Monster Anatomy

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Introduction

Mckay and Thompson made a remarkable observation between a modular object and Monster group M

[1] modular object: j-function

$$j(\tau) = \frac{12^3 E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)}$$

$$E_4(q) = 1 + 240q + 2160q^2 + 6720q^3 + \mathcal{O}(q^4)$$

$$E_6(q) = 1 - 504q - 16632q^2 - 122976q^3 + \mathcal{O}(q^4)$$

$$(q = e^{2\pi i\tau})$$

- invariant under SL(2,Z) under which

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \left[\begin{array}{c} a, b, c, d \in \mathbb{Z} \\ ad - bc = 1 \end{array} \right]$$

- partition function of a (chiral) RCFT with c=24, conjectured to describe the quantum theory of gravity in AdS3

Introduction

Mckay and Thompson made a remarkable observation between a modular object and Monster group M

[2] Monster group: the largest sporadic finite group of order

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \simeq 8 \times 10^{53}$$

[3] Monster Moonshine

$$j(\tau) - 744 = \frac{1}{q} + \overbrace{196884}^{=\,\mathbf{1} \oplus \mathbf{196883}} q + \underbrace{21493760}_{=\,\mathbf{1} \oplus \mathbf{196883} \oplus \mathbf{21296876}} q^2 + \mathcal{O}(q^3)$$

each coefficient of the above expansion can be expressed as a sum of dimensions of the irreducible representation of the monster group M.

Introduction

There exists a derivation of the Monster moonshine from an explicit construction of the c=24 chiral CFT based on the Leech lattice and Z_2 orbifold

[Frenkel,Lepowsky,Meurman]

GOAL Generalize the moonshine phenomena for a different sporadic group

- To the end, we need to find a relation between a modular object other than j-function and sporadic group other than the monster group M.

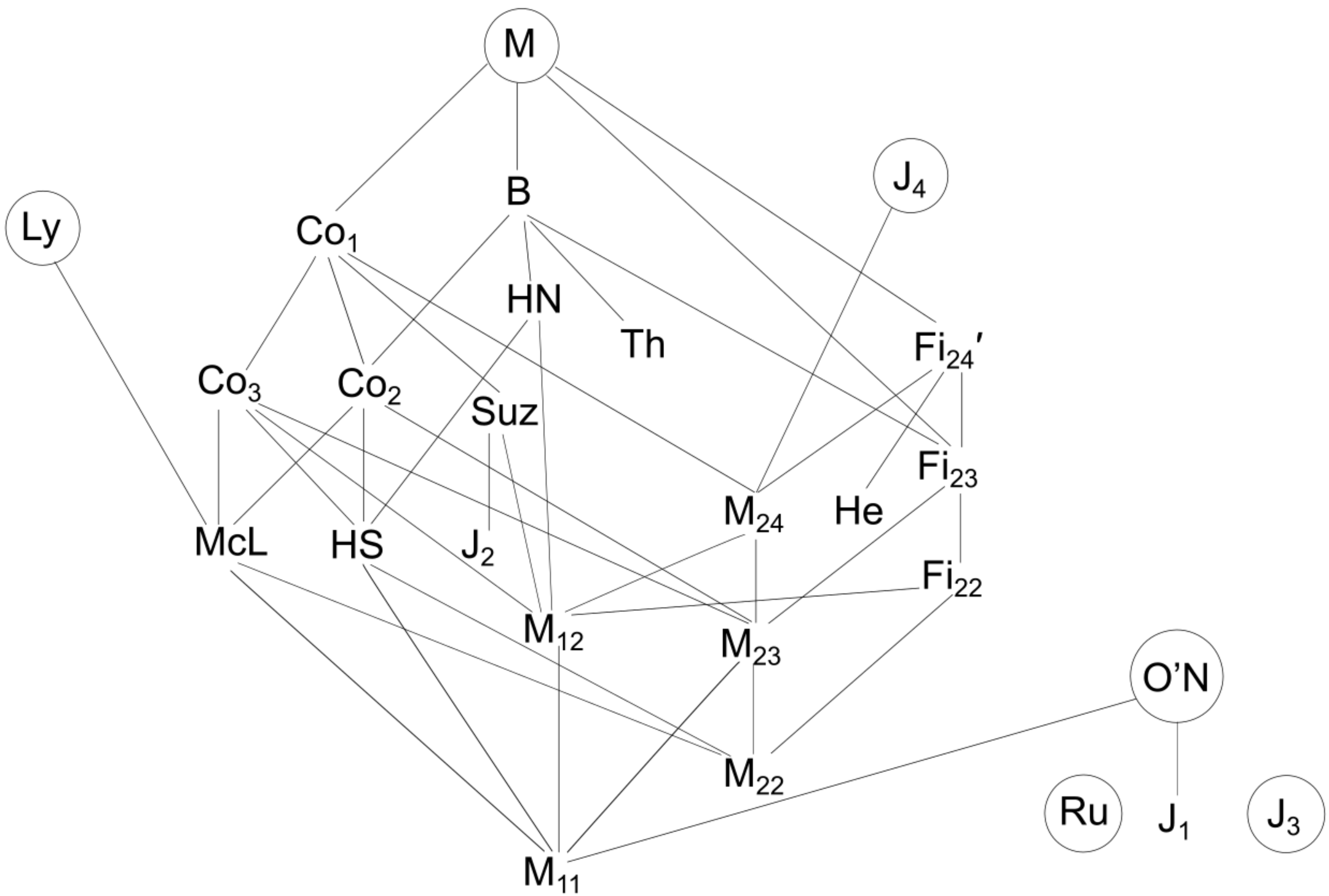
- e.g. Mathieu moonshine [Eguchi,Ooguri,Tachikawa]

[1] modular object: the elliptic genus on K3

[2] sporadic group: Mathieu group M_{24}

Sporadic Groups

Diagram of 26 sporadic simple groups, showing subquotient relationships.



