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Momentum-space entanglement in scalar field theory on fuzzy spheres

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1. Introduction

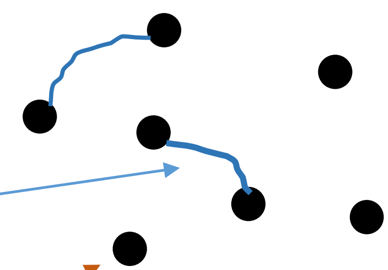
Quantum Gravity $\leftrightarrow$ Random Geometry/Matrix Models

String, M-theory, Membranes

Nonperturbative definition: Yang-Mills type (SUSY) matrix models

$$\mathcal{L} \sim \frac{1}{2} D\Phi_i D\Phi_i + \frac{1}{4}\left[\Phi_i, \Phi_j\right]^2 + (\text{fermions})$$

$$\Phi_i \sim \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix}$$



Matrix: Physical degrees of freedom (position and interaction)

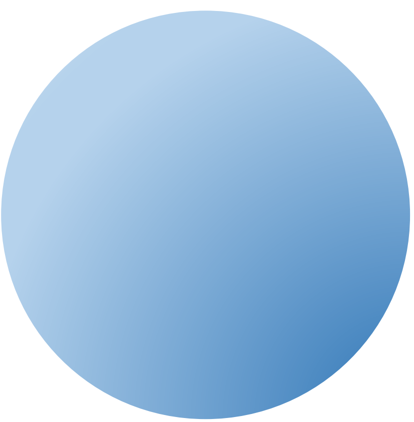
⇨ Not a smooth geometry (Quantum Geometry)

Today, I will consider one of the simplest examples of “quantum geometry”

A noncommutative geometry

$$[x_i, x_j] \neq 0$$

A fuzzy sphere: a finite dimensional approximation of a smooth sphere



$$S^2: \sum_{i=1}^{3} x_i^2 = R^2$$

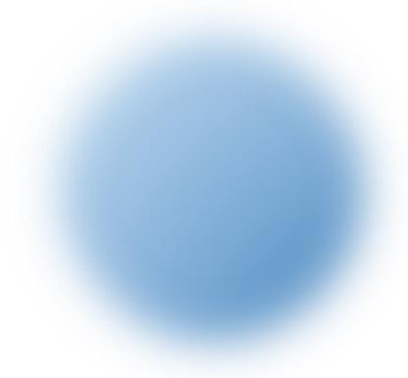
$$x_i \in \boldsymbol{R}$$

$$\sum_{i=1}^{3} \hat{x}_i^2 = R^2$$

$$\hat{x}_i = \alpha L_i$$

L_i: a **matrix**

Fuzzy S^2



2. Scalar Field Theory on Fuzzy Sphere

$$L = \int \frac{R^2 d\Omega}{4\pi} \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2R^2}(\mathcal{L}_i \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4}\phi^4 \right]$$

$$\phi(t,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{lm}(t) Y_{lm}(\theta,\varphi)$$

$$L = \frac{R^2}{N} \operatorname{tr}_N \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2\rho_N^2}[L_i, \phi]^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \right]$$

$$\phi = \sum_{l=0}^{2L} \sum_{m=-l}^{l} \phi_{lm}(t) T_{lm}$$

$N \times N$ matrix

$N = 2L + 1$

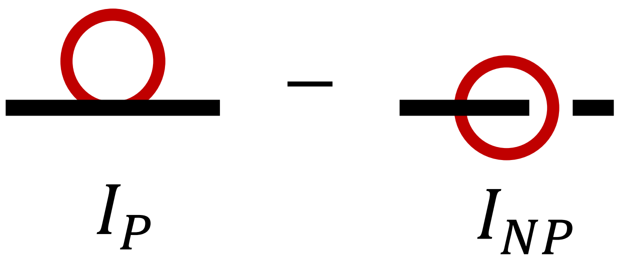
$\hat{x}_i = \alpha L_i$ $\qquad R^2 = \frac{\alpha^2 (N^2 - 1)}{4}$ $\qquad$ Matrix regularization of Spherical harmonics

L_i: N dimensional repr. Of SU(2)

Noncommutative Anomaly (NCA)

[Chu-Madore-Steinacker JHEP2001(08)]

Loop correction to a two-point function



$$\simeq \int_{-1}^{1} dt\, \frac{1 - P_L(t)}{1 - t} + \mathcal{O}(L^{-1})$$

There appears some “phase oscillation” but it is not singular.

