The geometry of optimal functionals

W. Wei Li and Guan-Lin Lin (1911.xxxx)

W. Wei Li, A. Sinha and A. Zahed (partial results)

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&Taiwan String workshop

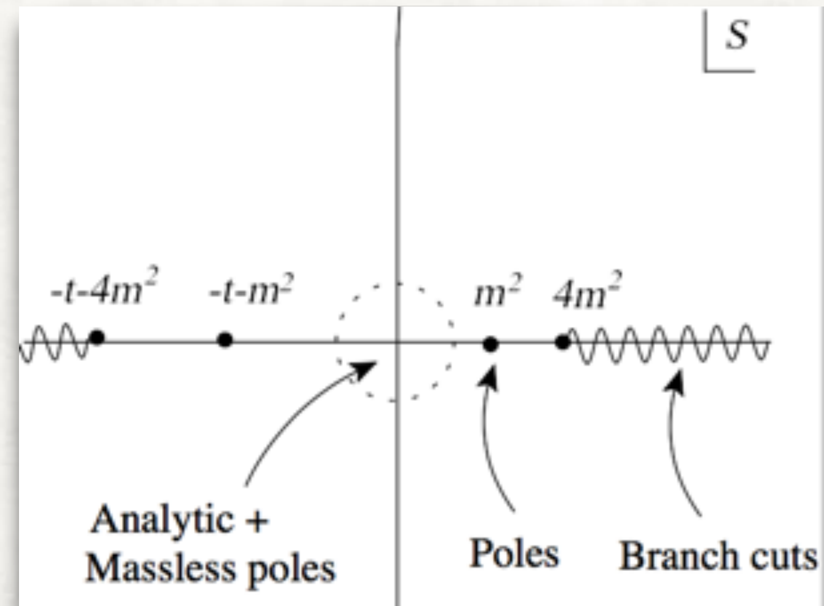
THE BOUNDARY OF THEORY SPACE

We are interested in the structure of theory space: the constraint on the parameters or quantum numbers of the theory derived from fundamental principles (consistency conditions)

- S-matrix bootstrap (60s):

Unitarity + Lorentz invariance + locality +causality,

The definition of some of these properties are not known non-perturbatively



- String Landscape/swampland:

Conjectural (WGC, distance, non-susy AdS, dS,…)

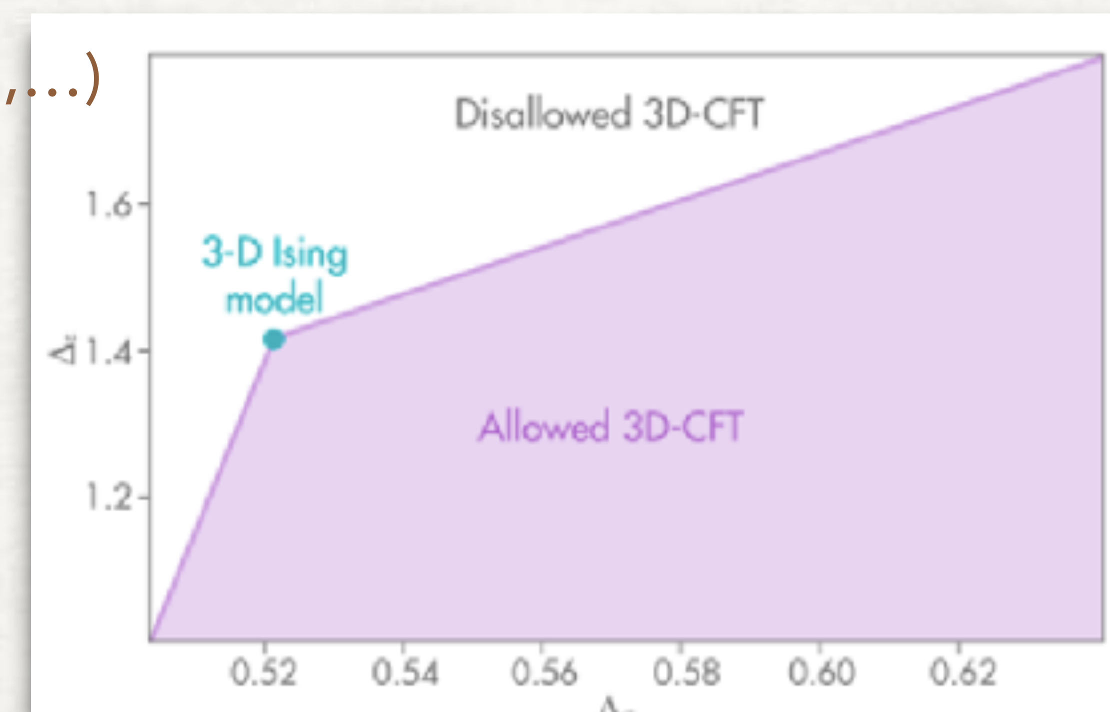
Concrete (anomaly cancellation….)

- CFT bootstrap:

Unitarity + conformal symmetry

Non-perturbative definition

S El-Showk, M. Paulos , D Poland , S Rychkov, D Simmons-Duffine , A. Vichi



THE BOUNDARY OF THEORY SPACE

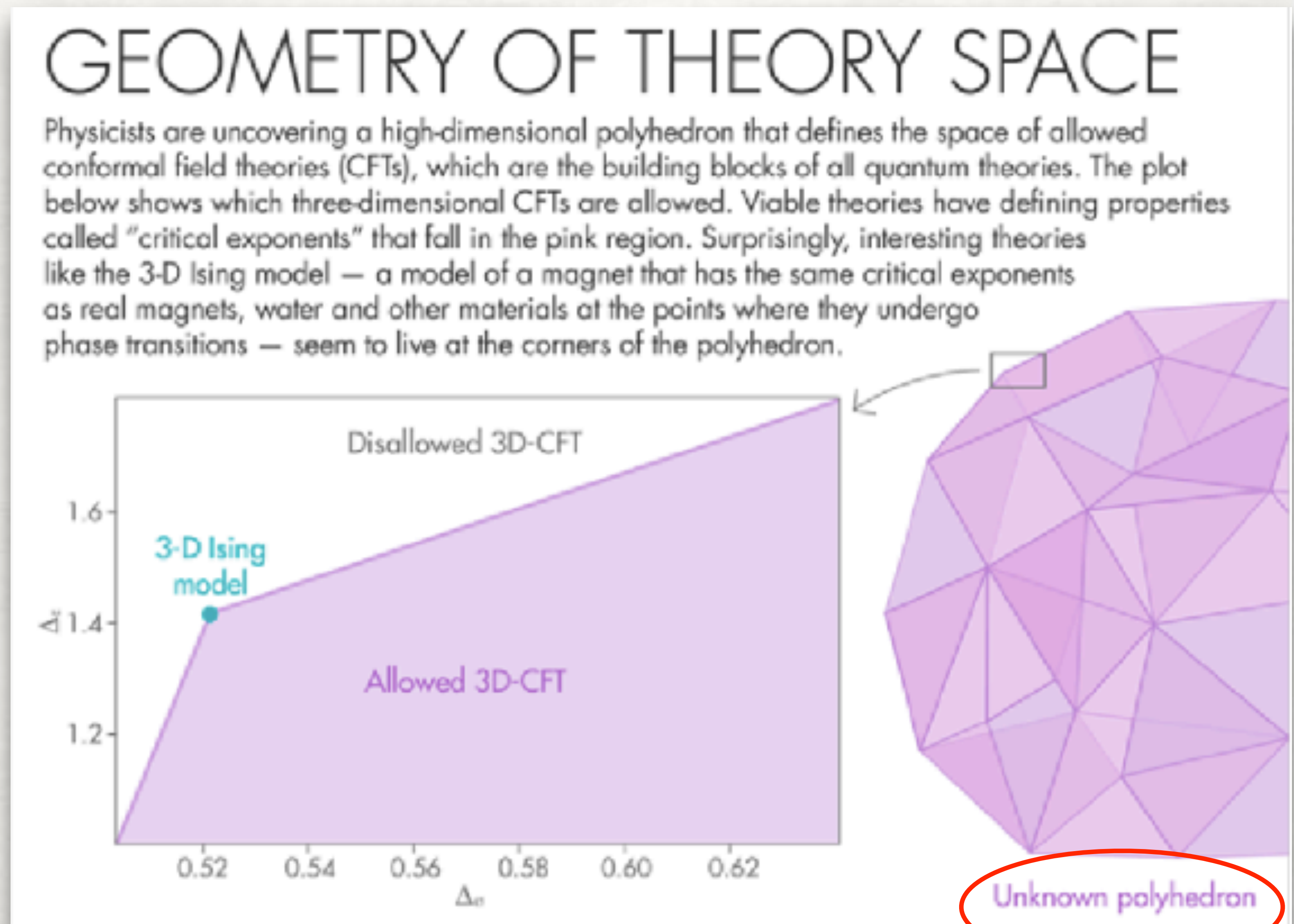
- CFT bootstrap: start with the 4-pt functions

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{G}(z)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}$$

Unitarity + conformal symmetry

$$Unitarity: \ \mathcal{G}(z) = \sum_{\Delta_O \in \phi\times\phi} c^2_{\Delta_O} G_{\Delta_O}(z)$$

$$Crossing: \ \mathcal{G}(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\phi} \mathcal{G}(1-z)$$



Quanta magazine
February 23, 2017

THE BOUNDARY OF THEORY SPACE

- How do we characterize the boundary?

Let's consider 1 D CFT as an example

$$G_\Delta(z) = z^\Delta {}_2F_1(\Delta, \Delta, 2\Delta, z)$$

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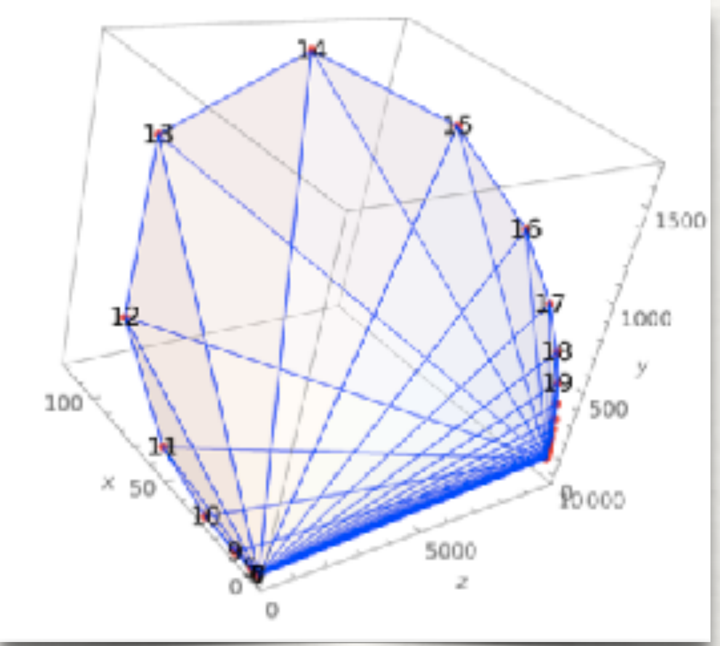
let's consider the Taylor expansion

$$\mathcal{G}(z) \to \vec{\mathcal{G}} = \begin{pmatrix} \mathcal{G}^0 \\ \mathcal{G}^1 \\ \mathcal{G}^2 \\ \vdots \\ \mathcal{G}^n \end{pmatrix}$$

$$G_\Delta(z) \to \vec{\mathbf{G}}_\Delta = \begin{pmatrix} G^0_\Delta \\ G^1_\Delta \\ G^2_\Delta \\ \vdots \\ G^n_\Delta \end{pmatrix}$$

The four point function lives in the convex hull of block vectors



Crossing says that the four-point function is on a sub plane

the crossing plane

$$\mathcal{G}^1 = 4\Delta_\phi \mathcal{G}^0$$

$$\mathcal{G}^3 = \frac{16}{3}(\Delta_\phi - 4\Delta^3)\mathcal{G}^0 + 4\Delta_\phi \mathcal{G}^2$$

$$\mathcal{G}^5 = \frac{64}{15}\Delta_\phi(32\Delta^4_\phi - 20\Delta^2_\phi + 3)\mathcal{G}^0 - \frac{16}{3}\Delta_\phi(4\Delta^2_\phi - 1)\mathcal{G}^2 + 4\Delta_\phi \mathcal{G}^4$$

