strings

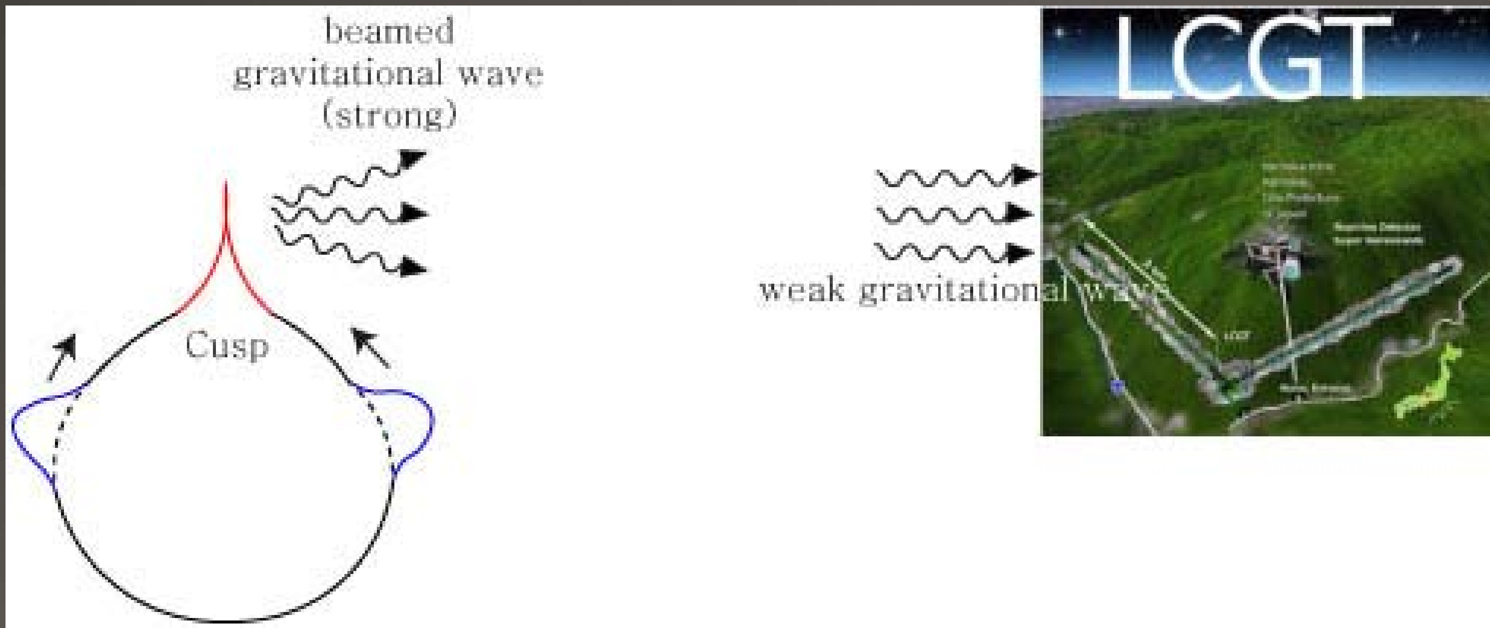




string network

Q. Cosmic string is dead by CMBR... Why again?

A. Gravitational wave from a cusp formed in cosmic string



Non BPS

n superimposed vortices

cylindrical coordinates (t, ρ, θ, z)

symmetric profiles of fields
$$\phi = e^{in\theta}|\phi|(\rho),$$
$$A^i = -\epsilon^{ij}\frac{x^j}{\rho^2}A(\rho)$$

