HOPF ALGEBRA, NICHOLS' ALGEBRA AND STRING BCJ RELATION

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BCJ Relation

Sum of Momentum Kernel $\times$ Amplitudes $= 0$

$$\sum_{\sigma \in s_{n-2}} \mathcal{S}\left(\tau^T, \sigma\right) A\left(1, \sigma, n\right) = 0$$

4pt Example

$$\sin\left(i\pi\alpha' s_{12}\right)\sin\left(i\pi\alpha'\left(s_{13}+s_{23}\right)\right)A\left(1,2,3,4\right)+\sin\left(i\pi\alpha' s_{12}\right)\sin\left(i\pi\alpha' s_{13}\right)A\left(1,3,2,4\right)=0$$

Bern,Carrasco,Johansson,08

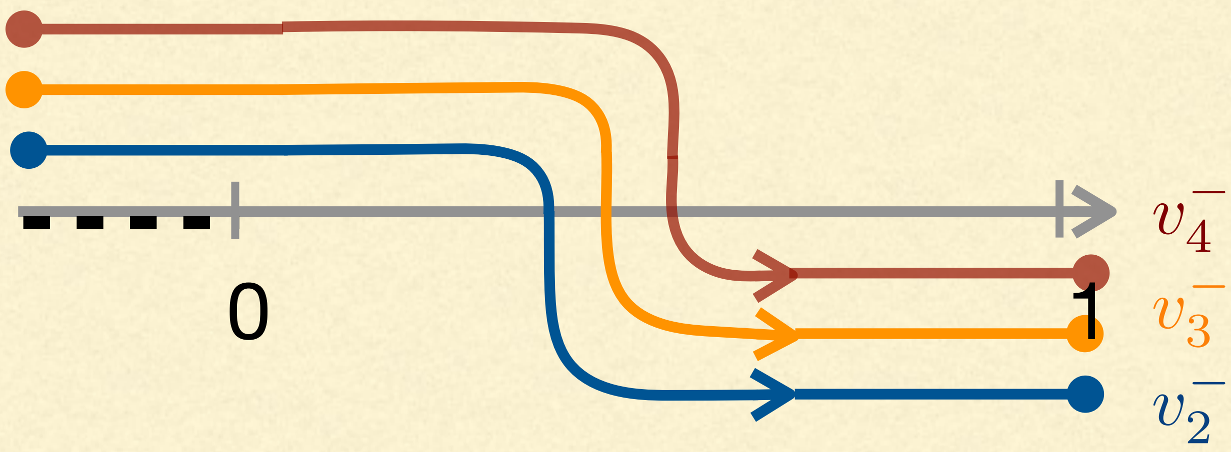
String Monodromy Relation

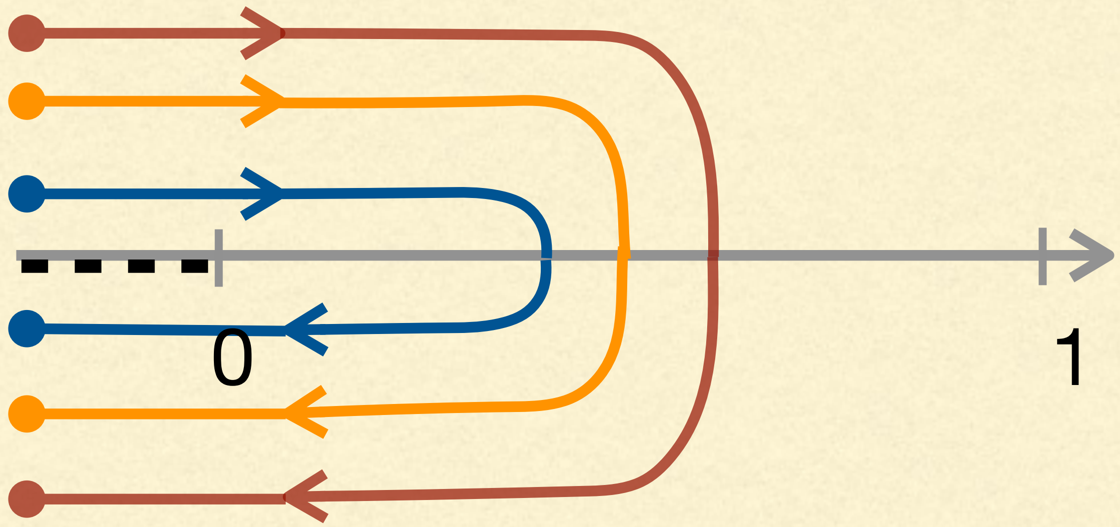
Sum of Braiding Phase $\times$ Amplitudes $= 0$

$$A(j, i_1, \ldots, i_{n-2}, n) + e^{i\pi\alpha' s_{ji_i}} A(i_1, j, \ldots, i_{n-2}, n) + \ldots + e^{i\pi\alpha' s_{ji_i}} A(i_1, j, \ldots, i_{n-2}, n) = 0$$