THE BOUNDARY OF THEORY SPACE

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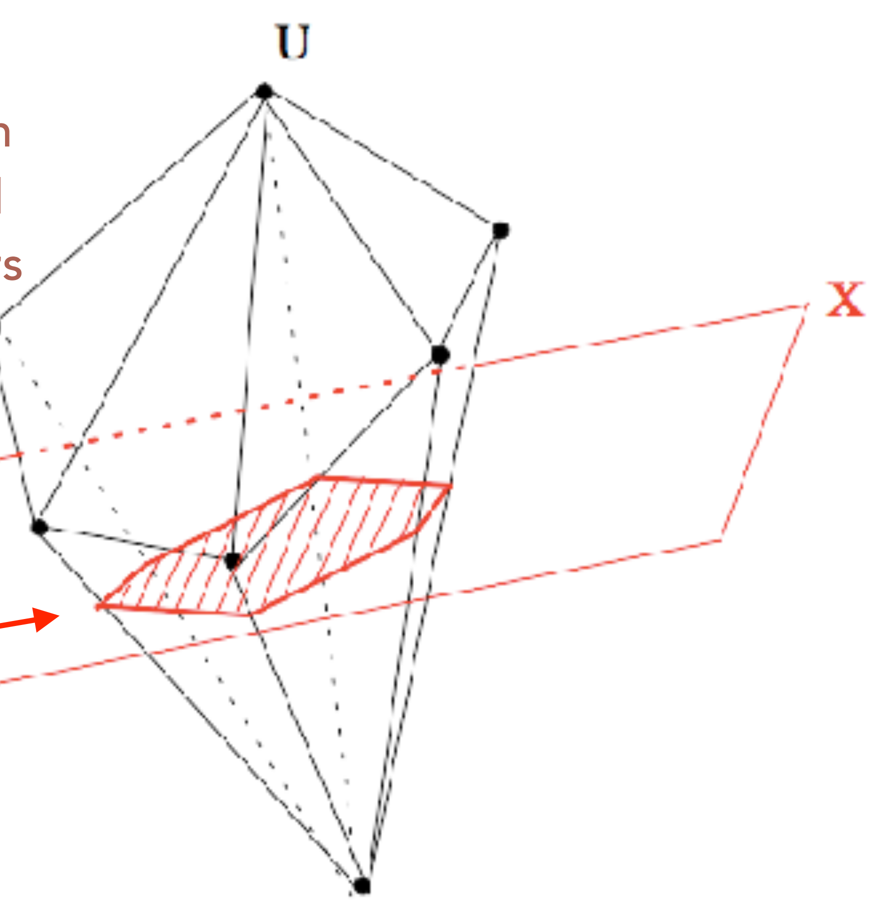

The four point function is consistent if it lies on the intersection between the crossing plane X[Δ**φ**] and the unitarity polytope U[{Δi}].

THE BOUNDARY OF THEORY SPACE

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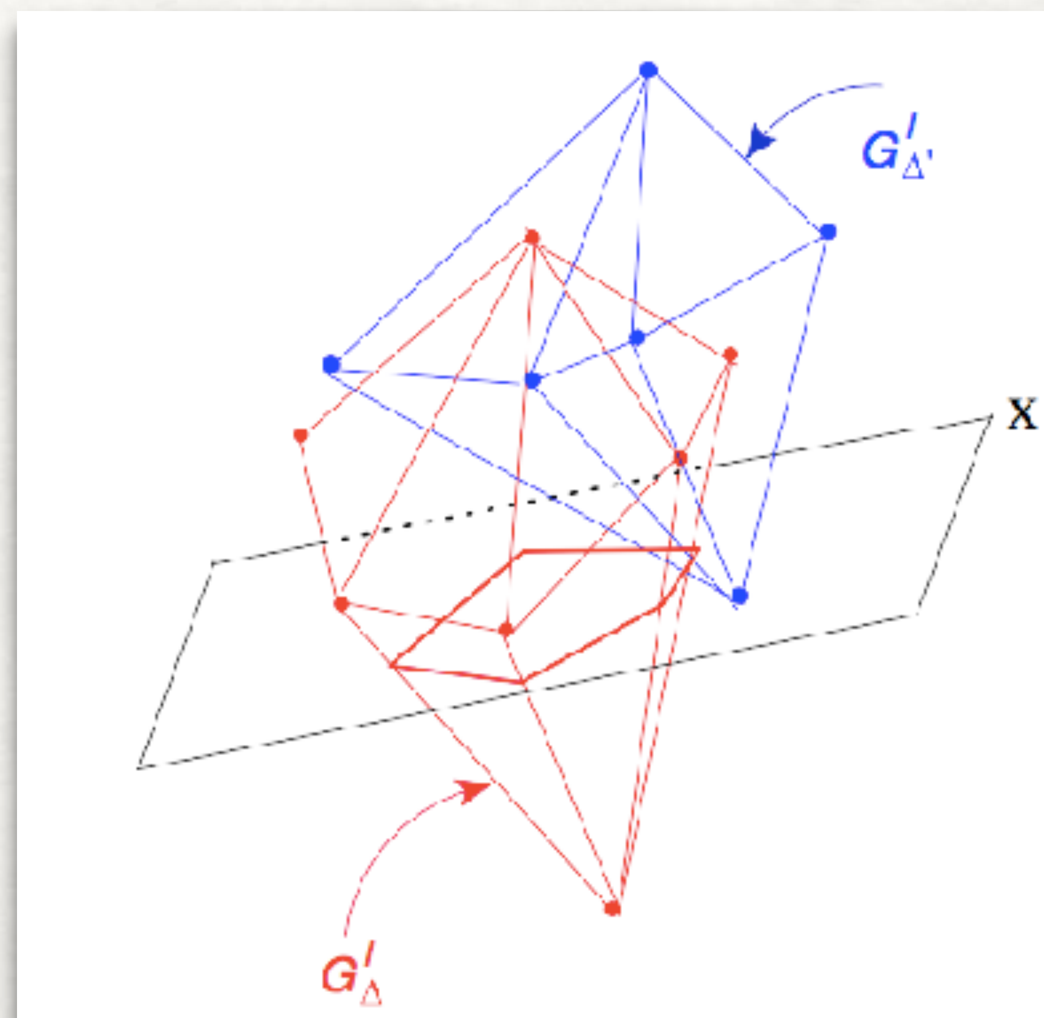


The problem is then finding the boundary of the spectrum for intersection

THE BOUNDARY OF THEORY SPACE

For numerical bootstrap, the boundary is defined through the root of optimal functionals. Reorganize crossing symmetry

$$\sum_{\Delta} c_{\Delta}^{2}\left[z^{-2\Delta_{\phi}}G_{\Delta}(z)-(1-z)^{-2\Delta_{\phi}}G_{\Delta}(1-z)\right]=0 \Rightarrow \sum_{\Delta} c_{\Delta}^{2}F_{\Delta}^{\Delta_{\phi}}(z)=0$$

Consider a linear functional

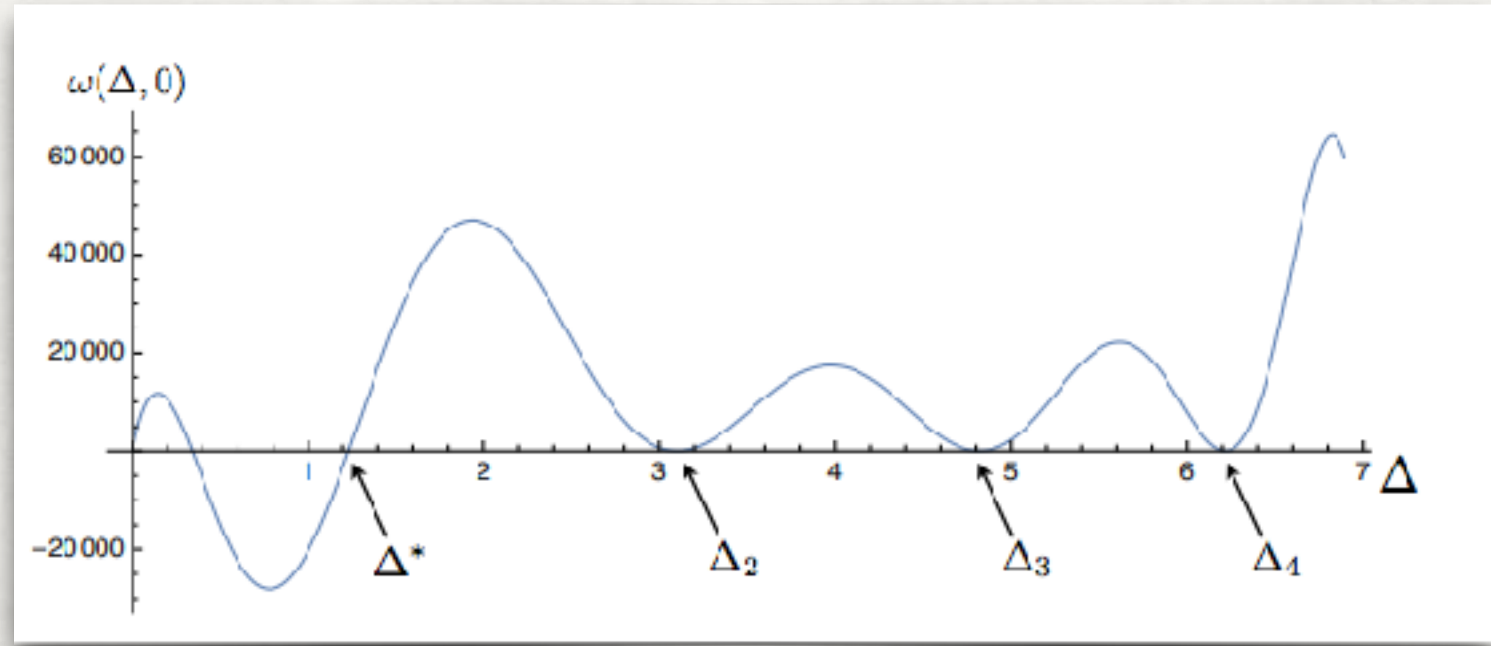
$$\omega(F_{\Delta}^{\Delta_{\phi}})=\left[a_{1}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{a_{3}}{3!}\frac{\mathrm{d}^{3}}{\mathrm{d}z^{3}}+\cdots+\frac{a_{2N+1}}{(2N+1)!}\frac{\mathrm{d}^{2N+1}}{\mathrm{d}z^{2N+1}}\right]F_{\Delta}^{\Delta_{\phi}}(z)\Big|_{z=1/2}=\alpha\cdot\mathbf{F}_{\Delta}^{\Delta_{\phi}}$$

when acted on crossing symmetry

$$\omega\left[\sum_{\Delta} c_{\Delta}^{2}F_{\Delta}^{\Delta_{\phi}}(z)\right]=\omega\left[F_{0}^{\Delta_{\phi}}\right]+\sum_{\Delta} c_{\Delta}^{2}\omega\left[F_{\Delta}^{\Delta_{\phi}}\right]=0$$

Thus

$$\omega(F_{\Delta}^{\Delta_{\phi}})=$$



The optimal functional is the functional whose last single root is the lowest. The optimal

$$\alpha=(a_{1},a_{3},\ldots,a_{2N+1})$$

Dalimil Mazac

THE BOUNDARY OF THEORY SPACE

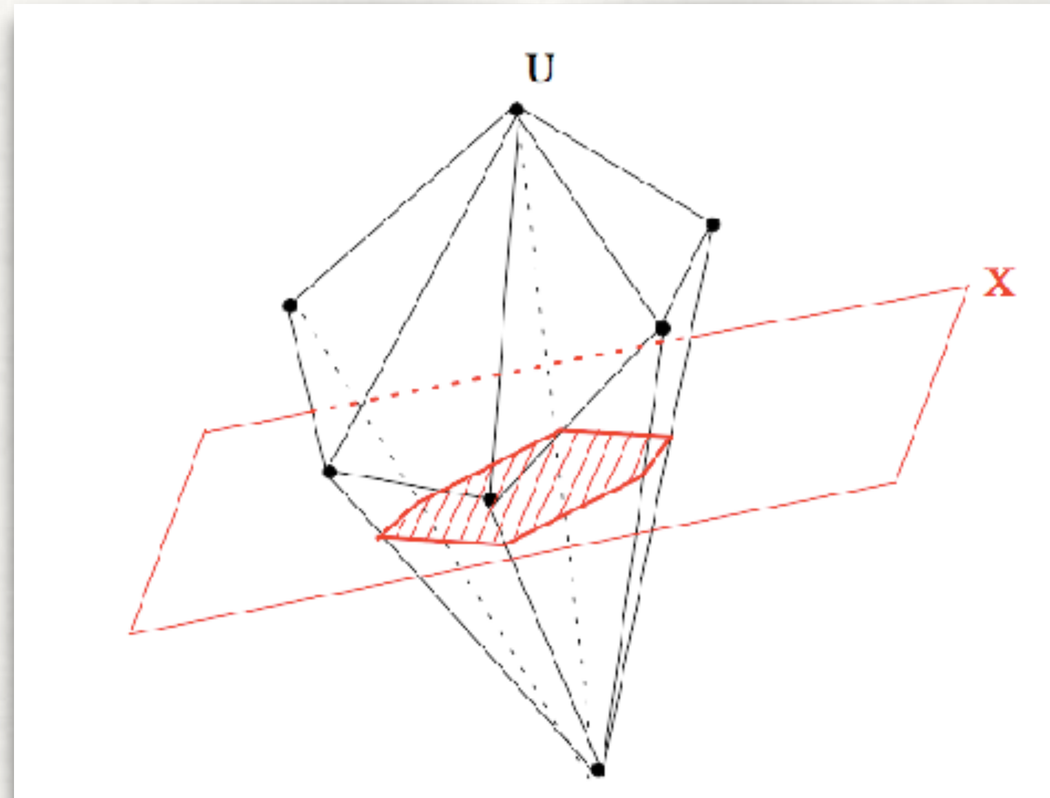
- The original problem is geometric, the boundary of its solution is non-geometric ??
- There are strong indications that there is deep structure behind the optimal functionals

Optimal functional for modular bootstrap chiral algebra $U(1)^c$ maps to the linear programing bound for sphere packing density with $d=2c$

Thomas Hartman, Dalimil Mazac, Leonardo Rastelli

THE GEOMETRY OF THE BOUNDARY

Let us go back to the original geometric picture: we are asking for a unitary polytope that intersects with the crossing plane



The unitary polytope is special, it is a cyclic polytope!