⇒ Noncommutative Plane limit: $[x, y] = i\theta$

UV/IR mixing: UV divergence $\Lambda \to \infty \leftrightarrow$ IR divergence $p \to 0$

$$I_{NP} \simeq 2\int_0^{\Lambda} dk\, \frac{kJ_0(\theta p k)}{k^2 + m^2} \leftrightarrow \frac{1}{2\pi}\int_0^{\Lambda} dk\, \frac{e^{i\theta p \times k}}{k^2 + m^2}$$

$$\Lambda_{eff}^2 = \frac{1}{\Lambda^{-2} + p\theta^2 p}$$

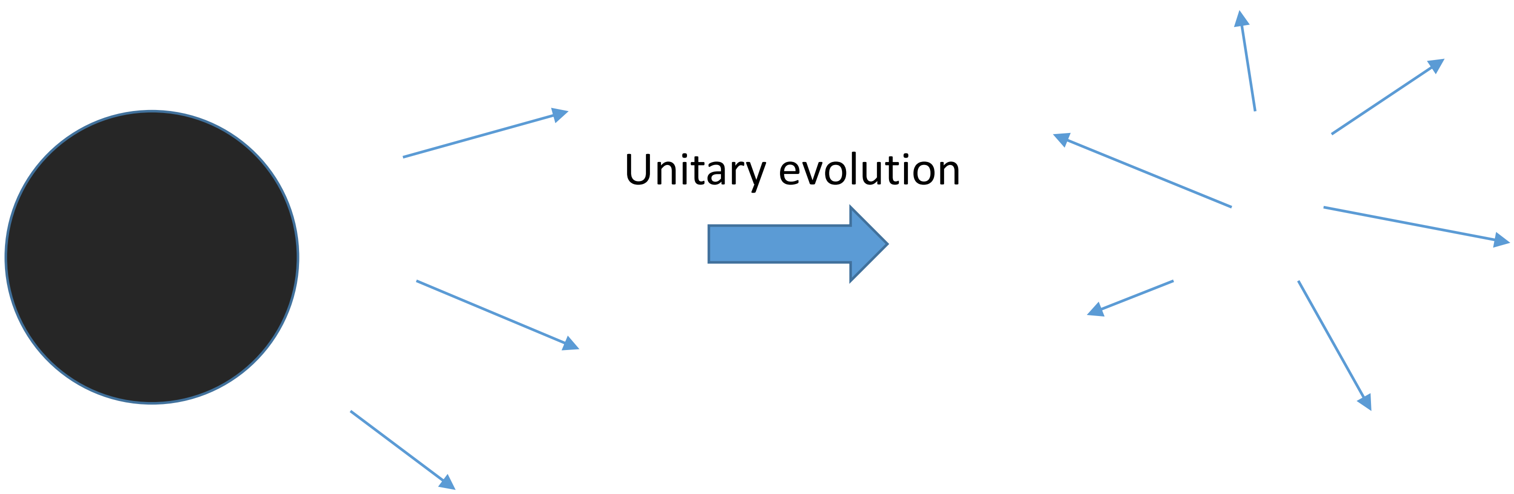
(Open-Closed String Interpretation: Seiberg-Minwalla-Van Raamsdonk JHEP2000]

UV/IR mixing: Some characteristic feature of Quantum Gravity/String Theory

Strong relation between
short distance physics (UV) and long distance physics (IR)

→ Nonlocal nature of quantum gravity/quantum geometry

Non-locality: May be a key feature to understand Black hole Information Problem.