4pt Example

$$A(2,1,3,4) + e^{i\pi\alpha' s_{12}} A(1,2,3,4) + e^{i\pi\alpha'(s_{12}+s_{23})} A(1,3,2,4) = 0$$





$$\sum_{\sigma\in s_{n-2}} \mathcal{S}\left(\tau^T,\sigma\right) A\left(1,\sigma,n\right) \;=\; \left\langle f\middle|[[[T_1,T_2]_{\alpha'},T_3]_{\alpha'}\ldots,T_{n-1}]_{\alpha'}\middle|vac\right\rangle.$$

$$T_i = \int_0^1 \frac{V\left(y_i\right)}{y_i}$$

$$[T_1,T_2]_{\alpha'} = T_1T_2 - e^{-i\pi\alpha' k_1\cdot k_2}T_2T_1$$

Quantum Algebra

$$[H_i,\, H_j] = 0$$

$$[H_i,\, X_j^{\pm}] = \pm(\alpha_i,\, \alpha_j)\, X_j^{\pm}$$

$$[X_i^{+},\, X_j^{-}] = \delta_{i\,j}\, \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}$$

$\longleftrightarrow$

$$F_i = X_i^{-} q^{-H_i/2}$$

$$E_i = X_i^{+} q^{H_i/2}$$

$$K_i = q^{H_i}$$

$$\sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix}_q q_i^{-k(m-k)/2} \left(X_i^{\pm}\right)^k X_j^{\pm} \left(X_i^{\pm}\right)^{m-k} = 0$$

q-Serre relation

$m = 1 - A_{i\,j}$

Or equivalently

$$\Big[[X_j, X_i]_q \cdots_{1-A_{ij}^{-1} times} \cdots X_i\Big]_q \qquad [X_i, X_j]_q = X_i \otimes X_j - q^{A_{ij}} X_j \otimes X_i$$

Universal Enveloping Algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/I$$

$$T(\mathfrak{g}) = K \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \ldots$$

I : Idea generated by elements of the form:

$$a \otimes b - b \otimes a - [a, b]$$

or

$a \otimes b - q^{(a,b)} b \otimes a$ and (quantum Serre relation)

Hopf Algebra

= Bialgebra + Antipode

$(\mathcal{H}, \cdot, I, \Delta, \epsilon)$ $\quad S$

$c_1 g_1 + c_2 g_2$

$g_1 \cdot g_2$

algebra

comultiplication $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

bialgebra

$\Delta(g_1 \cdot g_2) = \Delta(g_1) \cdot \Delta(g_2)$

Hopf algebra

antipode $S : \mathcal{H} \to \mathcal{H}$

$m(S \otimes id)\Delta(g) = \epsilon(g)\, I$

$S(g) \cdot g = I$

$S(g) = g^{-1}$

(generalized inverse)

Quantum Algebra as Hopf Algebra

$$\Delta(H_i) = H_i \otimes I + I \otimes H_i$$

$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{-H_i/2} + q^{H_i/2} \otimes X_i^{\pm}$$

$$S(H_i) = -H_i$$

$$S(X_i^{\pm}) = q^{-\sum_i H_i} X_i^{\pm} q^{\sum_i H_i}$$

Nichols' Algebra

V : a braided vector space

$\mathfrak{M}_n$: the quantum shuffle map

$$\mathfrak{M}_n := \sum_{\sigma \in \mathbb{S}_n} s(\sigma) : \quad V^{\otimes n} \to V^{\otimes n}$$

$\mathfrak{B}(V)$: Nichol's algebra

$$\mathfrak{B}(V) := \oplus_{n \geq 0} V^{\otimes n} / Ker(\mathfrak{M}_n)$$

Example: positive part of quantum algebra with Cartan matrix A_{ij}

V : vector space with braiding

$$s\left(\sigma_{i,j}\right)|i\rangle \otimes |j\rangle = q^{A_{ij}}\,|j\rangle \otimes |i\rangle$$

Kernel of the quantum shuffle map

$$\Big[[v_j, v_i]_q \cdots_{1-A_{ij}^{-1} times} \cdots v_i\Big]_q \qquad [v_i, v_j]_q = v_i \otimes v_j - q^{A_{ij}} v_j \otimes v_i$$

LHS of quantum Serre relation

$\mathfrak{B}\left(V\right)$: Nichol's algebra

Positive part of the quantum algebra

For String Amplitudes

BCJ relation

$$\left\langle f \left| [[[T_1, T_2]_{\alpha'}, T_3]_{\alpha'} \ldots, T_{n-1}]_{\alpha'} \right| vac \right\rangle . = 0$$

A generalized quantum Serre relation

Phys. **arXiv:[hep-th]**

numerator

String Numerator

↑

Screening of OPE
&Annihilation operator

↑

Algebra of Screening
Vertex operator

↑

String BCJ Relation

↑

monodromy sum

Maths. **arXiv:[math.QA]**

Product of structure constant

Product of successive q-structure
Constant in quantum algebra

↑

Full quantum algebra

↑ Drinfeld double

Positive part of quantum algebra

↑

Quantum Serre relation

↑

quantum symmetriser

Thank you~