N. Arkani-Hamed, S. H. Shao, YTH

The vectors of the convex hull form positive determinants when ordered

$$\langle \mathbf{G}_{\Delta_1}, \mathbf{G}_{\Delta_2}, \cdots, \mathbf{G}_{\Delta_n} \rangle > 0, \quad \Delta_1 < \Delta_2 < \cdots < \Delta_n$$

This immediately leads to the conclusion that the boundaries are known

$$\begin{aligned} d \in \ & odd: \quad (0, \Delta_i, \Delta_{i+1}, \Delta_j, \Delta_{j+1}, \ldots) \cup (\Delta_i, \Delta_{i+1}, \Delta_j, \Delta_{j+1}, \ldots, \infty) \\ d \in \ & even: \quad (\Delta_i, \Delta_{i+1}, \Delta_j, \Delta_{j+1}, \ldots) \end{aligned}$$

where

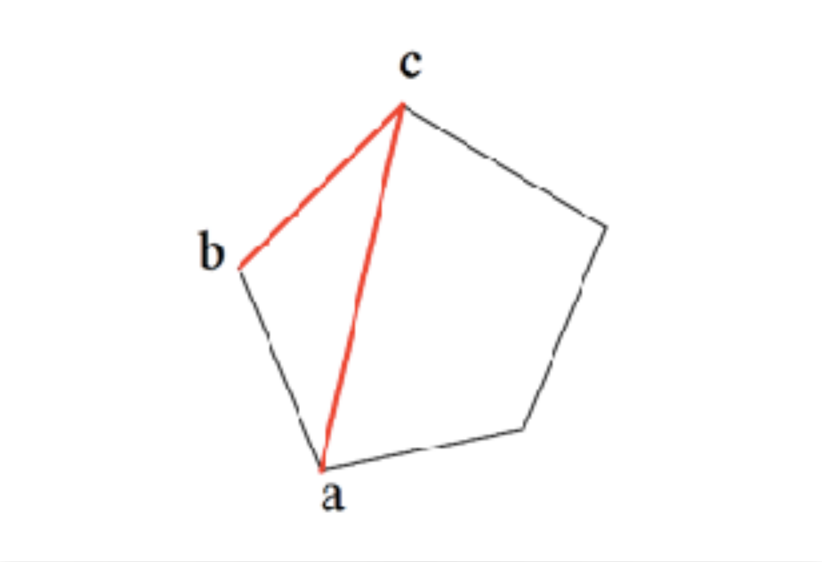
$$(\Delta_i, \Delta_{i+1}) \rightarrow (\Delta_i, \dot{\Delta}_i)$$

THE GEOMETRY OF THE BOUNDARY

The boundaries of a convex hull:

$$A^I = w_1 U_1^I + \cdots + w_n U_n^I, \quad w_i > 0, \sum_{i=1}^{n} w_i = 1$$

The line (bc) is a boundary but (ac) is not. This is because while the hull (A) is ONLY on one side of (bc), it can be on either side of (ac), including on (ac)



$$det[A, U_a, U_c] = 0$$

Thus for boundaries if must satisfy

$$det[A, U_i, U_j] > 0, \quad \forall A$$

THE GEOMETRY OF THE BOUNDARY

A new form of positivity trivializes the problem!

Let's say we have a set of vectors, with a well defined ordering. If its ordered determinant is positive:

$$det[U_{i_1}, U_{i_2}, \cdots, U_{i_k}] > 0, \quad \forall\, i_1 < i_2 < i_3 \cdots < i_k$$

The convex hull of U_i is a cyclic polytope

- It's boundaries are known

$$d = 2: \ (i, i+1), \quad d = 3: \ (0, i, i+1), (i, i+1, \infty), \quad d = 4: (i, i+1, j, j+1) \cdots$$

Exp: Let $A = aU_6 + bU_9$ with $a, b > 0$

$$\begin{aligned} Det[A, U_4, U_5, U_7, U_8] &= a\, Det[U_6, U_4, U_5, U_7, U_8] + b\, Det[U_9, U_4, U_5, U_7, U_8] \\ &= a\, Det[U_4, U_5, U_6, U_7, U_8] + b\, Det[U_4, U_5, U_7, U_8, U_9] \\ &= a\,(positive) + b\,(positive) \end{aligned}$$

THE GEOMETRY OF THE BOUNDARY

Let us go back to the original geometric picture: we are asking for a unitary polytope that intersects with the crossing plane

Consider d=2N+1, the crossing plane is N-dimensional

N=2

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 4\Delta_\phi & 0 & 0 \\ 0 & 1 & 0 \\ \frac{16}{3}(\Delta_\phi - 4\Delta_\phi^3) & 4\Delta_\phi & 0 \\ 0 & 0 & 1 \\ \frac{64}{15}\Delta_\phi(32\Delta_\phi^4 - 20\Delta_\phi^2 + 3) & \frac{16}{3}(\Delta_\phi - 4\Delta_\phi^3) & 4\Delta_\phi \end{pmatrix}$$

The crossing plane intersects the an N dimensional face of the polytope

$$(\Delta_{i_1}, \Delta_{i_1+1}, \Delta_{i_2}, \Delta_{i_3}, \ldots, \Delta_{i_N})$$

at a point $\mathbf{A}$,

$$\mathbf{A} = \mathbf{G}_{\Delta_{i_1}} \langle \Delta_{i_1+1}, \Delta_{i_2}, \ldots, \Delta_{i_N}, \mathbf{X} \rangle - \mathbf{G}_{\Delta_{i_1+1}} \langle \Delta_{i_1}, \Delta_{i_1+1} \ldots, \Delta_{i_N}, \mathbf{X} \rangle + \ldots$$
$$+ (-1)^N \mathbf{G}_{i_N} \langle \Delta_{i_1}, \Delta_{i_1+1}, \ldots, \Delta_{i_N-1}, \mathbf{X} \rangle$$

for the point to be inside the polytope

$$\langle \Delta_{i_1+1}, \Delta_{i_2}, \ldots, \Delta_{i_N}, \mathbf{X} \rangle,\ (-1)^N \langle \Delta_{i_2}, \Delta_{i_3} \ldots, \Delta_{i_N}, \Delta_{i_1}, \mathbf{X} \rangle,$$
$$\langle \Delta_{i_3}, \cdots, \Delta_{i_N}, \Delta_{i_1}, \Delta_{i_1+1}, \mathbf{X} \rangle, \quad \textit{same sign}$$

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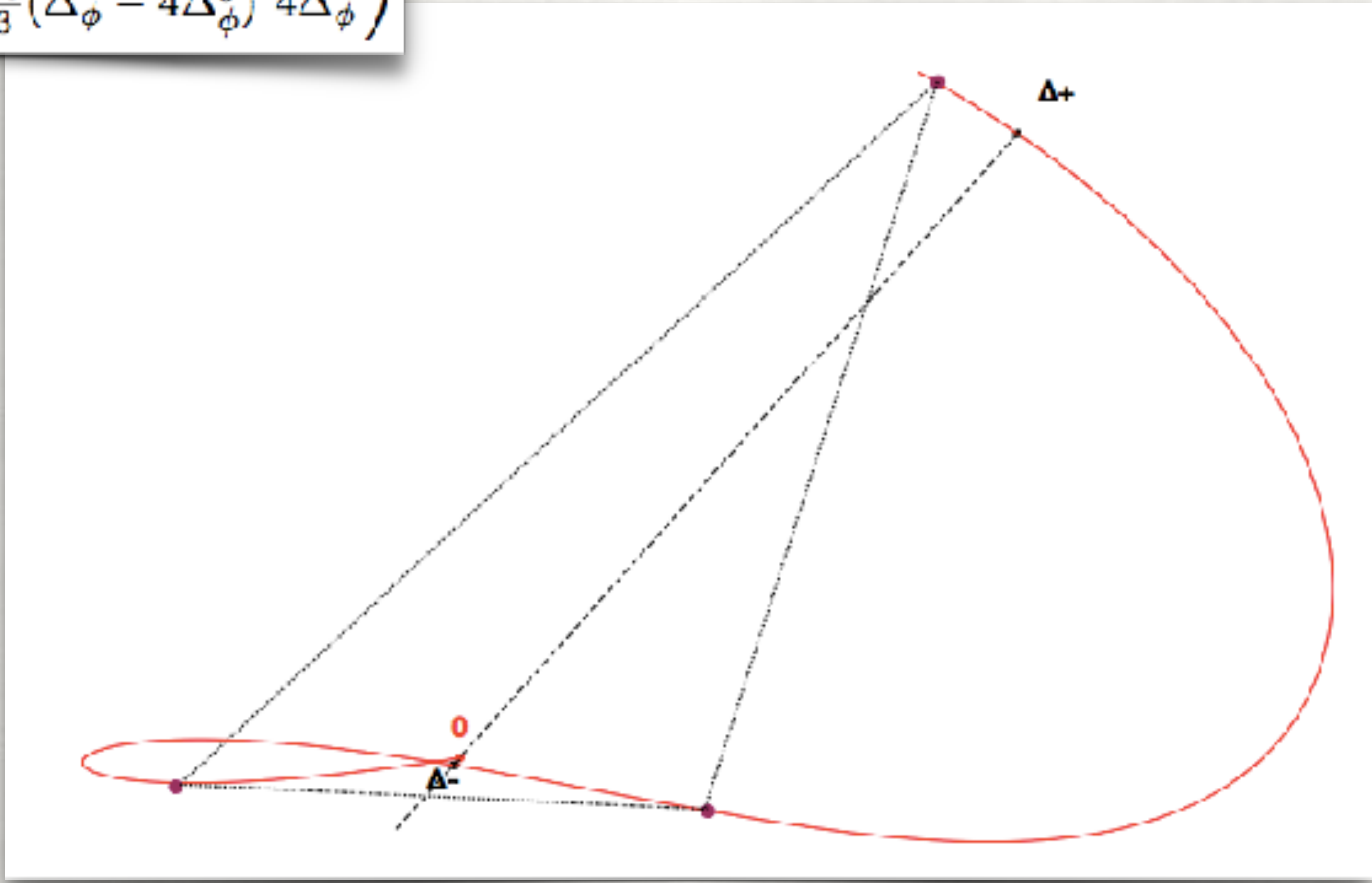
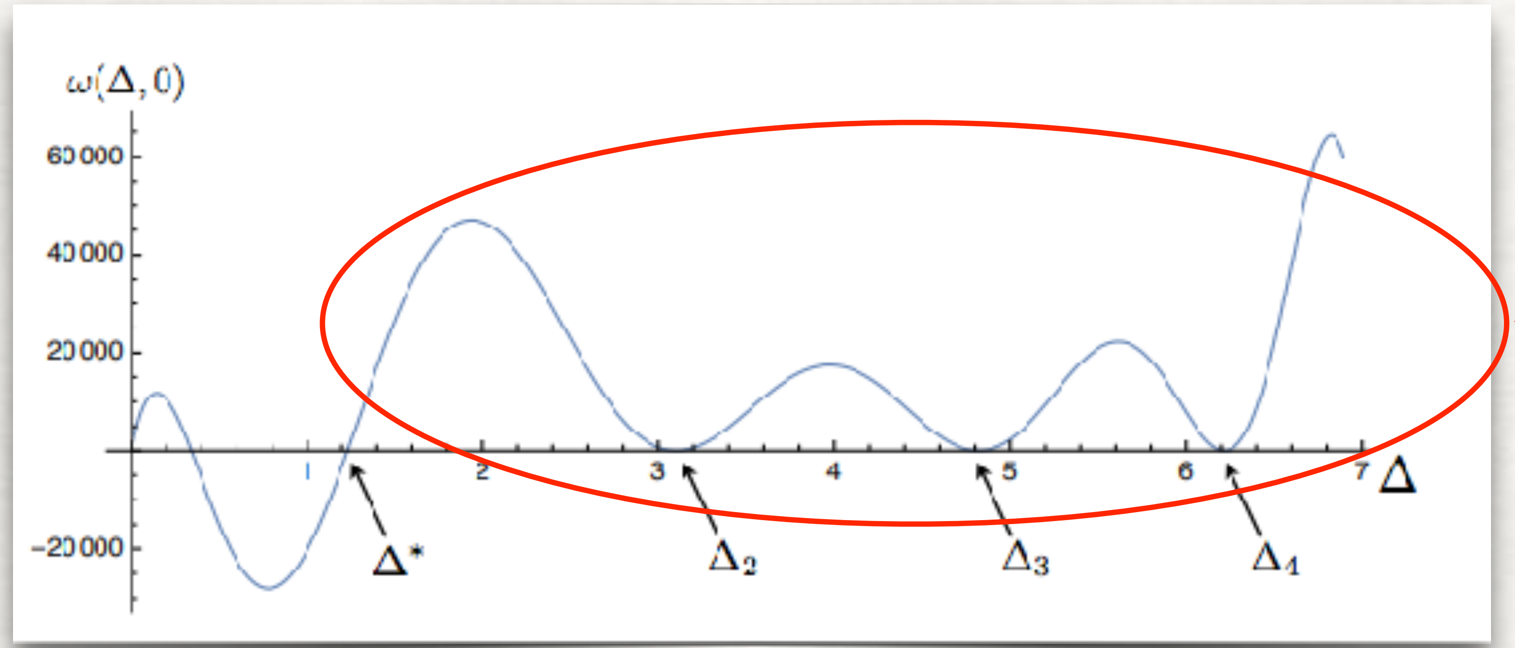
After projecting through (X,0), the geometry is N dim and we are seeking N+1 points on the block curve such that origin is enclosed.