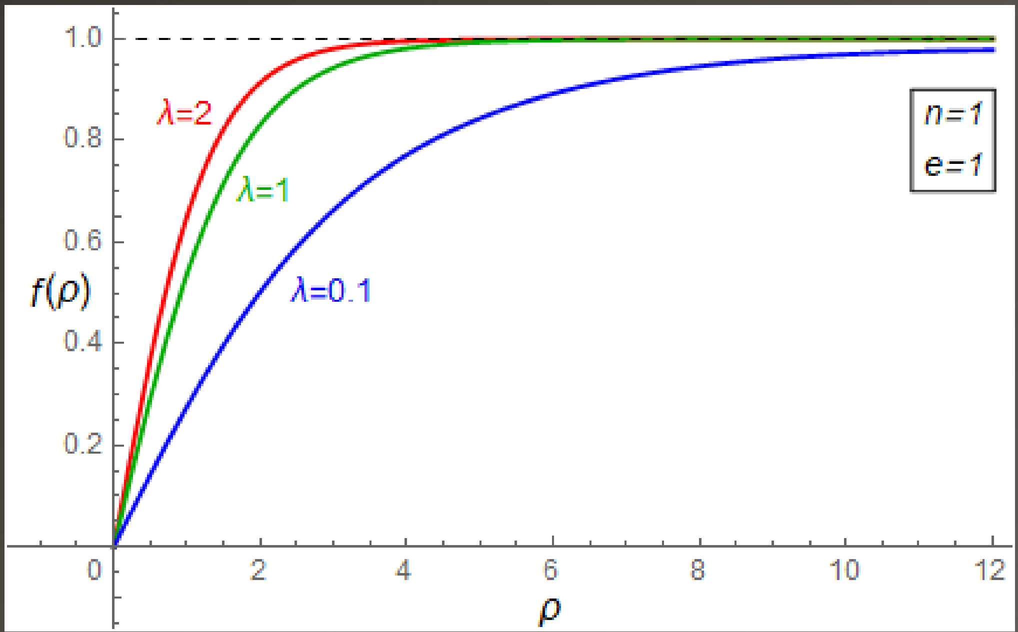
field equations
$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d|\phi|}{d\rho}\right) - \frac{e^2}{\rho^2}\left(\frac{n}{e} - A\right)^2|\phi| = \frac{\lambda}{2}(|\phi|^2 - v^2)|\phi|$$
$$\rho\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{dA}{d\rho}\right) + e^2|\phi|^2\left(\frac{n}{e} - A\right) = 0.$$

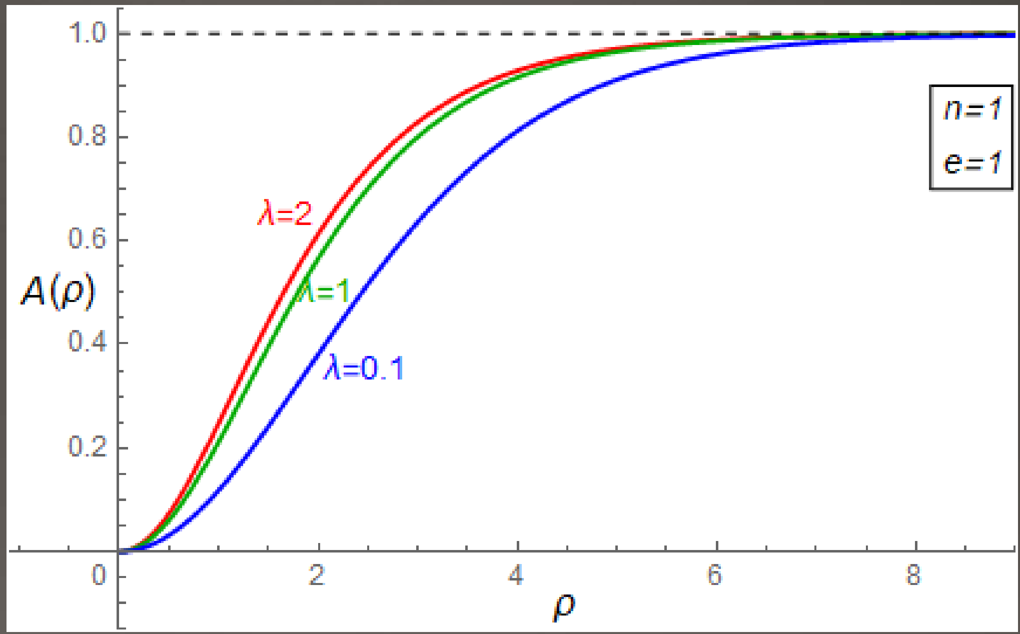
boundary conditions
$$|\phi|(0) = 0 \qquad |\phi|(\infty) = v$$
&
$$A(0) = 0 \qquad A(\infty) = \frac{n}{e}$$

near the origin
$$|\phi|(\rho) \approx v\phi_0\rho^n\left[1 - \frac{1}{4(n+1)}\left(2nea_0 + \frac{\lambda v^2}{2}\right)\rho^2 + \cdots\right]$$
$$A(\rho) \approx \frac{n}{e}\rho^2\left[a_0 - \frac{e^2v^2}{4n(n+1)}\phi_0^2\rho^{2n} + \cdots\right],$$