Sporadic Groups

name	order	factorization
Mathieu group M_{11}	7920	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
Mathieu group M_{12}	95040	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
Janko group J_1	175560	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
Mathieu group M_{22}	443520	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
Janko group $J_2 = HJ$	604800	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
Mathieu group M_{23}	10200960	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
Higman-Sims group HS	44352000	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
Janko group J_3	50232960	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
Mathieu group M_{24}	244823040	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
McLaughlin group McL	898128000	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Held group He	4030387200	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Rudvalis Group Ru	145926144000	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
Suzuki group Suz	448345497600	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
O'Nan group O'N	460815505920	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Conway group Co_3	495766656000	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Conway group Co_2	42305421312000	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Fischer group Fi_{22}	64561751654400	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Harada-Norton group HN	273030912000000	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
Lyons Group Ly	51765179004000000	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
Thompson Group Th	90745943887872000	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
Fischer group Fi_{23}	4089470473293004800	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Conway group Co_1	4157776806543360000	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Janko group J_4	86775571046077562880	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
Fischer group Fi'_{24}	1255205709190661721292800	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
baby monster group B	4154781481226426191177580544000000	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
monster group M	808017424794512875886459904961710757005754368000000000	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

Rational CFT

Any conformal field theory in 2 dimensions have two copies of infinite dimensional symmetry algebras, left and right moving Virasoro algebras

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \qquad (m \in \mathbb{Z})$$

- We focus on parity-preserving CFTs with $c_l = c_r = c$

One can decompose the Hilbert space of a given CFT into representations of the Virasoro algebras

$$\mathcal{H} = \bigoplus_{h,\bar{h}\geq 0} \overbrace{d_{h,\bar{h}}}^{\text{degeneracy}} \underbrace{V_h^l \otimes V_{\bar{h}}^r}_{\text{representation of Virasoro algebra}}$$

Rational CFT

Torus Partition Function CFT on a circle at $T = \dfrac{1}{2\pi\tau_2}$

$$Z(\tau,\bar\tau) = \mathrm{Tr}_{\mathcal{H}}\left[e^{-2\pi\tau_2 H} e^{-2\pi i \tau_1 J}\right]$$

$$= \mathrm{Tr}_{\mathcal{H}}\left[q^{L_0^l - \frac{c}{24}}\, \bar{q}^{L_0^r - \frac{c}{24}}\right]$$

$$L_0^l - c/24 = (H-J)/2$$
$$L_0^r - c/24 = (H+J)/2$$

$$= \sum_{h,\bar h} d_{h,\bar h} \underbrace{\mathrm{Tr}_{V_h^l}\left[q^{L_0^l - \frac{c}{24}}\right]}_{=\chi_h(\tau)} \underbrace{\mathrm{Tr}_{V_{\bar h}^r}\left[q^{L_0^r - \frac{c}{24}}\right]}_{=\bar\chi_{\bar h}(\tau)}$$

$$\mathcal{H} = \bigoplus_{h,\bar h \geq 0} d_{h,\bar h} V_h^l \otimes V_{\bar h}^r$$

Virasoro characters

- when c>1, $\chi_0(\tau) = q^{-\frac{c}{24}} \prod_{n=2} \dfrac{1}{1-q^n}$

$$\chi_h(\tau) = q^{h-\frac{c}{24}} \prod_{n=1} \frac{1}{1-q^n} \quad (h \neq 0)$$

Rational CFT

Torus Partition Function CFT on a circle at $T = \dfrac{1}{2\pi\tau_2}$

$$Z(\tau,\bar\tau) = \sum_{h,\bar h} d_{h,\bar h}\chi_h(\tau)\bar\chi_{\bar h}(\bar\tau)$$

Torus partition function is invariant under the modular transformation SL(2,Z), generated by T and S

[1] invariance under T requires states of integer spin

$$Z(\tau,\bar\tau) = \mathrm{Tr}\left[q^{L_0-\frac{c}{24}}\bar q^{\bar L_0-\frac{c}{24}}\right] \xrightarrow{\tau\to\tau+1} \mathrm{Tr}\left[e^{2\pi i J}q^{L_0-\frac{c}{24}}\bar q^{\bar L_0-\frac{c}{24}}\right]$$

[2] invariance under S then leads to strong constraints on the spectrum $d_{h,\bar h}$

$$Z(\tau,\bar\tau) = Z(-\frac{1}{\tau},-\frac{1}{\bar\tau}) \qquad \left(\chi_h(-\frac{1}{\tau}) = \sum_{h'\geq\frac{c-1}{24}} S(h,h')\chi_{h'}(\tau)\right)$$

Rational CFT

Modular Bootstrap $Z(\tau,\bar{\tau}) = Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}})$

[1] when c < 1, this consistency condition can be solved analytically. Those CFTs are classified and known as minimal models (m+1,m) with central charge [Zamolodchiov]…

$$c = 1 - \frac{6}{m(m+1)}$$

[2] when c > 1, one consequence of the constraint is that any unitary CFTs have infinite number of Virasoro primaries. However, it is extremely to difficult to solve it.

Rational CFT

Rational CFT is a special type of CFT with a finite number of conformal primaries, which implies that

$$Z(\tau,\bar\tau) = \sum_{i,j=0}^{n-1} M_{ji} f_i(\tau) \bar f_j(\bar\tau)$$

conformal character w.r.t. an extended chiral algebra that includes Virasoro alg.

When the partition function Z takes the above form, both central charge c and conformal weight h are rational [Anderson,Moore]

- e.g. Monster CFT with c=24: a single character RCFT

Virasoro characters

$$Z(\tau,\bar\tau) = |j(\tau) - 744|^2 \qquad f_0^{\mathbb{M}}(\tau) = j(\tau) - 744$$

$$= \chi_0(\tau) + 196883\chi_2(\tau) + 21296876\chi_3(\tau) + \cdots$$

Rational CFT

Note that, since the partition function is modular invariant, the conformal characters should be weight 0 vector-valued modular functions,

$$f_i(\tau+1) = \sum_{j=0}^{n-1} T_{ij} f_j(\tau)$$

$$f_i(-\frac{1}{\tau}) = \sum_{j=0}^{n-1} S_{ij} f_j(\tau)$$

$$S^2 = \left(ST\right)^3 = C \quad C^2 = 1$$

Conformal characters thus satisfy nth-order Modular Linear Differential Equation (MLDE).

$$\left[D_\tau^n + \sum_{k=0}^{n-1} \phi_k(\tau) D_\tau^k\right] f(\tau) = 0$$

$\phi_k(\tau)$: modular form of weight 2(n-k)

$$D_\tau = \partial_\tau - \frac{i\pi}{6} p E_2(\tau)$$

p: weight of a modular form on which the covariant derivative D acts

Rational CFT

MLDE is invariant under the modular transformation SL(2,Z), which implies that n independent solutions are vector-valued modular functions.

From the fact that the conformal characters have poles only at $\tau = i\infty$, one can show that n independent solutions can be expanded in powers of q as follows

$$f_i(\tau) = q^{h_i - \frac{c}{24}} \sum_m^{\infty} a_m q^m \qquad (a_m \in \mathbb{Z})$$

$$\sum_{i=0}^{n-1} \left(h_i - \frac{c}{24}\right) - \frac{n(n-1)}{12} = -\frac{l}{6}$$

$$(l \in \{0, 2, 3, 4, 6, 8, 9, 10, 12, ..\})$$

Remark use MLDEs to search for and classify possible characters of a new RCFT

[Mathur,Mukhi,Sen]