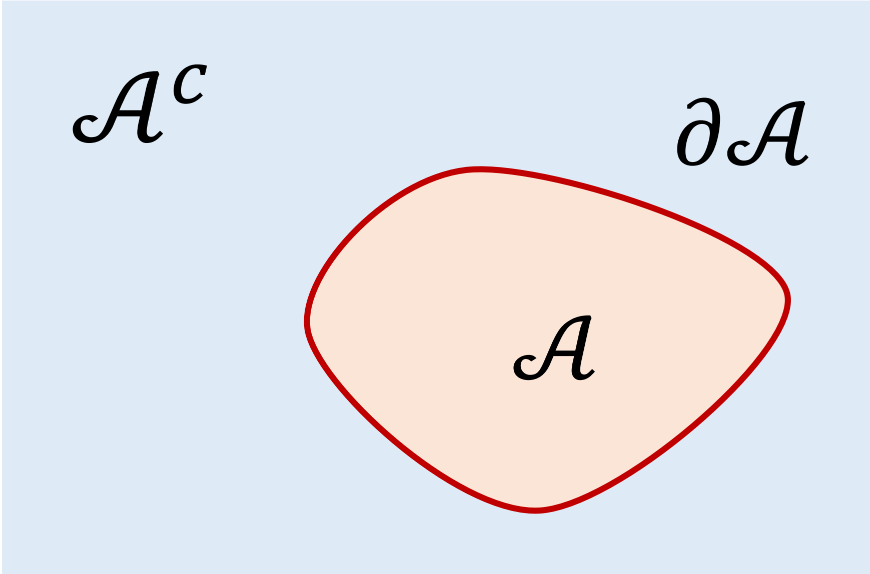


Gravitational field has nonlocal (quantum) interaction

[e.g. Osuga-Page PRD97]

Entanglement Entropy (EE) and QFT

Example: *Spatial* entanglement between $\mathcal{A}$ and $\mathcal{A}^c$.
How strongly different degrees of freedom are *correlated*.



Entanglement entropy (von Neumann entropy) is a good measure:

$$S_{EE} = -\mathrm{tr}_a \rho_a \ln \rho_a$$

ρ_a: Tracing out $\partial\mathcal{A}$ part degrees of freedom (reduced density matrix)

Entanglement Entropy in Momentum Space

- Free QFT: Hilbert space is *diagonal*

  $\rightarrow S_{EE} = 0$
- Interacting QFT:

  nontrivial S_{EE}

  similar structure to Wilsonian RG

[Balasubramanian-McDermott-Van Raamsdonk (2012)]

- QFT in a *noncommutative* space

  Nonlocal, Relation between UV and IR d.o.f.

3. Entanglement Entropy in Momentum Space

$$\rho = |\psi_0\rangle\langle\psi_0| \qquad |\psi_0\rangle : \text{ Ground state}$$

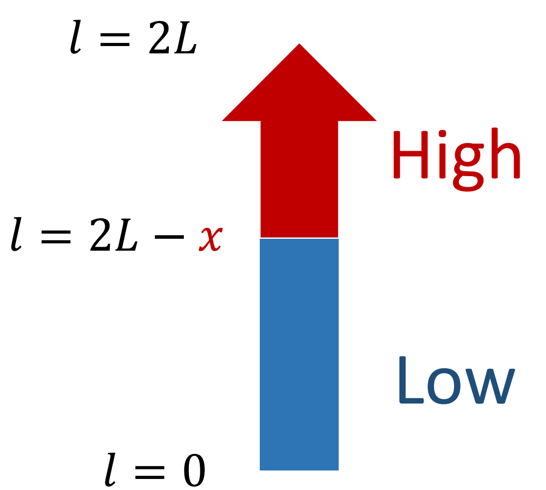
The matrix element of the density operator at temperature β

$$\langle\phi|\rho|\phi'\rangle = \frac{1}{Z}\lim_{\beta\to\infty}\langle\phi|e^{-\beta H}|\phi'\rangle = \frac{1}{Z}\lim_{\beta\to\infty}\int_{\phi(0)=\phi'}^{\phi(\beta)=\phi}\mathcal{D}\phi\; e^{-S}$$

Separate the degrees of freedom into two subsets.