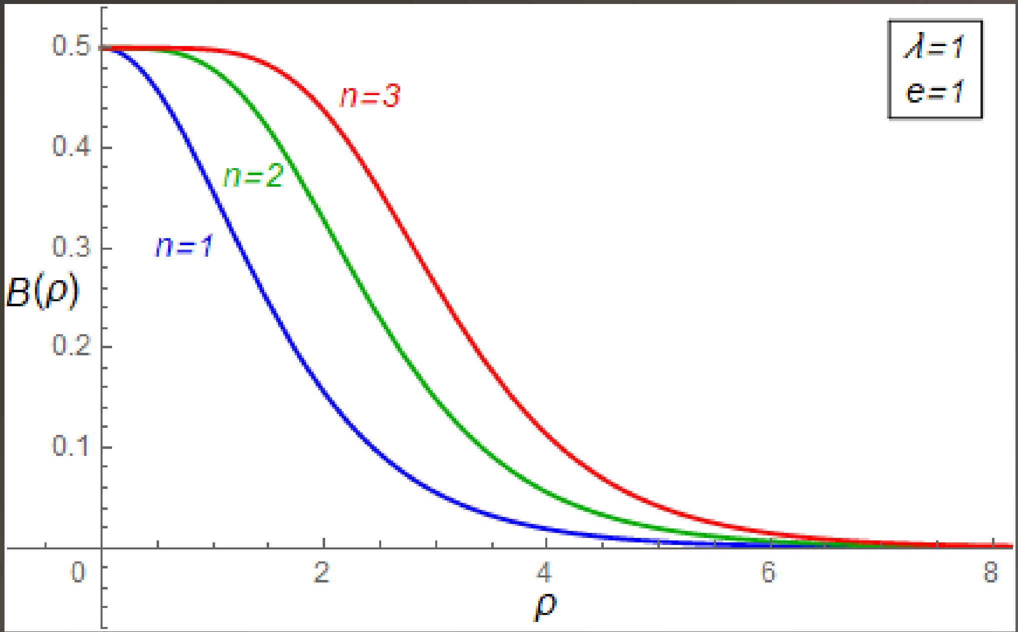
asymptotic region
$$|\phi|(\rho) \sim v\left[1 - \phi_\infty K_0(m_{\mathrm{H}}\rho)\right],$$
$$A(\rho) \sim \frac{n}{e}\left[1 - a_\infty m_{\mathrm{g}}\rho K_1(m_{\mathrm{g}}\rho)\right]$$