Rational CFT

Example: Ising model

[1] Ising model has the identity operator and two primaries of $h = 1/2, 1/16$

$$\chi_0(\tau) = \frac{1}{2}\left[\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right] = q^{-\frac{1}{24}\cdot\frac{1}{2}}\left(1 + q^2 + q^3 + 2q^4 + \cdots\right)$$

$$\chi_\epsilon(\tau) = \frac{1}{2}\left[\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right] = q^{\frac{1}{2}-\frac{1}{24}\cdot\frac{1}{2}}\left(1 + q + q^2 + q^3 + \cdots\right)$$

$$\chi_\sigma(\tau) = \frac{1}{\sqrt{2}}\sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}} = q^{\frac{1}{16}-\frac{1}{24}\cdot\frac{1}{2}}\left(1 + q + q^2 + 2q^3 + \cdots\right)$$

Rational CFT

Example: Ising model

[2] Three characters are solutions to the MLDE below

$$\left[(q\frac{d}{dq})^3 - \frac{1}{2}E_2(\tau)(q\frac{d}{dq})^2 + (\frac{1}{24}E_2(\tau)^2 - \frac{25}{768}E_4(\tau))(q\frac{d}{dq}) + \frac{23}{55296}\right] f(\tau) = 0$$

[3] Modular matrices S and T are given by

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \sqrt{\frac{1}{2}} \\ \frac{1}{2} & \frac{1}{2} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \end{pmatrix} \qquad T = e^{-\frac{2\pi i}{48}} \begin{pmatrix} 1 & & \\ & e^{\frac{2\pi i}{2}} & \\ & & e^{\frac{2\pi i}{16}} \end{pmatrix}$$

Rational CFT

Example: Ising model

[4] Modular invariant partition function becomes

$$Z(\tau,\bar{\tau}) = \Big|\chi_0(\tau)\Big|^2 + \Big|\chi_\epsilon(\tau)\Big|^2 + \Big|\chi_\sigma(\tau)\Big|^2$$

[5] Fusion algebra can read from the Verlinde formula

$$[1]\times[\sigma]=[\sigma] \qquad [\epsilon]\times[\epsilon]=[1] \qquad [\sigma]\times[\sigma]=[1]+[\epsilon]$$

$$[1]\times[\epsilon]=[\epsilon] \qquad [\epsilon]\times[\sigma]=[\sigma]$$

GOAL Generalize the moonshine phenomena for a different sporadic group

To the end, we need to find a relation between a modular object other than j-function and sporadic group other than the monster group M.

MODULAR OBJECT: Conformal characters of an RCFT, vector-valued modular function

Bilinear Relation

Mukhi *et al* observed recently that characters $f_i(\tau)(i = 0, 1, .., n-1)$ of a certain rational CFT with central charge c obey an intriguing bilinear relation giving a modular invariant

[Hampapura,Mukhi]

$$f_0^{\mathbb{M}}(\tau) = j(\tau) - 744 = \sum_{i=0}^{n-1} f_i(\tau)\tilde{f}_i(\tau)$$

$\tilde{f}_i(\tau)$ can be identified as characters of a dual rational CFT with central charge 24 - c

$$T^{\mathbb{M}} = \left[\alpha T^{\mathbb{M}} + \sum_{i=1}^{196883} \beta_i \varphi^i\right] + \left[(1-\alpha)T^{\mathbb{M}} - \sum_{i=1}^{196883} \beta_i \varphi^i\right]$$

The bilinear relation implies that

$$L_0^{\mathbb{M}} = L_0 + \tilde{L}_0$$

$$V^{\mathbb{M}} = \bigoplus_{a=0}^{n-1} V_a \otimes \tilde{V}_a$$

$$j(\tau) - 744 = \mathrm{Tr}_{V^{\mathbb{M}}}\left[q^{L_0^{\mathbb{M}} - \frac{1}{24}24}\right]$$

$$= \sum_{a=0}^{n-1} \mathrm{Tr}_{V_a}\left[q^{L_0 - \frac{c}{24}}\right] \mathrm{Tr}_{\tilde{V}_a}\left[q^{\tilde{L}_0 - \frac{24-c}{24}}\right]$$

Baby Monster Moonshine?

Let us consider characters for the Ising model and characters for a certain RCFT satisfying MLDE below

$$\left(D_\tau^3 + \frac{2315}{576}\pi^2 E_4(\tau) D_\tau - i\frac{27025}{6912}\pi^3 E_6(\tau)\right) f(\tau) = 0$$

$$\tilde{f}_0(\tau) = q^{-\frac{1}{24}\cdot\frac{47}{2}}\left(1 + 96256q^2 + 9646891q^3 + \cdots\right)$$

$$\tilde{f}_\epsilon(\tau) = q^{\frac{3}{2}-\frac{1}{24}\cdot\frac{47}{2}}\left(4371 + 1143745q + 64680601q^2 + \cdots\right)$$

$$\tilde{f}_\sigma(\tau) = q^{\frac{31}{16}-\frac{1}{24}\cdot\frac{47}{2}}\left(96256 + 10602492q + \cdots\right)$$

Dual RCFT has c=24-1/2 & primaries of h=3/2, 31/16

These characters obey the bilinear relation

$$j(\tau) - 744 = \chi_0(\tau)\tilde{f}_0(\tau) + \chi_\epsilon(\tau)\tilde{f}_\epsilon(\tau) + \chi_\sigma(\tau)\tilde{f}_\sigma(\tau)$$

(Note that the fusion algebra of the dual RCFT with c = 24 – ½ is well-defined)

Baby Monster Moonshine?

Note that **4371** and **96256**, the lowest order coefficients, are dimensions of irreducible representation of the double covering of baby Monster group $2.\mathbb{B}$. Other coefficients have decomposition into irreducible representations.

$$\tilde{f}_0(\tau) = q^{-\frac{1}{24}\cdot\frac{47}{2}}\left(1 + 96256q^2 + 9646891q^3 + \cdots\right)$$

$$\tilde{f}_\epsilon(\tau) = q^{\frac{3}{2}-\frac{1}{24}\cdot\frac{47}{2}}\left(4371 + 1143745q + 64680601q^2 + \cdots\right)$$

$$\tilde{f}_\sigma(\tau) = q^{\frac{31}{16}-\frac{1}{24}\cdot\frac{47}{2}}\left(96256 + 10602492q + \cdots\right)$$

$$92256 = \mathbf{1} \oplus \mathbf{96255}$$

$$1143745 = \mathbf{4371} \oplus \mathbf{1139374}$$

$$10602496 = \mathbf{96256} \oplus \mathbf{10506240}$$

Baby Monster Moonshine

It is known that the Ising model has Z_2 symmetry. The identity operator and the energy field are even while the spin field is odd under Z_2 action.

$$\mathbb{Z}_2 \;:\quad 1 \longrightarrow 1 \qquad \epsilon \longrightarrow \epsilon \qquad \sigma \longrightarrow -\sigma$$