Higher and lower modes

$$\phi_{lm}(t) \Rightarrow \begin{cases} \phi^{L}_{lm}(t) & (0 \le l \le 2L - x) \\ \phi^{H}_{lm}(t) & (2L - x + 1 \le l \le 2L) \end{cases}$$



Reduced density matrix

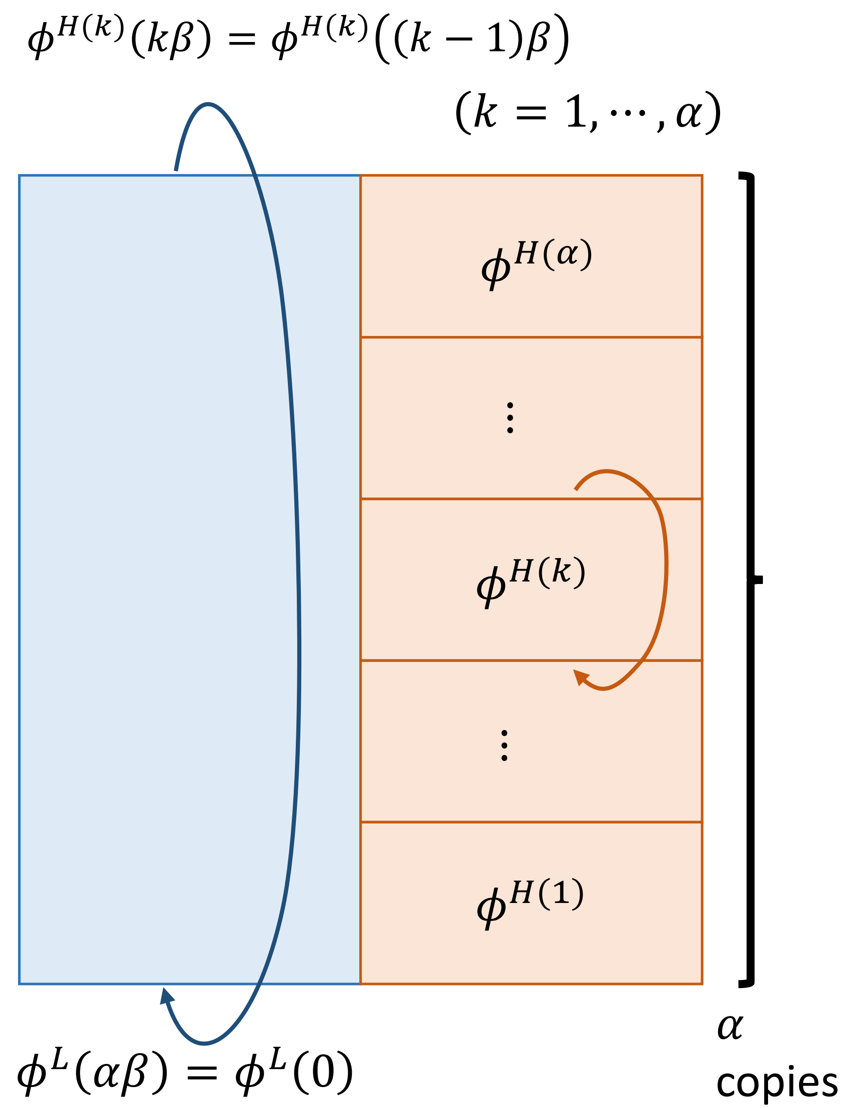
$$\rho_L = \langle \phi^L | \rho | \phi^{L\prime} \rangle = \frac{1}{Z} \int_{\phi^L(0)=\phi^{L\prime}, \phi^H(0)=\phi^H}^{\phi^L(\beta)=\phi^L, \phi^H(\beta)=\phi^H} \mathcal{D}\phi^H \, e^{-S}$$