field profiles

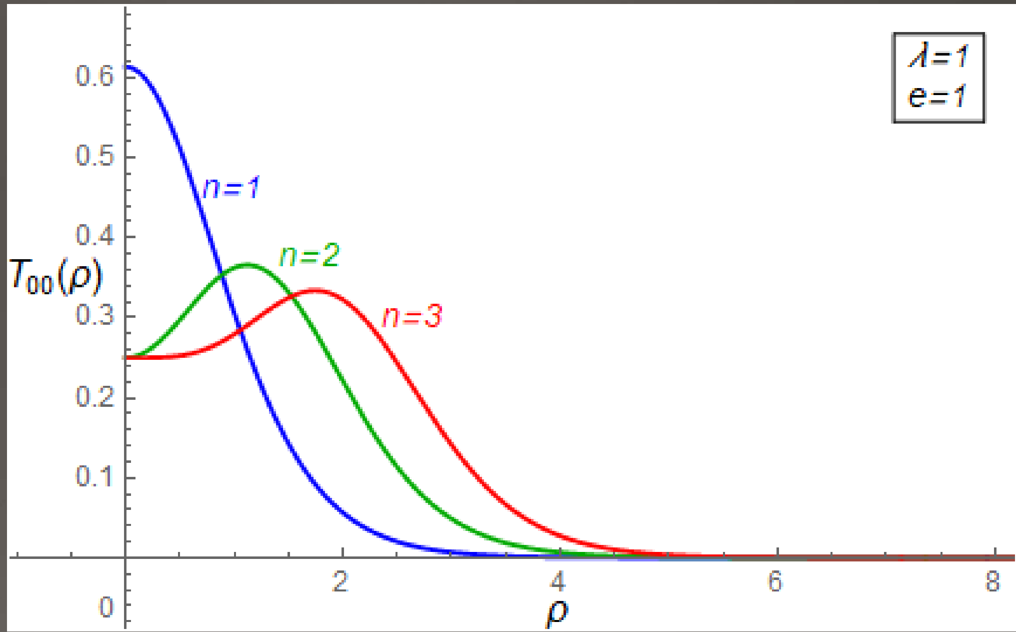




magnetic field

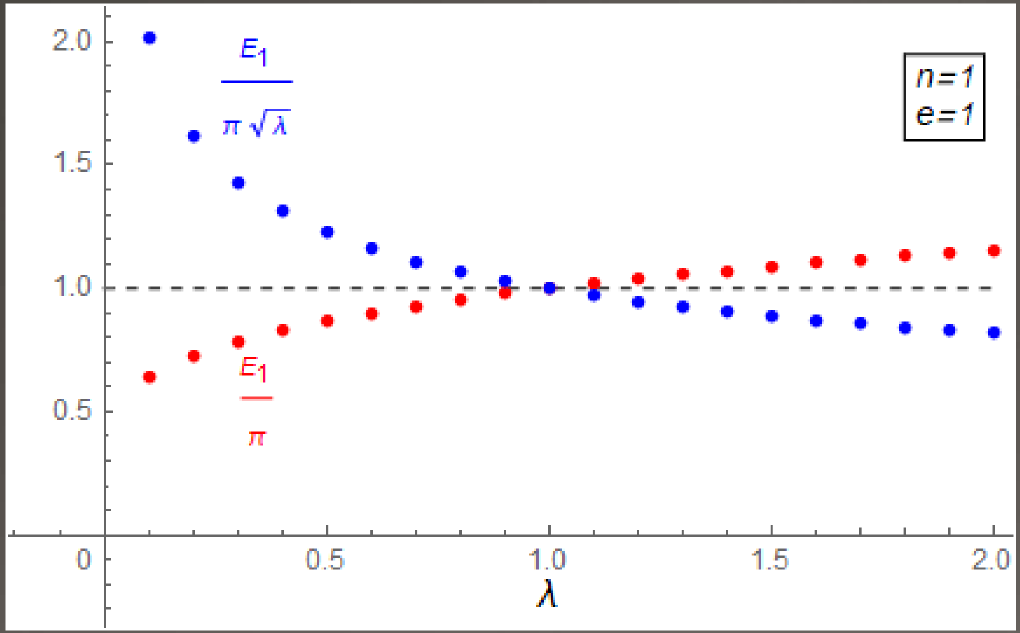


energy density



energy of single vortex

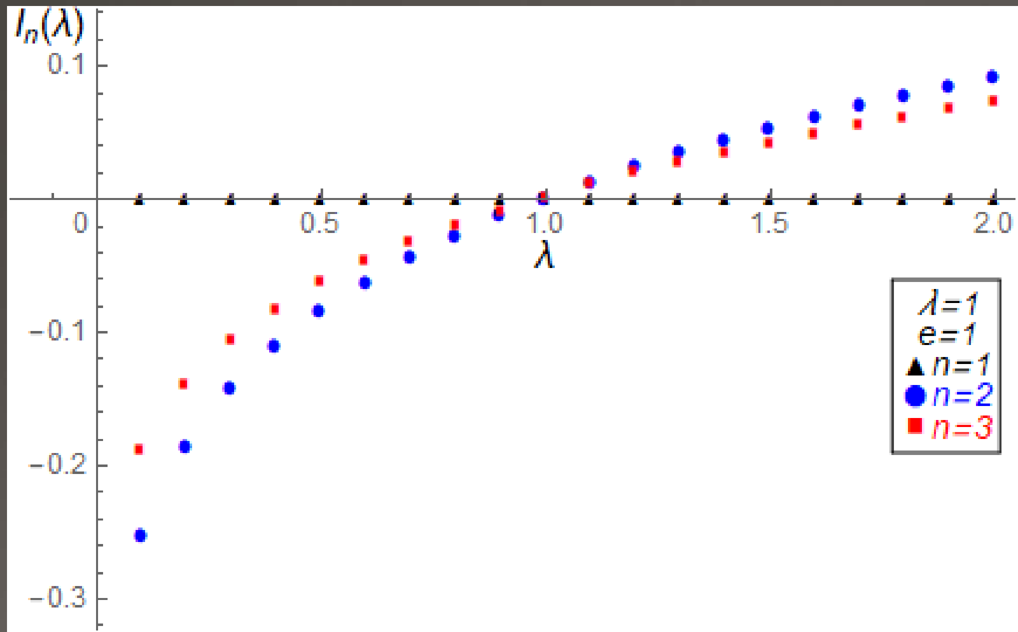
mass of n=1 vortex



Interaction among n superimposed vortices

interaction index

$$I_n(\lambda) \equiv \frac{E_n(\lambda) - nE_1(\lambda)}{{}_nC_2\, E_1(\lambda)}$$



interaction terms among n superimposed vortices

Introduce rescaled dimensionless fields

$$|\phi| = v(1-\tilde{\psi})$$

$$A_i = \frac{n}{e}(\tilde{A}_i - 1)$$

$$\begin{aligned}
\frac{E_n^{\mathrm{int}}}{\int dz} &= \frac{E_n}{\int dz} - n\frac{E_1}{\int dz} \\
&= n m_{\mathrm{BPS}} \int_0^\infty d\rho\, \rho \left[\frac{1}{\rho^2}(n\tilde{\psi}_n \tilde{A}_n^2 - \tilde{\psi}_1 \tilde{A}_1^2) + \frac{1}{2}\lambda v^2 \left(\frac{1}{n}\tilde{\psi}_n^3 - \tilde{\psi}_1^3 \right) \right. \\
&\qquad \left. - \frac{1}{\rho^2}(n\tilde{\psi}_n^2 \tilde{A}_n^2 - \tilde{\psi}_1^2 \tilde{A}_1^2) - \frac{1}{4}\lambda v^2 \left(\frac{1}{n}\tilde{\psi}_n^4 - \tilde{\psi}_1^4 \right) \right]
\end{aligned}$$

: Interaction terms are cubic or quartic interaction energy density

$$\frac{E_n^{\rm int}}{\int dz} \equiv \int dxdy\, \mathcal{E}_n^{\rm int} = 2\pi \int_0^\infty d\rho\, \rho\, \mathcal{E}_n^{\rm int}$$