THE GEOMETRY OF THE BOUNDARY

What does this have to do with optimal functionals ?

Recall that



$\omega(F_{\Delta}^{\Delta_\phi})$ >0

$\Delta > \Delta_*$,

From the fact that the vectors form a cyclic polytope, we conjecture the functional for $N = 2k+1$ is

$$\omega(F_{\Delta}^{\Delta_\phi}) = \langle \mathbf{X}, 0, \Delta_{i_1}, \Delta_{i_1}+1, \Delta_{i_2}, \Delta_{i_2}+1, \ldots, \Delta_{i_k}, \Delta_{i_k}+1, \Delta \rangle$$

We need a collection of k points to define the functional.

The geometry at fixed N is N-dimensional when projected through X,0. We are looking for a set of N+1 vertices to enclose the origin. For the gap, we must be on a degenerate simplex where the gap is one of it's vertices. Thus we are looking for k points and the gap (k+1 points on the curve) that form an k-dimensional sub plane!

THE GEOMETRY OF THE BOUNDARY

What does this have to do with optimal functionals ?

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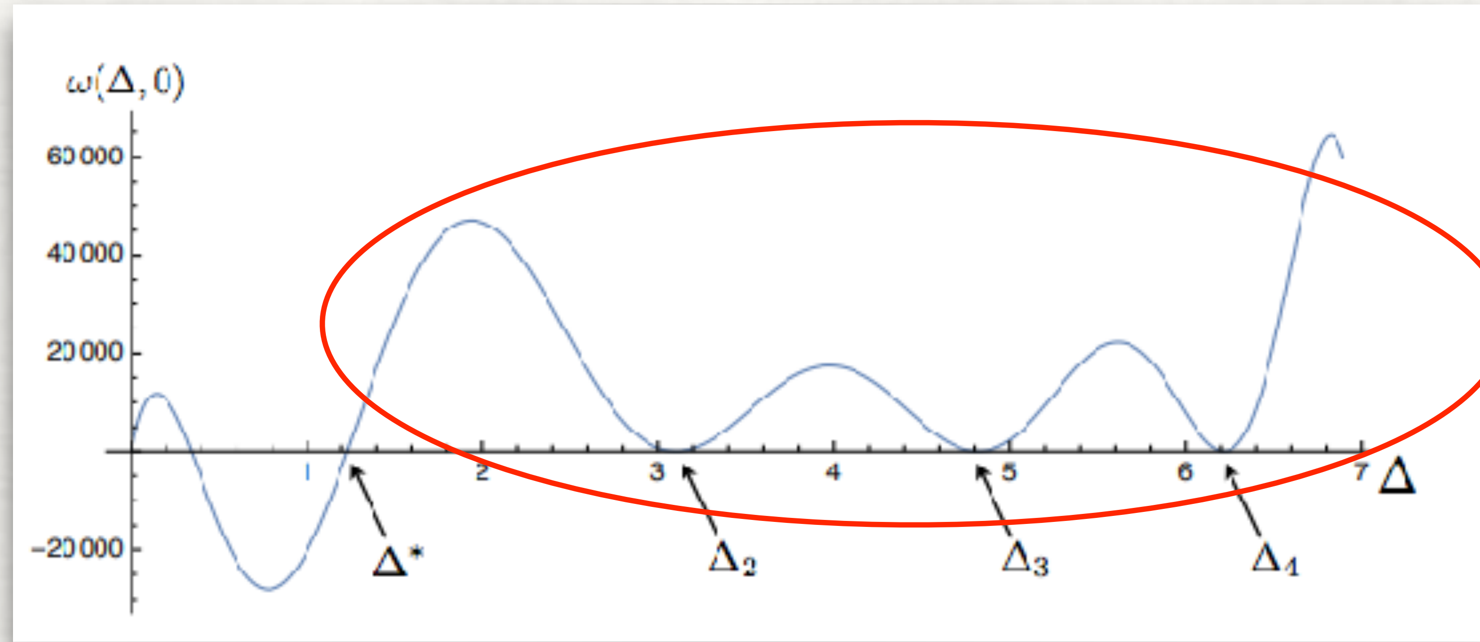

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We need a collection of k points to define the functional.

Thus we are looking for k points and the gap (k+1 points on the curve) that form an k-dimensional sub plane! This allows us to solve the functional at any derivative order with arbitrary precision!

$$For\ arbitrary\ \mathbf{v}_1\ , \mathbf{v}_2, \ldots, \mathbf{v}_k,\ \langle \mathbf{X}, 0, \Delta_{gap}, \Delta_1, \Delta_2, \ldots, \Delta_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \rangle = 0$$

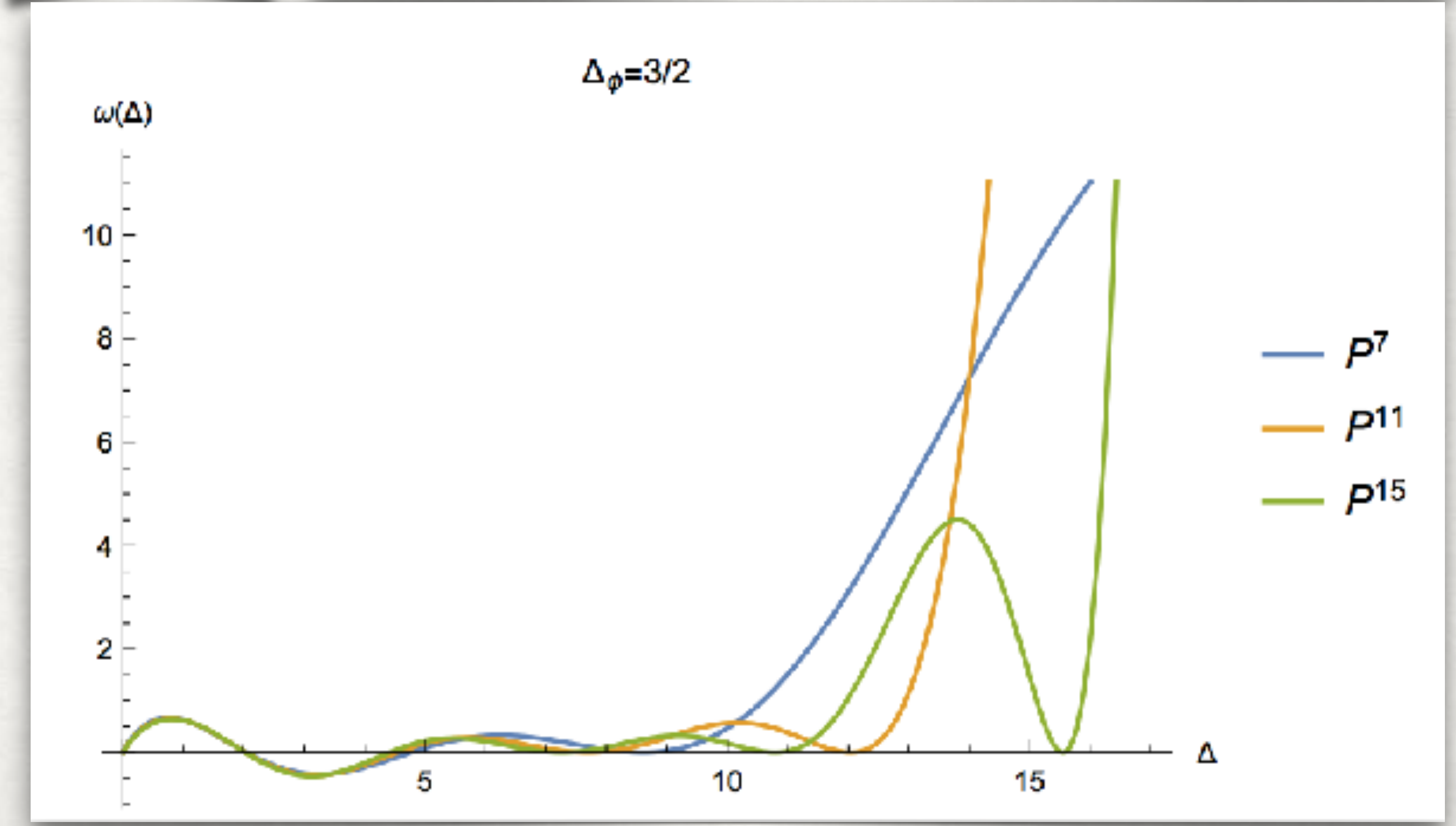
THE GEOMETRY OF THE BOUNDARY

For 2N+1 derivative expansion, after projecting through the crossing plane and 0, the collection of k+1 (N=2k+1) points that enclose the origin is unique, and gives the optimal functional

$$\omega(F_{\Delta}^{\Delta_\phi}) = \langle \mathbf{X}, 0, \Delta_{i_1}, \Delta_{i_1+1}, \Delta_{i_2}, \Delta_{i_2+1}, \ldots, \Delta_{i_k}, \Delta_{i_k+1}, \Delta \rangle$$

for N=2k the optimal functional is

$$\omega(F_{\Delta}^{\Delta_\phi}) = \langle \mathbf{X}, 0, \Delta_{i_1}, \Delta_{i_1+1}, \ldots, \Delta_{i_{k-1}}, \Delta_{i_{k-1}+1}, \infty, \Delta \rangle$$



Matches with optimal functional from numeric bootstrap

THE GEOMETRY OF THE BOUNDARY

For 2N+1 derivative expansion, after projecting through the crossing plane and 0, the collection of k+1 (N=2k+1) points that enclose the origin is unique, and gives the optimal functional

$$\omega(F_{\Delta}^{\Delta_\psi}) = \langle \mathbf{X}, 0, \Delta_{i_1}, \Delta_{i_1+1}, \Delta_{i_2}, \Delta_{i_2+1}, \ldots, \Delta_{i_k}, \Delta_{i_k+1}, \Delta \rangle$$

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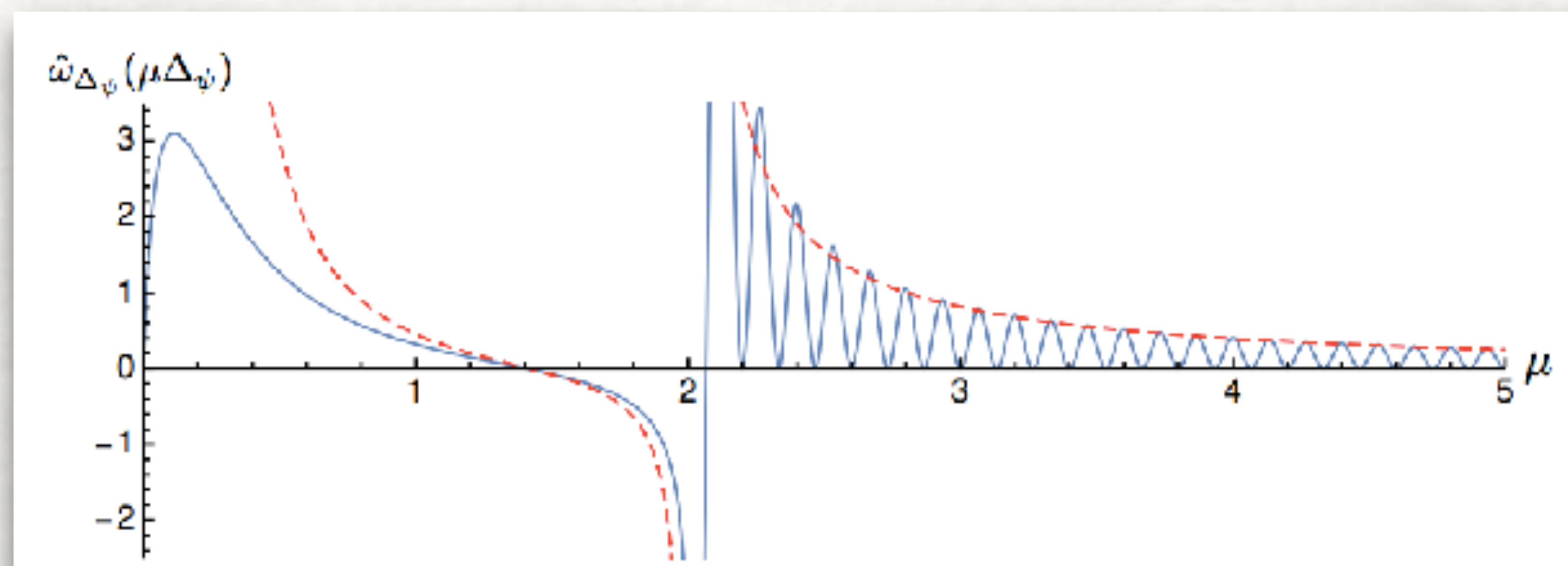
This predicts that at the infinite derivative order, the optimal functional should yield a series of double zero above the gap, which corresponds to the complete spectrum of the theory



Dalimil Mazac

THE GEOMETRY OF HIGHER DIM BOUNDARIES

This simplex picture also allows us to develop a recursive method to carve out the entire spectrum!