Entanglement Entropy

$$S_{EE}(\rho_L) = -\mathrm{Tr}_L[\rho_L \log \rho_L] = -\lim_{\alpha\to 1}\left[\frac{\partial}{\partial\alpha}\mathrm{Tr}(\rho_L^\alpha)\right]$$

Replica trick

$$\mathrm{Tr}_L(\rho_L^\alpha) = \frac{1}{(Z_1)^\alpha}\int \mathcal{D}\phi^L \mathcal{D}\phi^H \, e^{-S}$$

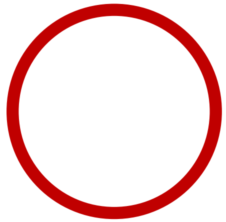


4. Perturbative Calculation

$$F_x^\alpha\Big|_{\lambda^0} = -\log\frac{Z_\alpha^{(0)}}{\left(Z_1^{(0)}\right)^\alpha}$$

$$\Omega_l^2 = \frac{l(l+1)}{R^2} + \mu^2$$

Tree level

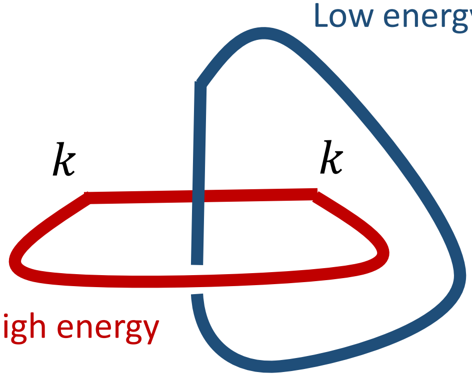


$$S_{EE}^{(0)} \sim e^{-\frac{\beta\Omega_{l_1}}{2}} \xrightarrow[\beta\to\infty]{} 0$$

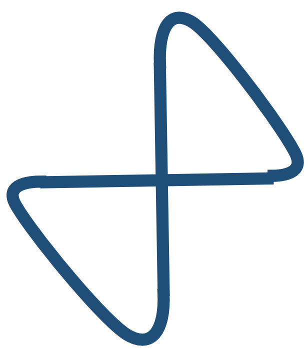
Zero temperature limit,

The Hamiltonian is diagonal !!

One Loop



Non-planar



Planar

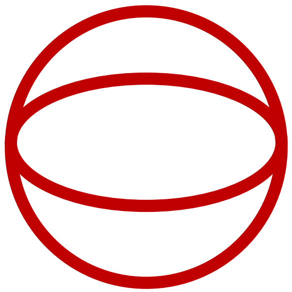
$$S_{EE}^{(1)} = \frac{\lambda\alpha\beta}{8R^2}\frac{df_1}{d\alpha}\Big|_{\alpha=1}$$

$$\xrightarrow[\beta\to\infty]{} \quad e^{-\frac{\beta\Omega_{l_1}}{2}} \to 0$$

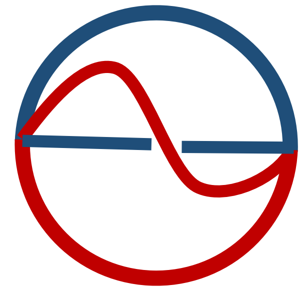
No contribution at this order.

Two Loop

$$\frac{(-1)^2}{2!}\langle S_{int}^2\rangle = \frac{\lambda^2 R^4}{32}\int_0^{\alpha\beta} dt dt' \left\langle \frac{1}{N}\mathrm{tr}(\phi(t)^4)\frac{1}{N}\mathrm{tr}(\phi(t')^4)\right\rangle$$



$\langle (HHHH)(HHHH)\rangle$ planar



$\langle (LHLH)(LLHH)\rangle$ Non-planar

The result is very complicated. We quote the final expression.