interaction energy density at large distance

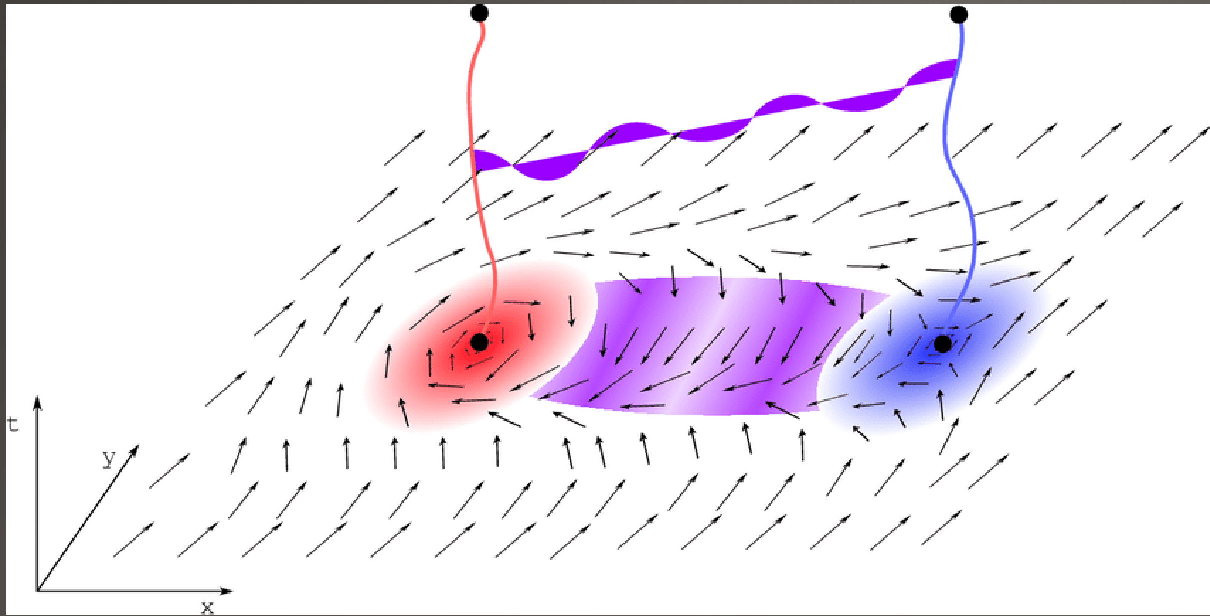
$$\mathcal{E}_n^{\rm int} \simeq \frac{n m_{\rm BPS}}{2\pi} K_0(m_{\rm H}\rho) \Big\{ m_g^2 \left[n\phi_\infty^n (a_\infty^n)^2 - \phi_\infty^1 (a_\infty^1)^2 \right] \left[K_1(m_{\rm g}\rho) \right]^2 + \frac{1}{2}\lambda v^2 \left[\frac{1}{n}(\phi_\infty^n)^3 - (\phi_\infty^1)^3 \right] \left[K_0(m_{\rm H}\rho) \right]^2 \Big\} + \mathcal{O}(K^4)$$

$\rightarrow$

$$\lambda > e^2 \;\Rightarrow\; m_{\rm H} > m_{\rm g} \;\Rightarrow e^{-m_{\rm g}\rho} > e^{-m_{\rm H}\rho} \;\Rightarrow K_1(m_{\rm g}\rho) > K_0(m_{\rm H}\rho)$$ repulsive : type II

$$\lambda < e^2 \;\Rightarrow\; m_{\rm H} < m_{\rm g} \;\Rightarrow e^{-m_{\rm g}\rho} < e^{-m_{\rm H}\rho} \;\Rightarrow K_1(m_{\rm g}\rho) < K_0(m_{\rm H}\rho)$$ attractive : type I

Two stacked vortices of vorticity n_1 and n_2 with large separation $s \equiv |\boldsymbol{x}_1 - \boldsymbol{x}_2| \gg \frac{1}{m_{\mathrm{v}}}$



Q. Interaction between two stacked vortices with large separation?

Abrikosov ansatz

$$|\phi| \simeq |\phi_1||\phi_2| \quad \longrightarrow \quad \tilde{\psi} \simeq \tilde{\psi}_1 + \tilde{\psi}_2$$

$$\Omega \simeq \Omega_1 + \Omega_2 = n_1 \tan^{-1}\frac{y-y_1}{x-x_1} + n_2 \tan^{-1}\frac{y-y_2}{x-x_2}$$

$$A_i \simeq A_i^1 + A_i^2$$

Interaction between two vortex stacks with large separation $s \equiv |\boldsymbol{x}_1 - \boldsymbol{x}_2| \gg \frac{1}{m_{\mathrm{v}}}$

$$\begin{aligned}\frac{E_{12}}{\int dz} &= \frac{E}{\int dz} - \frac{E_1}{\int dz} - \frac{E_2}{\int dz} \\ &\simeq \int dxdy\left[\frac{1}{2}F_{ij}^1 F_{ij}^2 + ev^2\tilde{A}_i^1\tilde{A}_i^2 + v^2\partial_i^2(\tilde{\psi}_1\tilde{\psi}_2) - v^2\partial_i\tilde{\psi}_1\partial_i\tilde{\psi}_2 - \lambda v^4\tilde{\psi}_1\tilde{\psi}_2 + \mathcal{O}(3,4)\right]\end{aligned}$$