For fixed N, a consistent CFT must contain N+1 set of operators that form a simplex containing the origin.

- Any set of operators $\{\Delta_1, \Delta_2, \cdots, \Delta_n, \Delta_{n+1}\}$ that contain the origin.
- There must exists an N subset $\{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ such that when combined with ∞ encloses the origin.

  Denote this set as $\mathbf{S}_n$
- For each element in $\mathbf{S}_n$, consider all Δ_{n+1} such that the resulting tuple enclosed the origin. Denote this set as $\mathbf{T}_n$. The space of all possible vertices for such simplex is $\mathbf{S}_n \cup \mathbf{T}_n$
- Now lets move to N+1. The set $\mathbf{S}_{n+1}$ is now the set of all $\{\Delta_1, \Delta_2, \cdots, \Delta_n, \Delta_{n+1}\}$ such that when combined with ∞, encloses zero

$$\langle \mathbf{X}, 0, \Delta_1, \Delta_2, \cdots, \Delta_n \rangle > 0$$

$$\langle \mathbf{X}, 0, \Delta_1, \Delta_2, \cdots, \Delta_{n-1}, \infty \rangle < 0, \quad \langle \mathbf{X}, 0, \Delta_2, \cdots, \Delta_{n+1}, \infty \rangle > 0, \cdots, \quad \text{e.t.c.}$$

THE GEOMETRY OF HIGHER DIM BOUNDARIES

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- Now lets move to N+1. The set $\mathbf{S}_{n+1}$ is now the set of all such that when combined with ∞ , encloses zero $\{\Delta_1, \Delta_2, \cdots, \Delta_n, \Delta_{n+1}\}$

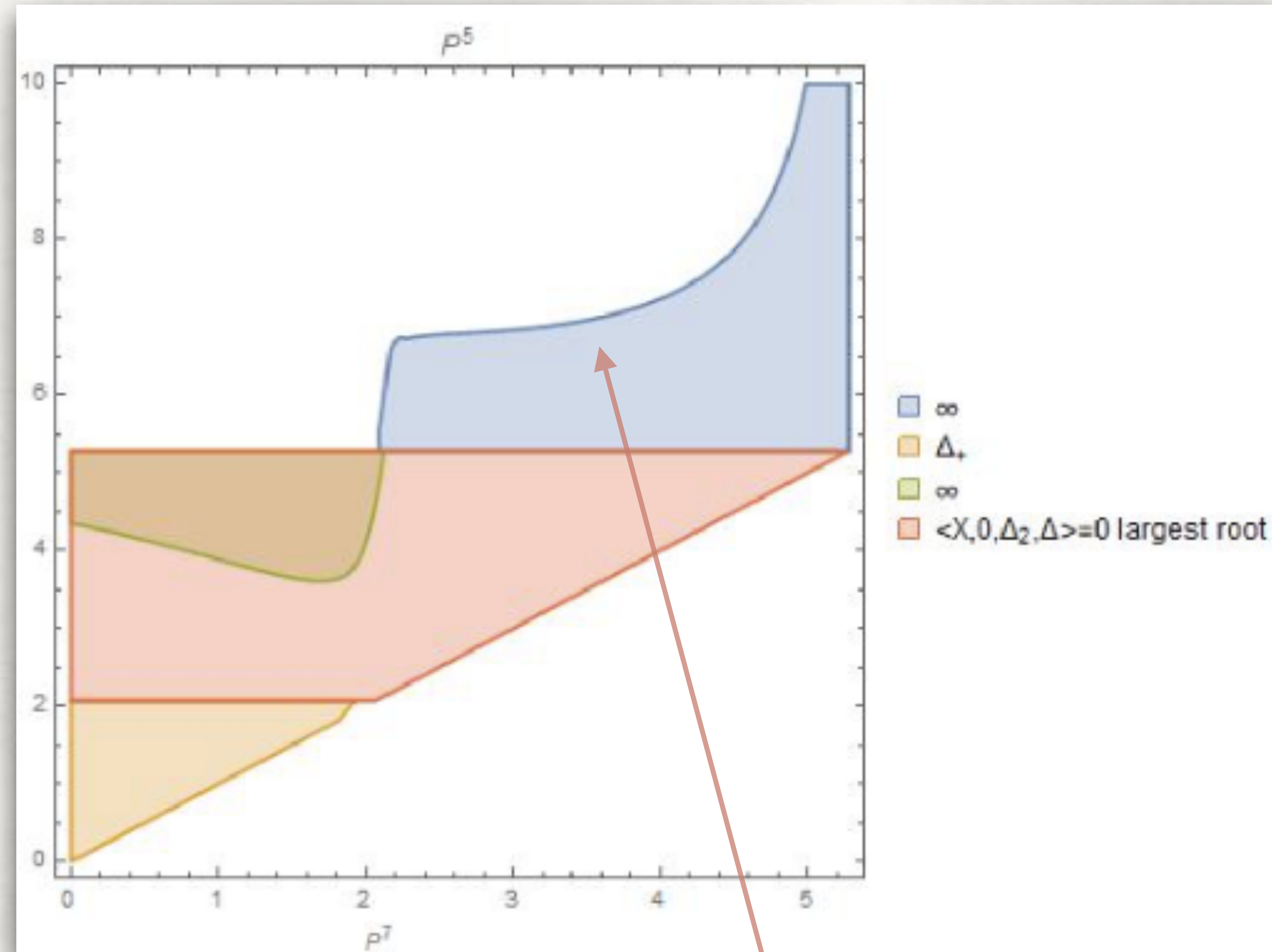
$$\langle \mathbf{X}, 0, \Delta_1, \Delta_2, \cdots, \Delta_n \rangle > 0$$

$$\langle \mathbf{X}, 0, \Delta_1, \Delta_2, \cdots, \Delta_{n-1}, \infty \rangle < 0, \quad \langle \mathbf{X}, 0, \Delta_2, \cdots, \Delta_{n+1}, \infty \rangle > 0, \cdots, \quad \text{e.t.c.}$$

The space of $\mathbf{S}_{n+1}$ is a subspace of $\mathbf{S}_n \cup \mathbf{T}_n$, i.e. $\mathbf{S}_n$ is further constrained

THE GEOMETRY OF HIGHER DIM BOUNDARIES

Let's consider explicit example, up to 5 Bounding ($\Delta_1, \Delta_2, \Delta_3$)