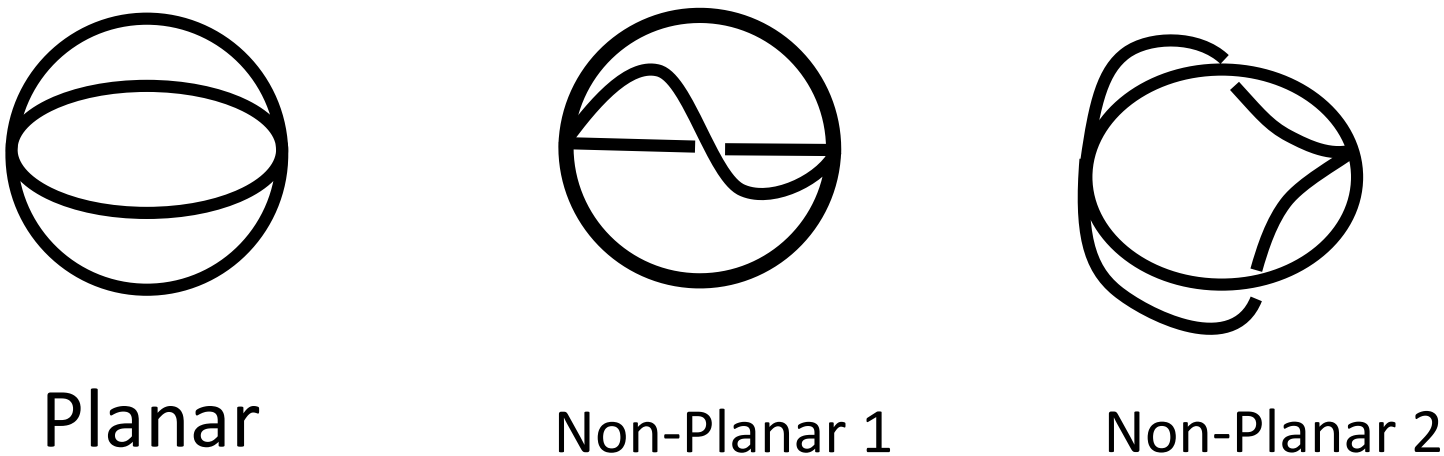
$$\begin{aligned} S_{EE} = -\frac{\lambda^2\beta}{2^5 R^4}\Bigg[& 2\sum_{l_1\in\Lambda_H}\sum_{l_2,l_3,l_4\in\Lambda_L}\frac{\Xi_{l_1l_2l_3l_4}}{\Omega_{l_1}\Omega_{l_2}\Omega_{l_3}\Omega_{l_4}}\frac{\Omega_{l_2}+\Omega_{l_3}+\Omega_{l_4}}{(\Omega_{l_1}+\Omega_{l_2}+\Omega_{l_3}+\Omega_{l_4})^2} \quad LLLH \\ & +2\sum_{l_1\in\Lambda_L}\sum_{l_2,l_3,l_4\in\Lambda_H}\frac{\Xi_{l_1l_2l_3l_4}}{\Omega_{l_1}\Omega_{l_2}\Omega_{l_3}\Omega_{l_4}}\frac{\Omega_{l_1}}{(\Omega_{l_1}+\Omega_{l_2}+\Omega_{l_3}+\Omega_{l_4})^2} \quad LHHH \\ & +\sum_{l_1,l_2\in\Lambda_L}\sum_{l_3,l_4\in\Lambda_H}\frac{\Xi_{l_1l_2l_3l_4}}{\Omega_{l_1}\Omega_{l_2}\Omega_{l_3}\Omega_{l_4}}\frac{\Omega_{l_1}+\Omega_{l_2}}{(\Omega_{l_1}+\Omega_{l_2}+\Omega_{l_3}+\Omega_{l_4})^2}\Bigg] + \mathcal{O}(\lambda^3) \end{aligned}$$

$LLHH$

Contains both lower and higher modes propagators. (No all H or all L)