When the Ising characters are replaced by their Z_2 twined characters, one obtains the McKay-Thompson series of class 2A of $\mathbb{M}$

$$q^{-1}\left[1 + 4372q^2 + 96256q^3 + \mathcal{O}(q^4)\right] = \chi_0(\tau)\tilde{f}_0(\tau) + \chi_\epsilon(\tau)\tilde{f}_\epsilon(\tau) - \chi_\sigma(\tau)\tilde{f}_\sigma(\tau)$$

$$||$$

$$j^{2\mathbb{A}}(\tau) = \mathrm{Tr}_{V^{\mathbb{M}}}\left[h' \cdot q^{L_0^{\mathbb{M}} - \frac{1}{24}24}\right] \qquad (h' \in [2A]_{\mathbb{M}}; h'^2 = 1)$$

$$||$$

$$\sum_{a=0}^{2} \mathrm{Tr}_{V_a}\left[h \cdot q^{L_0 - \frac{c}{24}}\right] \mathrm{Tr}_{\tilde{V}_a}\left[q^{\tilde{L}_0 - \frac{\tilde{c}}{24}}\right] \qquad (h \in \mathbb{Z}_2)$$

Baby Monster Moonshine

The above generalized bilinear relation then implies that Z_2 can be elevated into an element of 2A class of $\mathbb{M}$, which acts trivially on the Hilbert space $\tilde{V}_a$ (a=0,1,2) of dual RCFT.

$$j^{2\mathbb{A}}(\tau) = \mathrm{Tr}_{V^{\mathbb{M}}}\left[h' \cdot q^{L_0^{\mathbb{M}} - \frac{1}{24}24}\right] = \sum_{a=0}^{2} \mathrm{Tr}_{V_a}\left[h \cdot q^{L_0 - \frac{c}{24}}\right] \mathrm{Tr}_{\tilde{V}_a}\left[q^{\tilde{L}_0 - \frac{\tilde{c}}{24}}\right]$$

$$(h' \in [2A]_{\mathbb{M}}) \qquad\qquad (h \in \mathbb{Z}_2)$$

It implies that the dual RCFT with c = 24 – ½ has the centralizer of $[2A]_M$ as symmetry. Note that the centralizer of $[2A]_M$ is the double covering of the Baby Monster group.

$$\mathcal{C}([2A]_{\mathbb{M}}) = 2.\mathbb{B}$$

This explains why the RCFT dual to the Ising model can exhibit the moonshine phenomena for $2.\mathbb{B}$

Table 2a. Additional information for small order elements. [Conway,Norton]

	centraliser structure and order		class
(MONSTER)		8080 17424 79451 28758 86459 90496 17107 57005 75436 80000 00000	$1A$
(BABY)	$2.B$	8305 96296 24528 52382 35516 10880 00000	$2A$
(Conway)	$2^{1+24}C_1$	13 95118 39126 33632 81715 20000	$2B$
(Fischer)	$3.F'_{24}$	37656 17127 57198 51638 78400	$3A$
(Suzuki)	$3^{1+12}.2.Sz$	1429 61507 75402 49600	$3B$
(Thompson, Smith)	$3\times E$	272 23783 16636 16000	$3C$
(Conway)	$4.2^{22}.C_3$	8317 58427 33096 96000	$4A$
	$\{4\times F_4(2)\}.2$	26 48901 28269 31200	$4B$
	$4.2^{15}.2^8.S_6(2)$	4870 49291 36640	$4C$
	$4.2^{12}.G_2(4).2$	824 43239 42400	$4D$
(Harada, Norton)	$5\times F$	1 36515 45600 00000	$5A$
(Hall, Janko)	$5^{1+6}.2.HJ$	9 45000 00000	$5B$
(Fischer)	$3\times 2.F_{22}.2$	77474 10198 52800	$6A$
(Suzuki)	$6.Sz$	269 00729 85600	$6B$
	$2^{1+12}.3^2.U_4(3).2$	48 15794 99520	$6C$
	$2.3^{1+8}.2^{1+6}.U_4(2)$	13 06069 40160	$6D$
	$2.3^{1+4}.2^{1+6}.U_4(2)$	16124 31360	$6E$
	$3\times 2^{1+8}.A_9$	2786 91840	$6F$
(Held)	$7\times H$	2 82127 10400	$7A$
	$7^{1+4}.2.A_7$	847 07280	$7B$
	$8.2^7.2^6.U_3(3).2$	7927 23456	$8A$
(Mathieu)	$8.2^{10}.M_{12}$	7785 67680	$8B$
(Tits)	$8\times {}^2F'_4(2)$	1437 69600	$8C$
	$8.2^9.2^4.A_6$	235 92960	$8D$
	$[2^{22}3]$	125 82912	$8E$
	$8.2^6.U_3(3)$	30 96576	$8F$
	$9.3^{1+4}.S_4(3)$	566 87040	$9A$
	$[2^4 3^{11}]$	28 34352	$9B$
(Higman, Sims)	$5\times 2.HS.2$	8870 40000	$10A$
	$5\times 2^{1+8}.(A_5\times A_5).2$	184 32000	$10B$
	$2.5^{1+4}.2^{1+4}.A_5$	120 00000	$10C$
(Hall, Janko)	$5\times 2.HJ$	60 48000	$10D$
	$2.5^{1+2}.2^{1+4}.A_5$	4 80000	$10E$
(Mathieu)	$11\times M_{12}$	10 45440	$11A$

$g\in[3A]$ $g^3=1$

3.Fi_{24} Moonshine

Note that the centralizer of $[3A]_M$ is the triple covering of the largest Fischer group Fi_{24}

To look for a RCFT exhibiting the 3.Fi_{24} moonshine, all we need is to find a good RCFT with Z_3 symmetry. One such RCFT is the Z_k parafermion theory with k=3

Z_k parafermion theory $SU(2)_k/U(1)_k$ with $c_k = \frac{2(k-1)}{k+2}$

[1] Primaries $\phi^{(k)}_{l,m}$ of conformal weight

$$h^{(k)}_{l,m} = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k} \qquad \begin{gathered} 0 \le l \le k \\ -l+2 \le m \le l \\ l-m \in 2\mathbb{Z} \end{gathered}$$

[2] Z_k symmetry

$$g : \phi^{(k)}_{l,m} \to e^{2\pi i m/k} \phi^{(k)}_{l,m}$$

$3.Fi_{24}$ Moonshine

Z_3 parafermion theory, a.k.a. three-state Pott's model, has 6 primaries of conformal weight 0,2/3,2/3,2/5,2/5,1/15. Note that primaries with h=2/3 and h=2/5 appear twice, because they transform non-trivially under Z_3

$$f^{(3)}_{3,3}(\tau) = q^{-\frac{1}{24}\frac{4}{5}}\Big(1 + q^2 + 2q^3 + \mathcal{O}(q^4)\Big)$$

$$f^{(3)}_{3,1}(\tau) = f^{(3)}_{3,-1}(\tau) = q^{\frac{2}{3}-\frac{1}{24}\frac{4}{5}}\Big(1 + q + 2q^2 + 2q^3 + \mathcal{O}(q^4)\Big)$$

$$f^{(3)}_{1,1}(\tau) = f^{(3)}_{2,2}(\tau) = q^{\frac{2}{5}-\frac{1}{24}\frac{4}{5}}\Big(1 + 2q + 2q^2 + 4q^3 + \mathcal{O}(q^4)\Big)$$

$$f^{(3)}_{2,0}(\tau) = q^{\frac{1}{15}-\frac{1}{24}\frac{4}{5}}\Big(1 + q + 2q^2 + 3q^3 + \mathcal{O}(q^4)\Big)$$