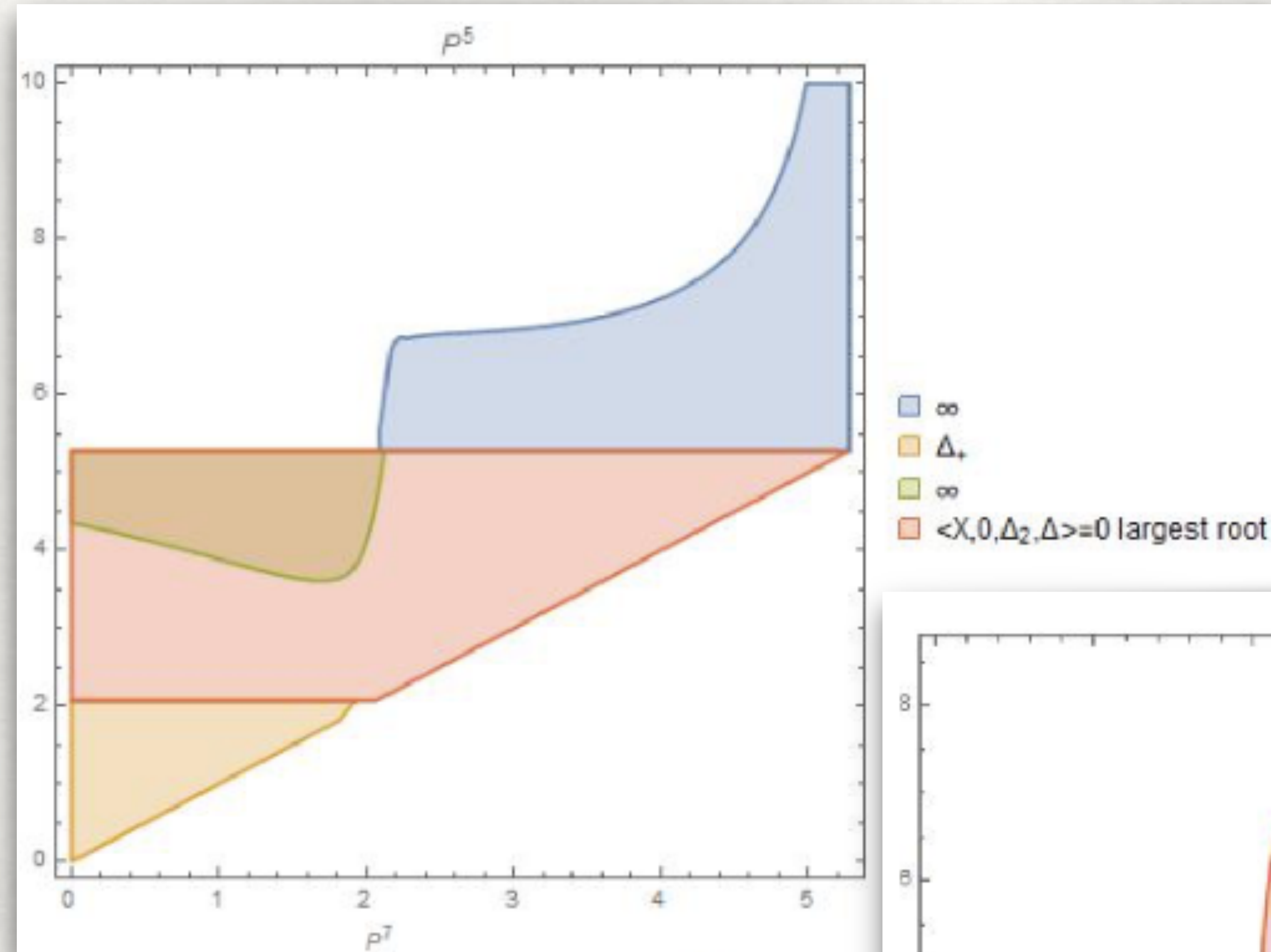


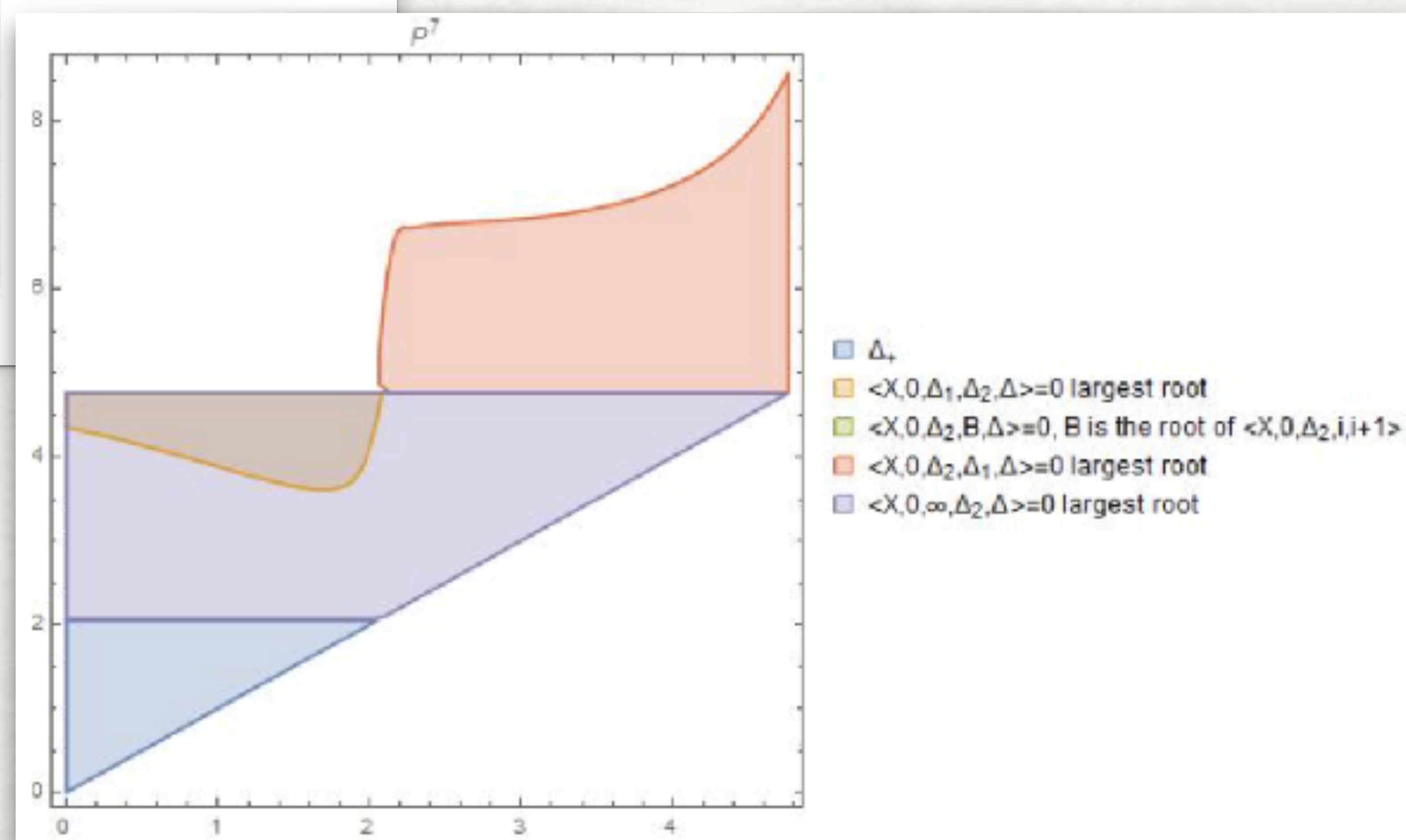
This analytically gives the optimal functional for the next operator

$$sign\left[\frac{\omega\left(F^{\Delta_\phi}_{\Delta_1}\right)}{\omega\left(F_0\right)}\right] = sign\left[\frac{\omega\left(F^{\Delta_\phi}_{\Delta_2}\right)}{\omega\left(F_0\right)}\right] = \cdots = sign\left[\frac{\omega\left(F^{\Delta_\phi}_{\Delta_n}\right)}{\omega\left(F_0\right)}\right] = sign\left[\frac{\omega\left(F^{\Delta_\phi}_{\infty}\right)}{\omega\left(F_0\right)}\right]$$

THE GEOMETRY OF HIGHER DIM BOUNDARIES

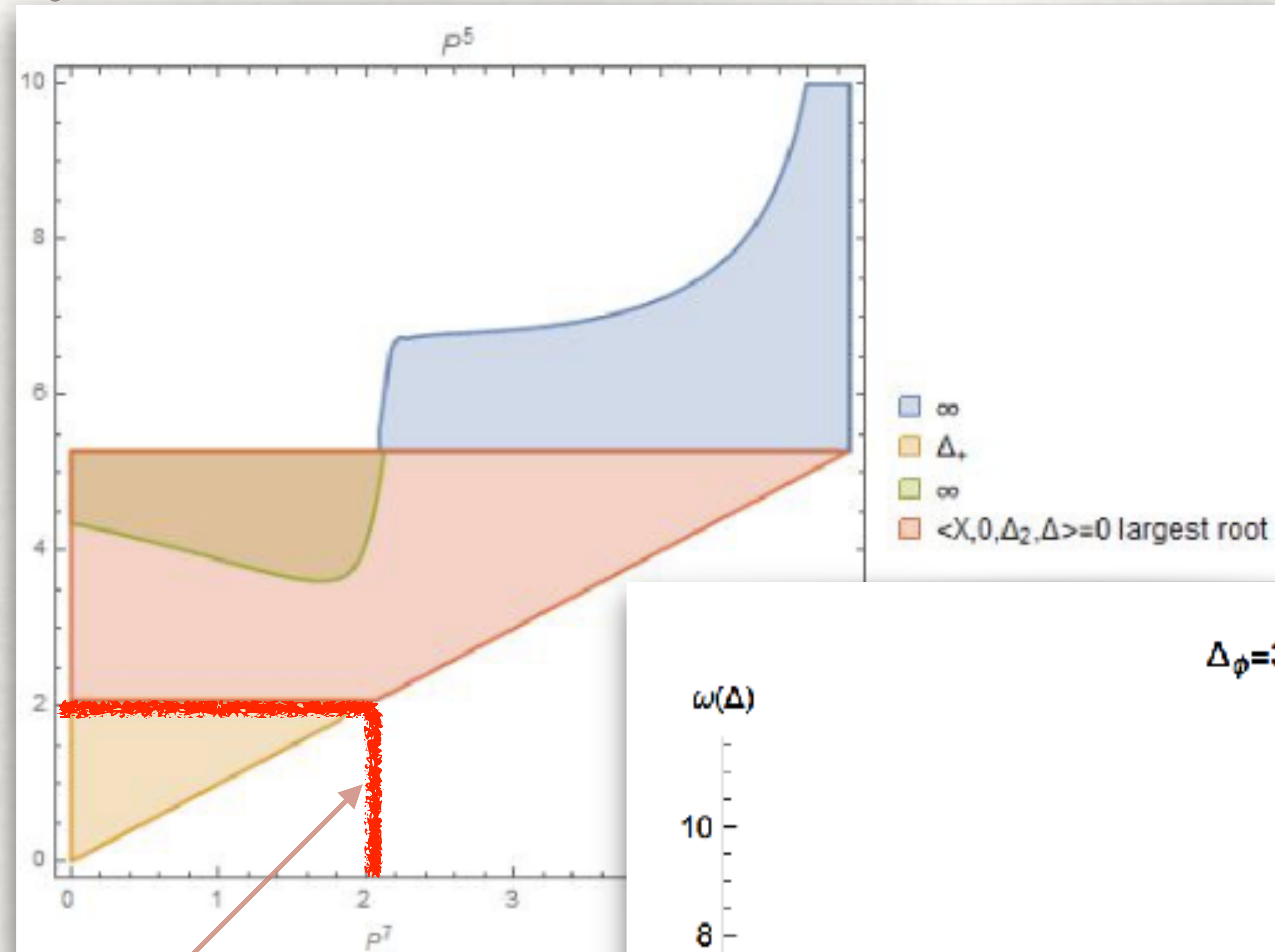
Let's consider explicit example, up to 5, 7 deriv. Bounding (Δ_1,Δ_2,Δ_3)



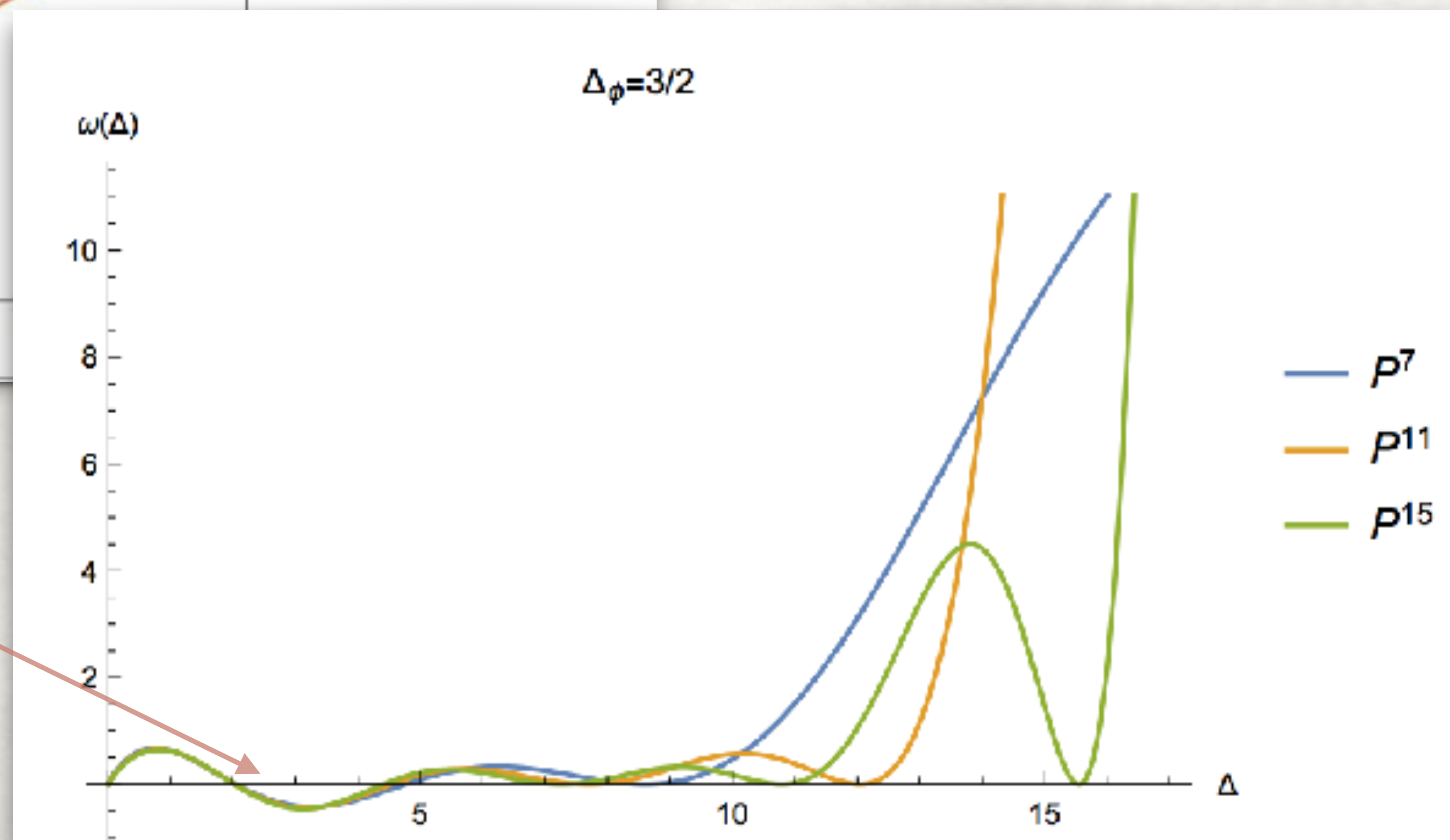


THE GEOMETRY OF HIGHER DIM BOUNDARIES

Let's consider explicit example, up to 5. Bounding ($\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_5\ldots$)