Matrix part (Schematically)

$$\Xi_{l_1 l_2 l_3 l_4} \equiv \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \sum_{m_3=-l_3}^{l_3} \sum_{m_4=-l_4}^{l_4} (-1)^{m_1+m_2+m_3+m_4}$$
$$\times \frac{1}{N} \mathrm{tr}\left(T_{l_1 m_1} T_{l_2 m_2} T_{l_3 m_3} T_{l_4 m_4}\right)$$
$$\times \frac{1}{N} \mathrm{tr}\left[\underline{T_{l_1\,-m_1} T_{l_4\,-m_4} T_{l_3\,-m_3} T_{l_2\,-m_2}} + 4\underline{T_{l_1\,-m_1} T_{l_2\,-m_2} T_{l_4\,-m_4} T_{l_3\,-m_3}} + \underline{T_{l_1\,-m_1} T_{l_2\,-m_2} T_{l_3\,-m_3} T_{l_4\,-m_4}}\right]$$



5. Entanglement Entropy and Noncommutativity

Effect of NC geometry → Difference of Planar and Non-Planar

Commutation relation
$$\left[T_{l_1m_1}, T_{l_2m_2}\right] = \sum_{l_3m_3} G^{l_3m_3}_{l_1m_1;l_2m_2} T_{l_3m_3}$$

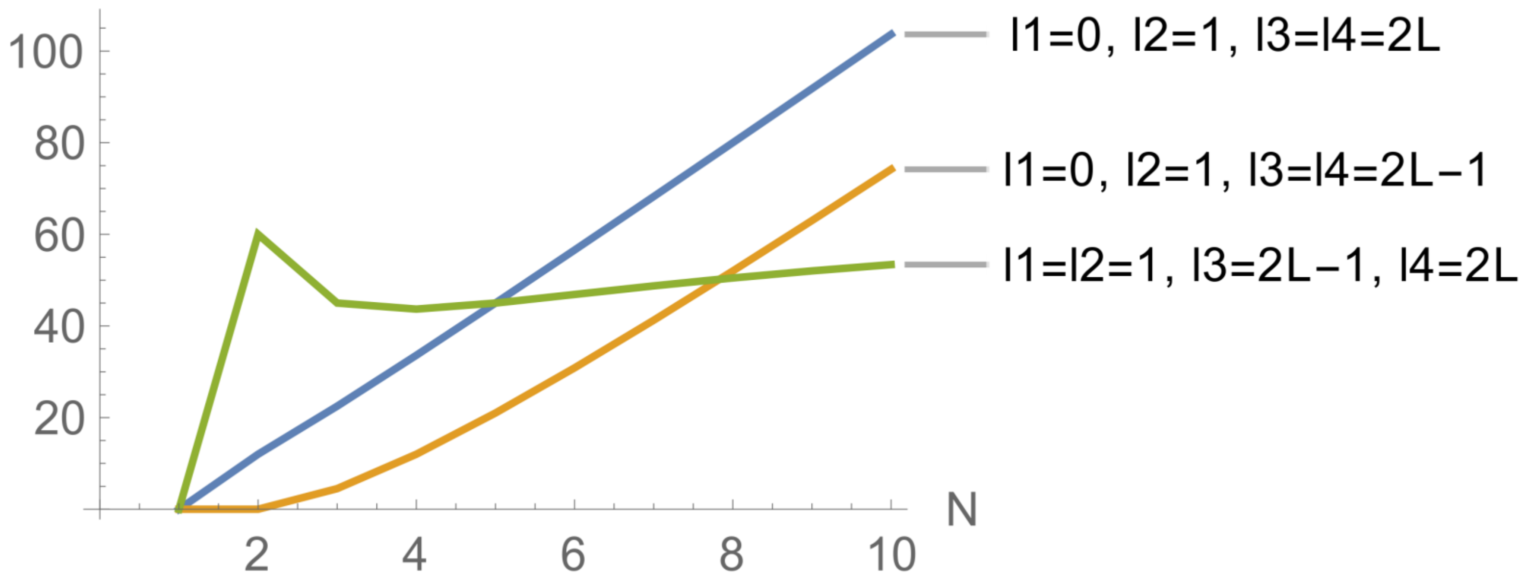
Structure constant:
$$G^{l_3m_3}_{l_1m_1;l_2m_2} = F_{l_1m_1\ l_2m_2}{}^{l_3m_3} - F_{l_2m_2\ l_1m_1}{}^{l_3m_3}$$