The characters $\tilde{f}(\tau)$ of the dual RCFT with c = 116/5 could exhibit the moonshine for $3.Fi_{24}$, which is the centralizer of $[3A]_M$

$$\begin{aligned} j(\tau) - 744 = {} & f^{(3)}_{1,1}(\tau)\tilde{f}_{1,1}(\tau) + f^{(3)}_{2,2}(\tau)\tilde{f}_{2,2}(\tau) + f^{(3)}_{2,0}(\tau)\tilde{f}_{2,0}(\tau) + f^{(3)}_{3,1}(\tau)\tilde{f}_{3,1}(\tau) \\ & + f^{(3)}_{3,-1}(\tau)\tilde{f}_{3,-1}(\tau) + f^{(3)}_{3,3}(\tau)\tilde{f}_{3,3}(\tau) \end{aligned}$$

3.Fi_{24} Moonshine

One can show that the bilinear relations can be satisfied with parafermion characters and their dual characters, independent solutions to the 4^{th} order MLDE below

$$\left[D_\tau^4 + \frac{907\pi^2}{225}E_4(\tau)D_\tau^2 - i\frac{4289\pi^3}{675}E_6(\tau)D_\tau - \frac{175769\pi^4}{50625}E_4(\tau)^2\right]\tilde{f}(\tau) = 0$$

$$\tilde{f}_{1,1}(\tau) = \tilde{f}_{2,2}(\tau) = q^{\frac{29}{15}-\frac{1}{24}\frac{116}{5}}\left(64584 + 6789393q + 261202536q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_{2,0}(\tau) = q^{\frac{8}{5}-\frac{1}{24}\frac{116}{5}}\left(8671 + 1675504q + 83293626q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_{3,1}(\tau) = \tilde{f}_{3,-1}(\tau) = q^{\frac{4}{3}-\frac{1}{24}\frac{116}{5}}\left(783 + 306936q + 19648602q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_{3,3}(\tau) = q^{-\frac{1}{24}\frac{116}{5}}\left(1 + 57478q^2 + 5477520q^3 + \mathcal{O}(q^4)\right)$$

Note also that one can utilize the Hecke operator method to obtain the dual characeters.

[Harvey, Wu]

$3.Fi_{24}$ Moonshine

Note that **783**, **8671** and **64584**, the lowest order coefficients, are indeed dimensions of irreducible representation of 3.Fi24. Other coefficients have decomposition into irreducible representations.

$$\tilde{f}_{1,1}(\tau) = \tilde{f}_{2,2}(\tau) = q^{\frac{29}{15}-\frac{1}{24}\frac{116}{5}}\left(64584 + 6789393q + 261202536q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_{2,0}(\tau) = q^{\frac{8}{5}-\frac{1}{24}\frac{116}{5}}\left(8671 + 1675504q + 83293626q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_{3,1}(\tau) = \tilde{f}_{3,-1}(\tau) = q^{\frac{4}{3}-\frac{1}{24}\frac{116}{5}}\left(783 + 306936q + 19648602q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_{3,3}(\tau) = q^{-\frac{1}{24}\frac{116}{5}}\left(1 + 57478q^2 + 5477520q^3 + \mathcal{O}(q^4)\right)$$

$$57478 = \mathbf{1} \oplus \mathbf{57477} \qquad 306936 = \mathbf{783} \oplus \mathbf{306153}$$

$$1675504 = \mathbf{8671} \oplus \mathbf{1666833} \qquad 6789393 = \mathbf{64584} \oplus \mathbf{6724809}$$

3.Fi_{24} Moonshine

When the three-state Pott's model characters are replaced by their Z_3 twined characters, one obtains the McKay-Thompson series of class 3A of $\mathbb{M}$

$$q^{-1}\left(1+783q^2+8672q^3+\mathcal{O}(q^4)\right) = wf^{(3)}_{1,1}(\tau)\tilde{f}_{1,1}(\tau)+w^2f^{(3)}_{2,2}(\tau)\tilde{f}_{2,2}(\tau)+f^{(3)}_{2,0}(\tau)\tilde{f}_{2,0}(\tau)$$
$$+wf^{(3)}_{3,1}(\tau)\tilde{f}_{3,1}(\tau)+w^2f^{(3)}_{3,-1}(\tau)\tilde{f}_{3,-1}(\tau)+f^{(3)}_{3,3}(\tau)\tilde{f}_{3,3}(\tau)$$

$$||\qquad\qquad\qquad\qquad ||$$

$$j^{3\mathbb{A}}(\tau)=\mathrm{Tr}_{V^{\mathbb{M}}}\left[h\cdot q^{L_0^{\mathbb{M}}-\frac{1}{24}24}\right] \quad (h\in[3A]_{\mathbb{M}})$$

$$\sum_{a=0}^{5}\mathrm{Tr}_{V_a}\left[h\cdot q^{L_0-\frac{c}{24}}\right]\mathrm{Tr}_{\tilde{V}_a}\left[q^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right] \quad (h\in\mathbb{Z}_3)$$