"event horizon"
for operators

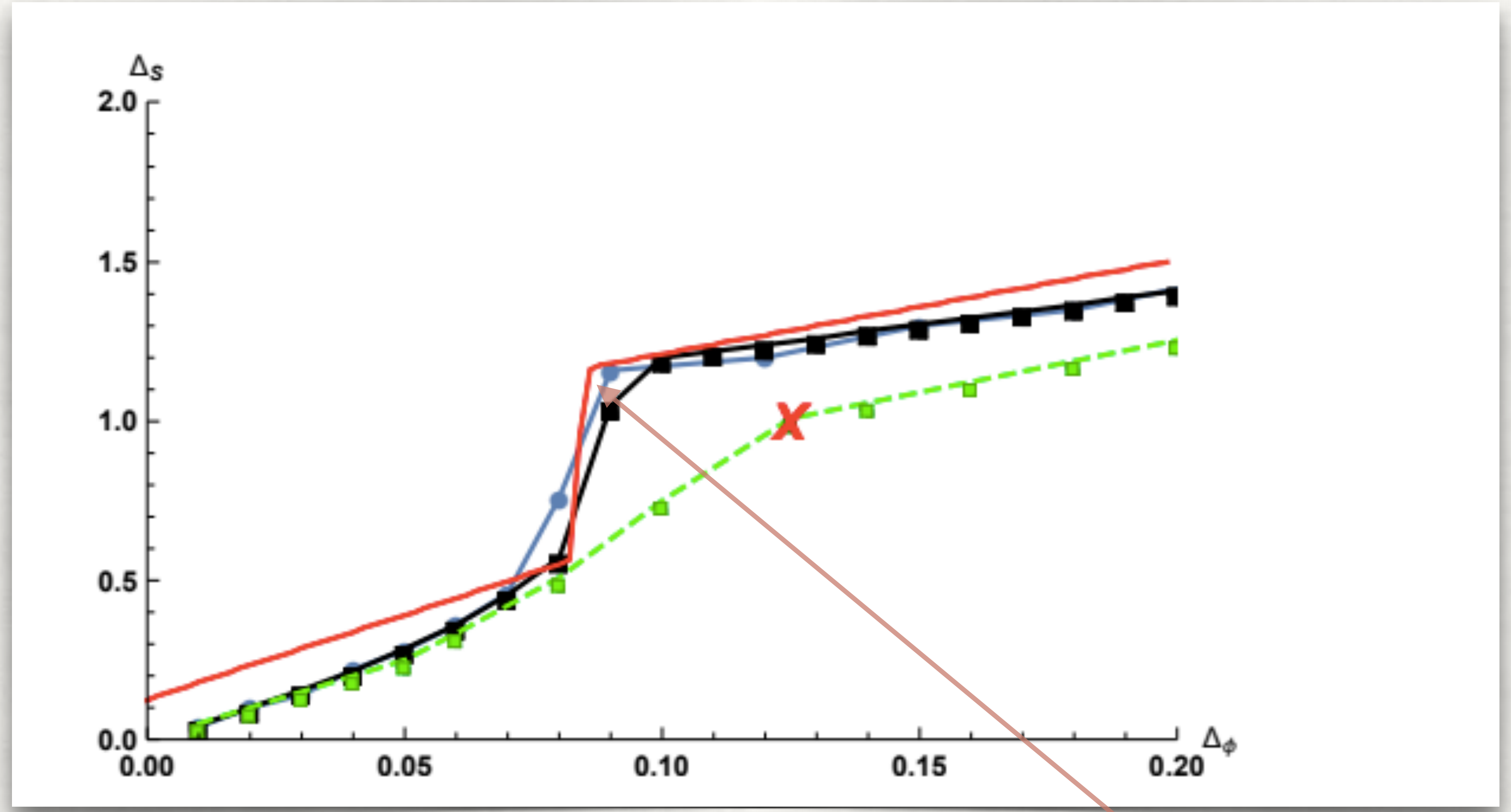


EXTENSIONS

The convex hull being a cyclic polytope can be found in

- Diagonal limit of 2D global blocks K. Sen, A. Sinha and A. Zahed

$$G_{d,\Delta,0}(z) = \left(\frac{z^2}{1-z}\right)^{\frac{\Delta}{2}} {}_3F_2\left[\begin{matrix}1+\frac{\Delta-d}{2}, \frac{\Delta}{2}, \frac{\Delta}{2} \\ \frac{\Delta+1}{2}, \Delta+1-\frac{d}{2}\end{matrix}; \frac{z^2}{4(z-1)}\right]$$

- 

$$when\ \Delta_\phi < \Delta_{kink}, \omega(\Delta) = \langle \mathbf{X}, 0, \Delta^*, \Delta \rangle$$

$$when\ \Delta_\phi > \Delta_{kink}, \omega(\Delta) = \langle \mathbf{X}, 0, \infty, \Delta \rangle$$

$$For\ arbitrary\ \mathbf{v}_1\ , \langle \mathbf{X}, 0, \Delta_+, \mathbf{v}_1 \rangle = 0$$

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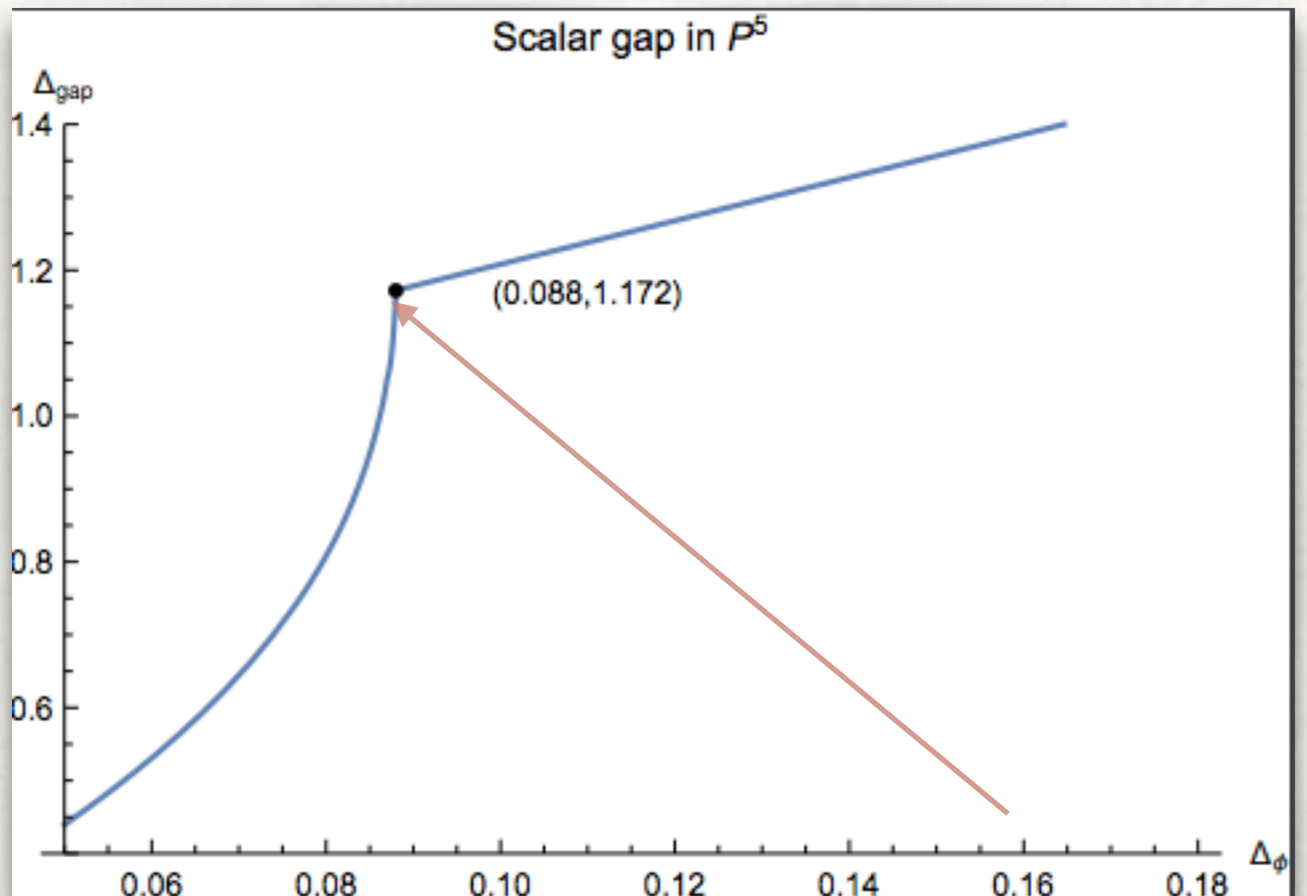
EXTENSIONS

The convex hull being a cyclic polytope can be found in

- Diagonal limit of 2D global blocks K. Sen, A. Sinha and A. Zahed

$$G_{d,\Delta,0}(z) = \left(\frac{z^2}{1-z}\right)^{\frac{\Delta}{2}} {}_3F_2\left[\begin{matrix} 1+\frac{\Delta-d}{2}, \frac{\Delta}{2}, \frac{\Delta}{2} \\ \frac{\Delta+1}{2}, \Delta+1-\frac{d}{2} \end{matrix}; \frac{z^2}{4(z-1)}\right]$$

-



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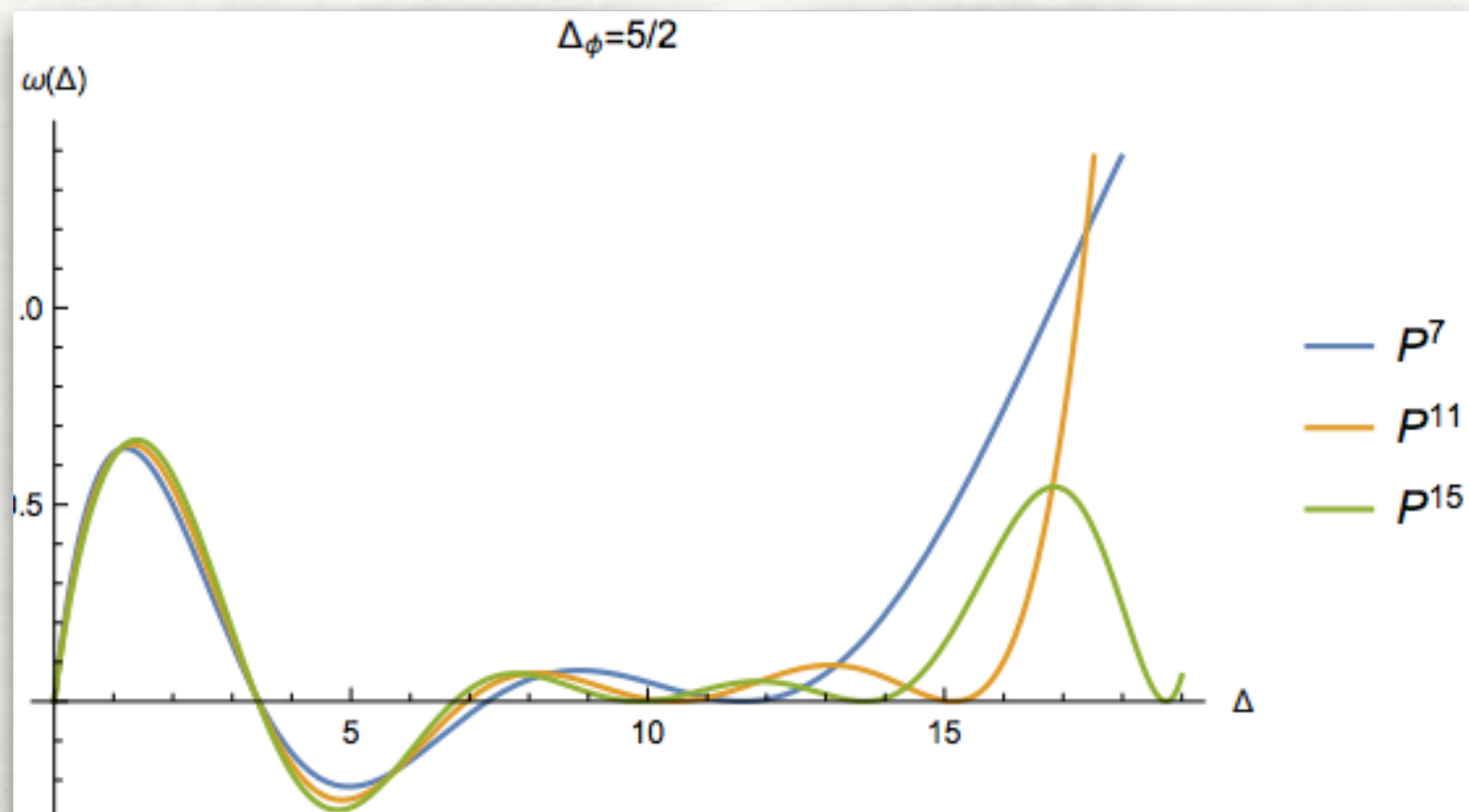
We again find, for 2N+1 derivative expansion, after projecting through the crossing plane and 0, the collection of k+1 (N=2k+1) points that enclose the origin is unique, and gives the optimal functional

- 

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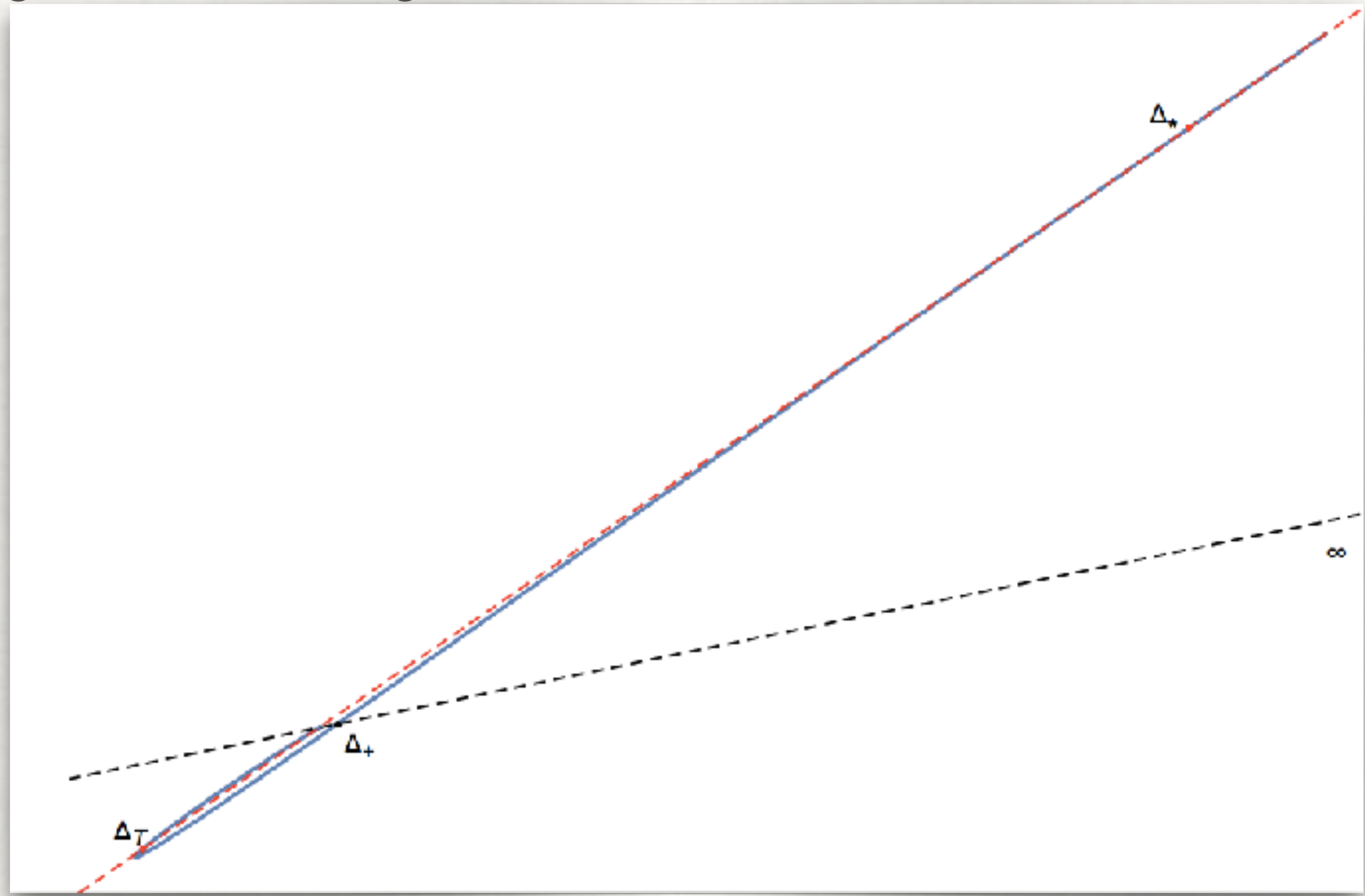