:Leading interaction terms are quadratic interaction energy density between two vortex stacks

$$\frac{E_{12}}{\int dz} = \int dxdy\ \mathcal{E}_{12}$$

$$\rightarrow \quad \mathcal{E}_{12} \simeq \partial_i[\text{total derivative terms}] + 2\pi\delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_2)v^2\left[n_1 n_2 a_\infty^{n_1} a_\infty^{n_2} K_0(m_\mathrm{g}\rho_1) - v^2 \phi_\infty^{n_1}\phi_\infty^{n_2} K_0(m_\mathrm{H}\rho_1)\right]$$

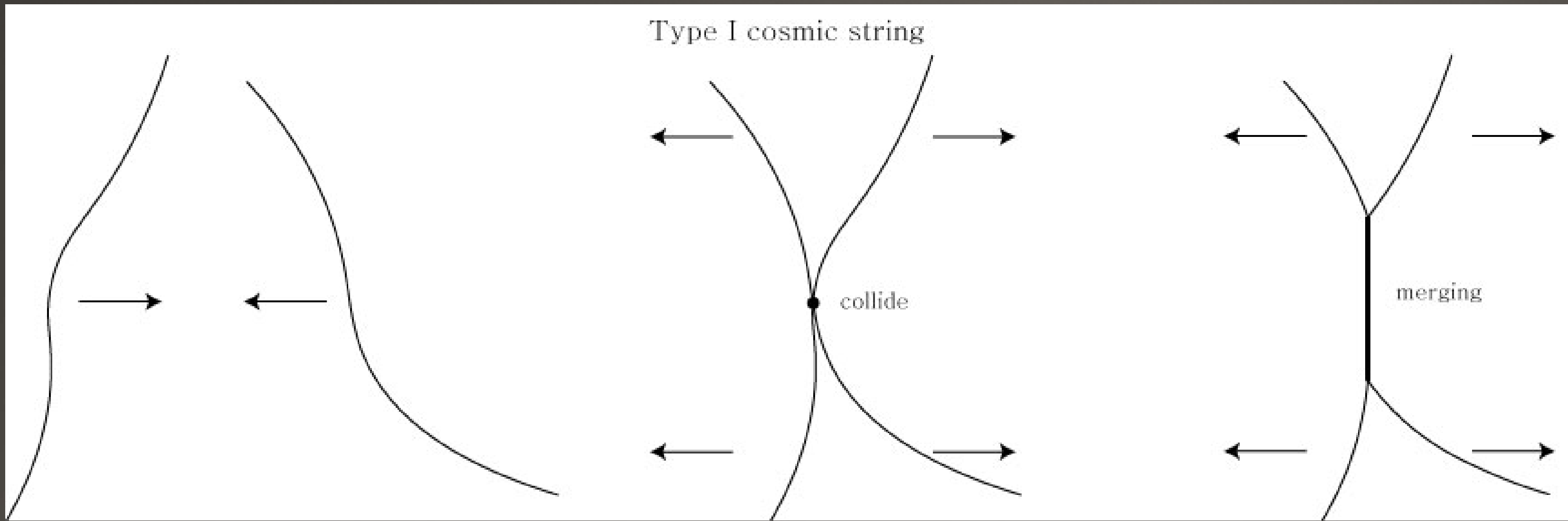
interaction energy as the sum of repulsive and attractive forces

$$\frac{E_{12}}{\int dz} \simeq 2m_\mathrm{BPS}\left[n_1 n_2 a_\infty^{n_1} a_\infty^{n_2} K_0(m_\mathrm{g}s) - \phi_\infty^{n_1}\phi_\infty^{n_2} K_0(m_\mathrm{H}s)\right]$$

$\downarrow$ ←numerical work for arbitrary separation

Vortex-vortex interaction potential V (s)

Implication to cosmic string



Type I cosmic string
→ merging
→ low number density but energetic gravitational waves

To be continued ...