$$(w=e^{2\pi i/3})$$

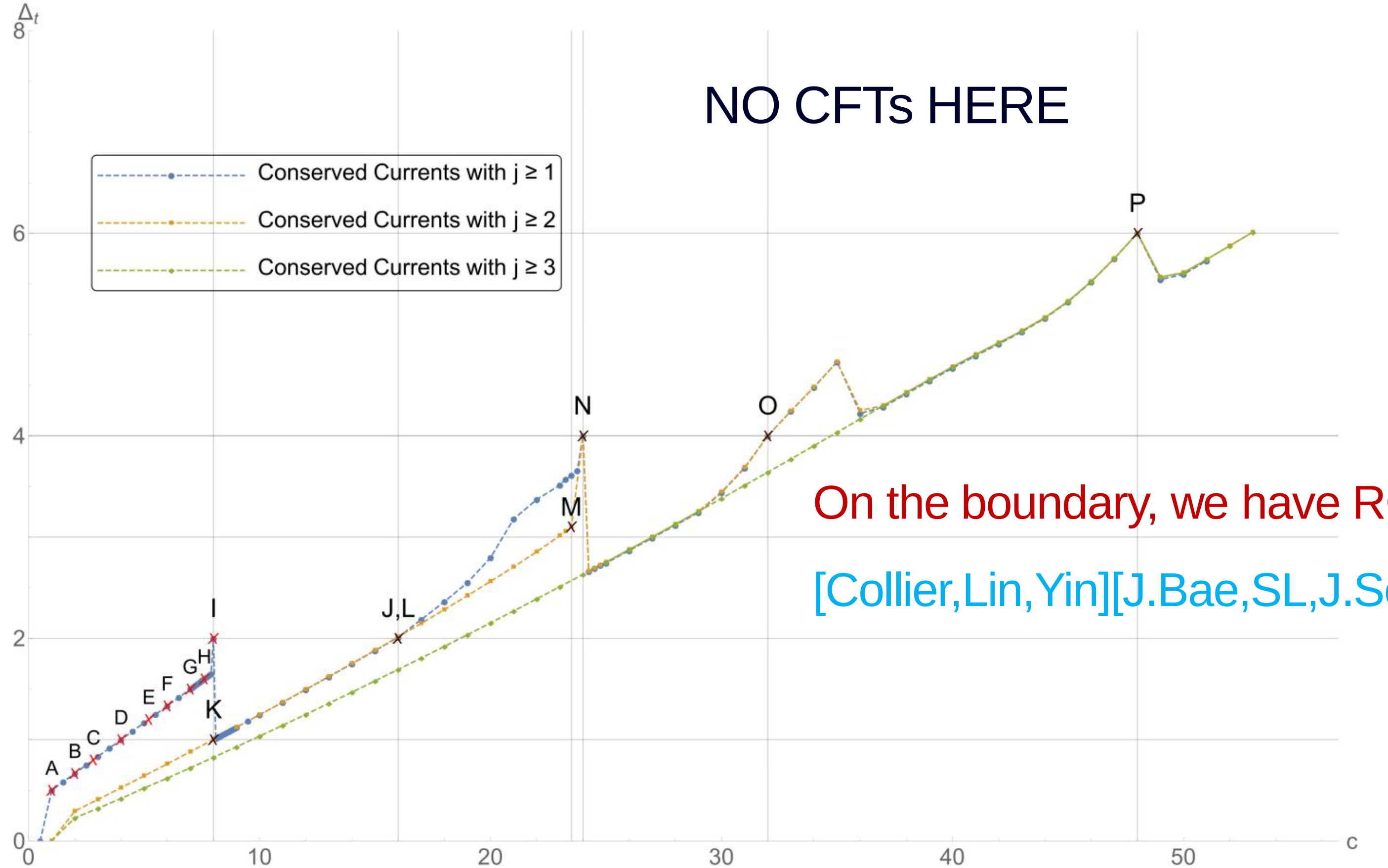
Conclusion

Group	$(X, C_M(X))$	c	Comments
M	$(1A, M)$	24	
B	$(2A, 2.B)$	$47/2$	Hecke dual of $c = 1/2$
Th	$(3C, 3 \times Th)$	$248/11$	dual of extension of $\mathbb{Z}_9$ pf
Co_1	$(2B, 2^{1+24}.Co_1)$		dual of $Ising \otimes Ising$?
Co_2	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 2.2^{1+22}Co_2)$		dual of $Ising \otimes Ising$
Co_3	$(4A, 4.2^{22}.Co_3)$	23	Hecke dual of $\mathbb{Z}_4$ pf
HN	$(5A, 5 \times HN)$		dual $(4,3) \otimes (8,7) \otimes (8,7)$, $(\mathbb{Z}_5 pf)^{\otimes 2}$
Suz	$(6B, 6.Suz)$		CFT with $\mathbb{Z}_6$, maybe $\mathbb{Z}_{36}$ pf?
Fi'_{24}	$(3A, 3.Fi'_{24})$		Hecke dual of $\mathbb{Z}_3$ pf
Fi_{23}	$(S_3, ??)$?		dual of $(5,4) \otimes (6,5)$ minimal models
Fi_{22}	$(6A, 3 \times 2.Fi_{22}.2)$		dual $(5,4) \otimes (6,5) \otimes (7,6)$?
McL	$(D_8, 2^{1+22}McL)$		Find CFT with D_8 symmetry
HS	$(10A, 5 \times 2.HS.2)$		Look at $\mathbb{Z}_{25}$ pf?
HJ	$(5B, 5^{1+6}.2.HJ)$		Look at $\mathbb{Z}_{25}$ pf?
He	$(7A, 7 \times He)$		Hecke dual of $\mathbb{Z}_7$ pf
M_{24}	$(\mathbb{Z}_2 \times \mathbb{Z}_2, 2^{2+11+22}.M_{24})$		
M_{23}		maybe $2.M_{22}$	dual of $(4,3) \otimes (5,4) \otimes (6,5)$
M_{22}	$(2 \times S_5, 2M_{22})$		
M_{12}	$(11A, 11 \times M_{12})$		
M_{11}	Many		
T	$(8C, 8 \times T)$		27th sporadic, Try $\mathbb{Z}_8$ or $\mathbb{Z}_{64}$ pf?

Rational CFT

[3] There has been a recent development solving the constraint numerically.

e.g. upper bound on the twist gap of the lowest primary above the vacuum

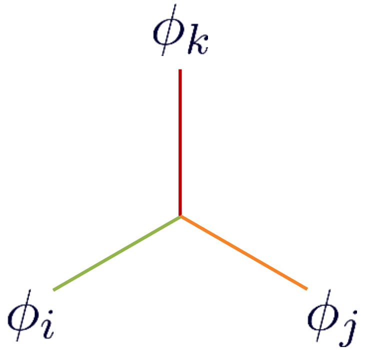


On the boundary, we have RCFTs!

[Collier,Lin,Yin][J.Bae,SL,J.Song]

Rational CFT

For a consistent unitary RCFT, each fusion rule coefficient N_{ij}^{k} has to be a non-negative integer.



$$[\phi_i] \times [\phi_j] = N_{ij}^{k} [\phi_k]$$

$$N_{ij}^{k} \in \mathbb{Z}_{\geq}$$

Fusion rule coefficients can be computed from the Verlinde formula

$$N_{ij}^{k} = \sum_{l} \frac{S_{il} S_{jl} \bar{S}_{kl}}{S_{0l}} \in \mathbb{Z}_{\geq}$$

[Verlinde]

When a solution to MLDE have a negative fusion rule coefficient, a corresponding candidate RCFT is not well-defined.