$$F_{l_1m_1\ l_2m_2}{}^{l_3m_3} = \sqrt{N(2l_1+1)(2l_2+1)(2l_3+1)}\,(-1)^{2L+l_1+l_2+l_3+m_3}\left(1-(-1)^{l_1+l_2+l_3}\right)\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}\begin{Bmatrix} l_1 & l_2 & l_3 \\ L & L & L \end{Bmatrix}$$

Fusion coefficient

$$\Xi_{l_1l_2l_3l_4} - \Xi_{l_1l_2l_3l_4}\Big|_{\mathrm{Planar}} \sim \sum_{\{m_i\}} (-1)^{m_1+m_2+m_3+m_4} \cdot \frac{1}{N}\mathrm{tr}\left(T_{l_1m_1}T_{l_2m_2}T_{l_3m_3}T_{l_4m_4}\right)$$
$$\times \sum_{l_5,m_5} (-1)^{m_5} G^{l_5m_5}_{l_3-m_3\ l_4-m_4}\left[6F_{l_1-m_1\ l_2-m_2}{}^{l_5-m_5} + G^{l_5-m_5}_{l_1-m_1\ l_2-m_2}\right]$$

Behavior of $\Xi_{l_1 l_2 l_3 l_4} - \Xi_{l_1 l_2 l_3 l_4}\big|_{\text{Planar}}$ for some typical l_i .



Comments:

- Difference between planar and non-planar part
  → A measure of NCA for two-point functions
- Opening a loop for 2PI diagrams, we obtain 1PI two-point function → More direct relation between NCA and S_{EE}?

6. Summary

- $S_{EE} \neq 0$ for mixed graphs of low/high modes at two loop level
- Noncommutative effect is from non-planar diagrams
- Matrix part is separated.
- Evaluated by
  - Replica Method (Path-integral)
  - Hamitonian formalism (skipped, ongoing)

Comments:

- Asymptotic (analytic) evaluation is required.

$$\begin{Bmatrix} L & L & l \\ L & L & 2L-n \end{Bmatrix} = \frac{(-1)^n}{N}\left[1 - \frac{l(l+1)}{N}(2n+1) + \frac{l(l+1)}{4N^2}\left(\left(l(l+1)-1\right)(2n+1)^2+3\right) + \mathcal{O}(N^{-3})\right]$$

$$\begin{Bmatrix} L & L & 2L-n \\ L & L & 2L-m \end{Bmatrix} = 2^{-4L-2}\sqrt{\frac{2\pi}{L}}\cdot 4^{n-m}L^{n-2m}\sum_{t=0}^{m}\frac{(-1)^{t-m}(16L^3)^t n!m!}{[(t+n-m)!]^2(t!)^2(m-t)!}\left(1+\mathcal{O}(L^{-1})\right)$$

$$\begin{Bmatrix} 2L-m & 2L & 2L \\ L & L & L \end{Bmatrix} = (-1)^{2L-m}3^{\frac{3}{4}}(2\pi)^{\frac{1}{4}}L^{\frac{m}{2}-\frac{3}{4}}(8\sqrt{m!})^{-1}\left(\frac{3}{4}\right)^{3L-\frac{m}{2}}\left(1+\mathcal{O}(L^{-1})\right)$$

$$\begin{Bmatrix} l_1 & l_2 & l \\ L & L & 2L-n \\ L & L & 2L-n \end{Bmatrix} = \frac{(-1)^{N-1+l}\sqrt{2}}{2N}\begin{Bmatrix} l & l_1 & l_2 \\ L & L & L \end{Bmatrix}\left[1 - \frac{2n+1}{4}\frac{2l_1(l_1+1)+2l_2(l_2+1)-l(l+1)-1}{N} + \mathcal{O}(N^{-2})\right]$$