Generalized Bilinear Relation

Replacing the dual characters with the twined characters, defined below, gives the Mckay-Thompson series of the Monster. For instance,

$$q^{-1}\left[1+276q^2-2048q^3+\mathcal{O}(q^4)\right]=\chi_0(\tau)\tilde{f}_0^{2D}(\tau)+\chi_\epsilon(\tau)\tilde{f}_\epsilon^{2D}(\tau)+\chi_\sigma(\tau)\tilde{f}_\sigma^{2D}(\tau)$$

$$\|$$

$$\sum_{a=0}^{2}\mathrm{Tr}_{V^a}\left[q^{L_0-\frac{c}{24}}\right]\mathrm{Tr}_{\tilde{V}^a}\left[g\cdot q^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right]$$

$$(g\in[2D]_{2.\mathbb{B}})$$

where

$$\tilde{f}_0^{2\mathbf{D}}(\tau)=\mathrm{Tr}_{\tilde{V}_0}\left[gq^{\tilde{L}_0-\frac{24-c}{24}}\right]=q^{-\frac{1}{24}\frac{47}{2}}\left(1+2048q^2+37675q^3+\cdots\right)$$

$$\tilde{f}_\epsilon^{2\mathbf{D}}(\tau)=\mathrm{Tr}_{\tilde{V}_\sigma}\left[gq^{\tilde{L}_0-\frac{24-c}{24}}\right]=q^{\frac{3}{2}-\frac{1}{24}\frac{47}{2}}\left(275+9153q+144025q^2+\cdots\right)$$

$$\tilde{f}_\sigma^{2\mathbf{D}}(\tau)=\mathrm{Tr}_{\tilde{V}_\sigma}\left[gq^{\tilde{L}_0-\frac{24-c}{24}}\right]=-q^{\frac{31}{16}-\frac{1}{24}\frac{47}{2}}\left(2048+47104q+\cdots\right)$$

Generalized Bilinear Relation

This generalized bilinear relation implies that an element of 2B class of $\mathbb{M}$ could reduce to an element of 2D class of $2.\mathbb{B}$

$$q^{-1}\left[1+276q^2-2048q^3+\mathcal{O}(q^4)\right]=\chi_0(\tau)\tilde{f}_0^{2D}(\tau)+\chi_\epsilon(\tau)\tilde{f}_\epsilon^{2D}(\tau)+\chi_\sigma(\tau)\tilde{f}_\sigma^{2D}(\tau)$$

$$||\qquad\qquad\qquad\qquad ||$$

$$j^{2B}(\tau)\equiv \mathrm{Tr}_{V^{\mathbb{M}}}\left[g\cdot q^{L_0^{\mathbb{M}}-\frac{c^{\mathbb{M}}}{24}}\right] \qquad \sum_{a=0}^{2}\mathrm{Tr}_{V^a}\left[q^{L_0-\frac{c}{24}}\right]\mathrm{Tr}_{\tilde{V}^a}\left[g\cdot q^{\tilde{L}_0-\frac{\tilde{c}}{24}}\right]$$

$$(g\in[2B]_{\mathbb{M}}) \qquad (g\in[2D]_{2.\mathbb{B}})$$

Generalized Bilinear Relation

We have generalized bilinear relations for other classes as well, which strongly suggests that the RCFT with c = 24-1/2, dual to the Ising model, has $2.\mathbb{B}$ as symmetry.

$$j^{2B}(\tau) \equiv \mathrm{Tr}_{V^{\mathbb{M}}}\left[g \cdot q^{L_0^{\mathbb{M}} - \frac{c^{\mathbb{M}}}{24}}\right] = \sum_{a=0}^{2} \mathrm{Tr}_{V^a}\left[q^{L_0 - \frac{c}{24}}\right] \mathrm{Tr}_{\tilde{V}^a}\left[g \cdot q^{\tilde{L}_0 - \frac{\tilde{c}}{24}}\right]$$

$g_{\mathbb{M}}$	$\mathbf{2B}$	$\mathbf{3A}$	$\mathbf{3B}$	$\mathbf{4B}$
$g_{2.\mathbb{B}}$	$\mathbf{2D}$	$\mathbf{3A}$	$\mathbf{3B}$	$\mathbf{4A}$

Table 2a. *Additional information for small order elements.*

	centraliser structure and order		class
(MONSTER)		8080 17424 79451 28758 86459 90496 17107 57005 75436 80000 00000	1*A*
(BABY)	2.B	8305 96296 24528 52382 35516 10880 00000	2*A*
(Conway)	$2^{1+24}C_1$	13 95118 39126 33632 81715 20000	2*B*
(Fischer)	$3.F'_{24}$	37656 17127 57198 51638 78400	3*A*
(Suzuki)	$3^{1+12}.2.Sz$	1429 61507 75402 49600	3*B*
(Thompson, Smith)	$3\times E$	272 23783 16636 16000	3*C*
(Conway)	$4.2^{22}.C_3$	8317 58427 33096 96000	4*A*
	$\{4\times F_4(2)\}.2$	26 48901 28269 31200	4*B*
	$4.2^{15}.2^8.S_6(2)$	4870 49291 36640	4*C*
	$4.2^{12}.G_2(4).2$	824 43239 42400	4*D*
(Harada, Norton)	$5\times F$	1 36515 45600 00000	5*A*
(Hall, Janko)	$5^{1+6}.2.HJ$	9 45000 00000	5*B*
(Fischer)	$3\times 2.F_{22}.2$	77474 10198 52800	6*A*
(Suzuki)	$6.Sz$	269 00729 85600	6*B*
	$2^{1+12}.3^2.U_4(3).2$	48 15794 99520	6*C*
	$2.3^{1+8}.2^{1+6}.U_4(2)$	13 06069 40160	6*D*
	$2.3^{1+4}.2^{1+6}.U_4(2)$	16124 31360	6*E*
	$3\times 2^{1+8}.A_9$	2786 91840	6*F*
(Held)	$7\times H$	2 82127 10400	7*A*
	$7^{1+4}.2.A_7$	847 07280	7*B*
	$8.2^7.2^6.U_3(3).2$	7927 23456	8*A*
(Mathieu)	$8.2^{10}.M_{12}$	7785 67680	8*B*
(Tits)	$8\times {}^2F'_4(2)$	1437 69600	8*C*
	$8.2^9.2^4.A_6$	235 92960	8*D*
	$[2^{22}3]$	125 82912	8*E*
	$8.2^6.U_3(3)$	30 96576	8*F*
	$9.3^{1+4}.S_4(3)$	566 87040	9*A*
	$[2^4 3^{11}]$	28 34352	9*B*
(Higman, Sims)	$5\times 2.HS.2$	8870 40000	10*A*
	$5\times 2^{1+8}.(A_5\times A_5).2$	184 32000	10*B*
	$2.5^{1+4}.2^{1+4}.A_5$	120 00000	10*C*
(Hall, Janko)	$5\times 2.HJ$	60 48000	10*D*
	$2.5^{1+2}.2^{1+4}.A_5$	4 80000	10*E*
(Mathieu)	$11\times M_{12}$	10 45440	11*A*

Thompson Moonshine

Note that the centralizer of $[3C]_M$ is the Thompson group Th

To look for a RCFT exhibiting the Th moonshine, all we need is to find a good RCFT with Z_3 symmetry. However we already used up Z_3 parafermion theory to obtain an RCFT exhibiting $3.Fi_{24}$ Moonshine.

Instead, let us consider Z_9 parafermion theory. One can show that a set of linear combination of Z_9 parafermion characters is closed under SL(2,Z) action.

$$f_0(\tau) = \phi^{(9)}_{9,9}(\tau) + \phi^{(9)}_{9,3}(\tau) + \phi^{(9)}_{9,-3}(\tau)$$

$$f_1(\tau) = \phi^{(9)}_{2,0}(\tau) + \phi^{(9)}_{7,3}(\tau) + \phi^{(9)}_{7,-3}(\tau)$$

$$f_2(\tau) = \phi^{(9)}_{4,0}(\tau) + \phi^{(9)}_{5,3}(\tau) + \phi^{(9)}_{5,-3}(\tau)$$

$$f_3(\tau) = \phi^{(9)}_{6,0}(\tau) + \phi^{(9)}_{3,3}(\tau) + \phi^{(9)}_{3,-3}(\tau)$$

$$f_4(\tau) = \phi^{(9)}_{8,0}(\tau) + \phi^{(9)}_{8,6}(\tau) + \phi^{(9)}_{8,-6}(\tau)$$

Thompson Moonshine

Since the linear combination of characters all have m=0 mod 3, Z_9 symmetry reduces to Z_3 symmetry on these characters f_i (i=0,1,2,3,4).

$$f_0(\tau) = \phi^{(9)}_{9,9}(\tau) + \phi^{(9)}_{9,3}(\tau) + \phi^{(9)}_{9,-3}(\tau)$$

$$f_1(\tau) = \phi^{(9)}_{2,0}(\tau) + \phi^{(9)}_{7,3}(\tau) + \phi^{(9)}_{7,-3}(\tau)$$

$$f_2(\tau) = \phi^{(9)}_{4,0}(\tau) + \phi^{(9)}_{5,3}(\tau) + \phi^{(9)}_{5,-3}(\tau)$$

$$f_3(\tau) = \phi^{(9)}_{6,0}(\tau) + \phi^{(9)}_{3,3}(\tau) + \phi^{(9)}_{3,-3}(\tau)$$

$$f_4(\tau) = \phi^{(9)}_{8,0}(\tau) + \phi^{(9)}_{8,6}(\tau) + \phi^{(9)}_{8,-6}(\tau)$$

Thus one can naturally expects that the characters $f_i(\tau)$ of the dual RCFT with c = 248/11 could exhibit the moonshine for the Thompson group, which is the centralizer of [3C]

$$j(\tau) - 744 = \sum_{i=0}^{4} f_i(\tau)\tilde{f}_i(\tau)$$

Thompson Moonshine

One can show that the dual characters are solutions to the 5th MLDE below

$$\left[D_\tau^5 - \frac{413\pi^2}{396}E_4(\tau)D_\tau^3 + \frac{845\pi^3}{792}E_6(\tau)D_\tau^2 - \frac{861871\pi^4}{1724976}E_4^2(\tau)D_\tau + \frac{3912448\pi^5}{39135393}E_4(\tau)E_6(\tau)\right]\tilde{f}(\tau) = 0$$

Five independent solutions can be expanded in powers of q as follows

$$\tilde{f}_0(\tau) = q^{-\frac{1}{24}\frac{248}{11}}\left(1 + 30876q^2 + 2634256q^3 + \mathcal{O}(q^4)\right)$$

$$\tilde{f}_1(\tau) = q^{\frac{20}{11}-\frac{1}{24}\frac{248}{11}}\left(30628 + 3438240q + 132944368q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_2(\tau) = q^{\frac{16}{11}-\frac{1}{24}\frac{248}{11}}\left(4123 + 961248q + 49925748q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_3(\tau) = q^{\frac{21}{11}-\frac{1}{24}\frac{248}{11}}\left(61256 + 5955131q + 216162752q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_4(\tau) = q^{\frac{13}{11}-\frac{1}{24}\frac{248}{11}}\left(248 + 147498q + 10107488q^2 + \mathcal{O}(q^3)\right)$$

Thompson Moonshine

As expected, the lowest order coefficients **248**, **4123, 30628** and **61256** are indeed dimensions of irreducible representation of the Thompson group. Other coefficients have decomposition into irreducible representations.

$$\tilde{f}_0(\tau) = q^{-\frac{1}{24}\frac{248}{11}}\left(1 + 30876q^2 + 2634256q^3 + \mathcal{O}(q^4)\right)$$

$$\tilde{f}_1(\tau) = q^{\frac{20}{11}-\frac{1}{24}\frac{248}{11}}\left(30628 + 3438240q + 132944368q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_2(\tau) = q^{\frac{16}{11}-\frac{1}{24}\frac{248}{11}}\left(4123 + 961248q + 49925748q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_3(\tau) = q^{\frac{21}{11}-\frac{1}{24}\frac{248}{11}}\left(61256 + 5955131q + 216162752q^2 + \mathcal{O}(q^3)\right)$$

$$\tilde{f}_4(\tau) = q^{\frac{13}{11}-\frac{1}{24}\frac{248}{11}}\left(248 + 147498q + 10107488q^2 + \mathcal{O}(q^3)\right)$$

$$30876 = \mathbf{1} \oplus \mathbf{30875} \qquad 961248 = \mathbf{4123} \oplus \mathbf{957125}$$

Thompson Moonshine

Replacing the parafermion characters with their Z_3 twined ones $f_i^w(\tau)$ leads to the Mckay-Thompson series of $[3C]_M$, which strongly suggests that the dual RCFT has the Thompson group as symmetry.

$$q^{-1}\Big(1+248q^3+4124q^6+\mathcal{O}(q^9)\Big)=\sum_{i=0}^{4} f_i^w(\tau)\tilde{f}_i(\tau) \qquad (w=e^{2\pi i/3})$$

$$||$$

$$j^{3C}(\tau)=\mathrm{Tr}_{V^{\mathbb{M}}}\Big[h\cdot q^{L_0^{\mathbb{M}}-\frac{1}{24}24}\Big]$$

$$(h\in[3C]_{\mathbb{M}})$$

$$f_0^w(\tau)=\phi_{9,9}^{(9)}(\tau)+w\phi_{9,3}^{(9)}(\tau)+w^2\phi_{9,-3}^{(9)}(\tau)$$

$$f_1^w(\tau)=\phi_{2,0}^{(9)}(\tau)+w\phi_{7,3}^{(9)}(\tau)+w^2\phi_{7,-3}^{(9)}(\tau)$$

$$f_2^w(\tau)=\phi_{4,0}^{(9)}(\tau)+w\phi_{5,3}^{(9)}(\tau)+w^2\phi_{5,-3}^{(9)}(\tau)$$

$$f_3^w(\tau)=\phi_{6,0}^{(9)}(\tau)+w\phi_{3,3}^{(9)}(\tau)+w^2\phi_{3,-3}^{(9)}(\tau)$$

$$f_4^w(\tau)=\phi_{8,0}^{(9)}(\tau)+w^2\phi_{8,6}^{(9)}(\tau)+w\phi_{8,-6}^{(9)}(\tau)$$