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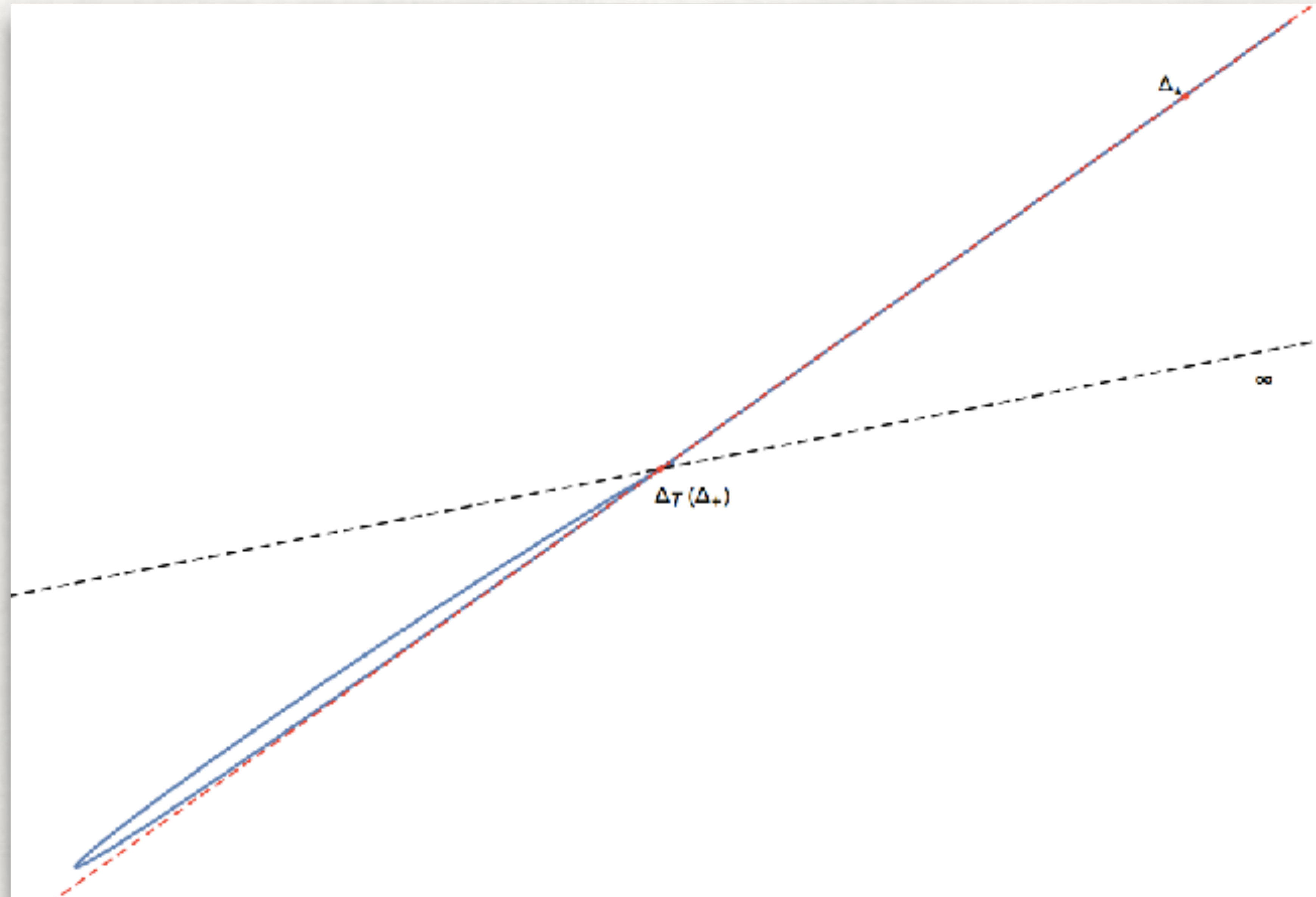

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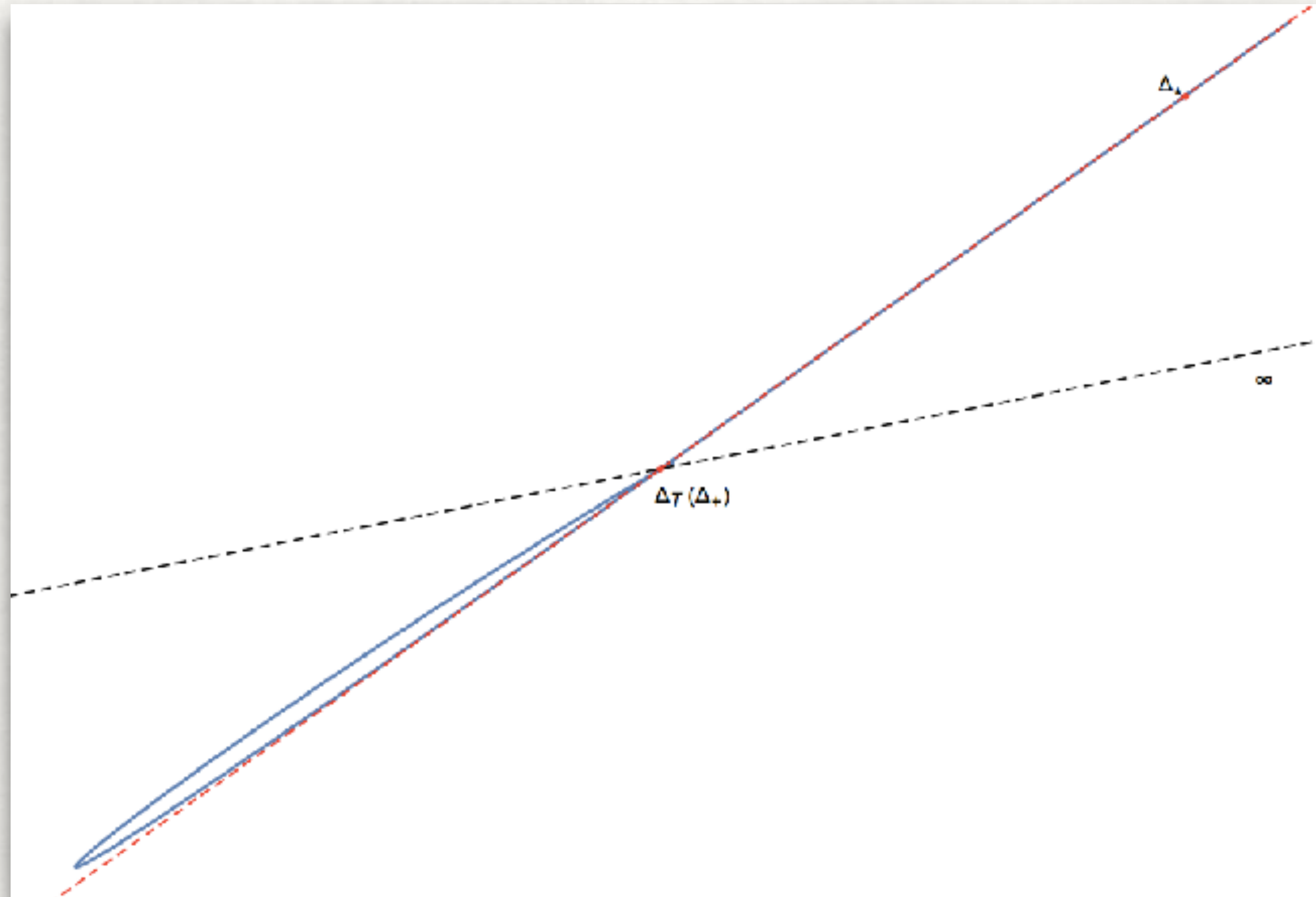

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$$\text{when } \Delta_\phi < \Delta_{kink}, \omega(\Delta) = \langle \mathbf{X}, 0, \Delta^*, \Delta \rangle$$
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$$\text{For arbitrary } \mathbf{v}_1 \;, \langle \mathbf{X}, 0, \Delta_{gap}, \infty, \mathbf{v}_1 \rangle = 0$$

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EXTENSIONS

The convex hull being a cyclic polytope can be found in

- Modular bootstrap:

$$\boxed{F(\zeta) = \sum_{d>0} N_d \exp(-d\exp(\zeta) + d)\,, \quad N_d \geq 0\,,}$$

$$q = e^{-2\pi e^{\zeta}}$$

$$S:\ \zeta \to -\zeta$$

expanding around $\zeta = 0$

$$F(\zeta) = \sum_{n=0}^{\infty} F_n \zeta^n\,,$$

$$\exp(-d\exp(\zeta) + d) = \sum_{n=0}^{\infty} c_n(d)\zeta^n$$

$$\begin{aligned}
&F_0\,,\\
&F_1 = -\frac{\pi(c-1)+3}{6}F_0 - \frac{\pi}{6}(1-q_0^2)^2 c - \frac{1}{6}(1-q_0)(3q_0 - 3 - 25\pi q_0 + \pi)\,,\\
&F_2\,,\\
&F_3 = -\frac{\pi(c-1)+3}{6}F_2 + \left[\frac{\pi^3}{648}c^3 - \frac{\pi^2(\pi-3)}{216}c^2 + \frac{\pi(\pi^2 - 6\pi + 3)}{216}c + \frac{27 - 9\pi + 9\pi^2 - \pi^3}{648}\right]F_0\\
&+ \left[\frac{\pi^3(1-q_0)^2}{648}c^3 - \frac{\pi^2(\pi-3)(1-q_0)^2}{216}c^2\right.\\
&+ \frac{\pi((3 - 6\pi + \pi^2) + 2(-3 - 30\pi + 71\pi^2)q_0 + (3 + 66\pi - 287\pi^2)q_0^2)}{216}c\\
&\left.+ \frac{(27 - 9\pi + 9\pi^2 - \pi^3) - 2(27 + 531\pi - 2043\pi^2 + 1079\pi^3)q_0 + (27 + 1071\pi - 7983\pi^2 + 7775\pi^3)q_0^2}{648}\right]
\end{aligned}$$

$$\begin{aligned}
c_0 &= 1\,,\\
c_1 &= -d\,,\\
c_2 &= \frac{1}{2}(d^2 - d)\,,\\
c_3 &= -\frac{1}{6}(d^3 - 3d^2 + d)\,,\\
c_4 &= \frac{1}{24}(d^4 - 6d^3 + 7d^2 - d)
\end{aligned}$$

EXTENSIONS

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$$q = e^{-2\pi e^{\zeta}}$$

$$S:\ \zeta \to -\zeta$$

expanding around $\zeta = 0$

$$\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ \vdots \end{pmatrix} = \sum_d N_d \begin{pmatrix} 1 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

We again have a convex hull problem, where the vertices are simply a GL transformation from the moment curve, i.e. a canonical cyclic polytope!

$$\begin{aligned} c_0 &= 1\,, \\ c_1 &= -d\,, \\ c_2 &= \frac{1}{2}(d^2 - d)\,, \\ c_3 &= -\frac{1}{6}(d^3 - 3d^2 + d)\,, \\ c_4 &= \frac{1}{24}(d^4 - 6d^3 + 7d^2 - d) \end{aligned}$$

$\xrightarrow{\text{GL transform}}$

$$\begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^d \end{pmatrix} \in \mathbb{P}^d, \quad x \in \mathbb{R}\,.$$

EXTENSIONS

The convex hull being a cyclic polytope can be found in

- Modular bootstrap:

$$F(\zeta) = \sum_{d>0} N_d \exp(-d\exp(\zeta) + d)\,, \quad N_d \geq 0\,,$$

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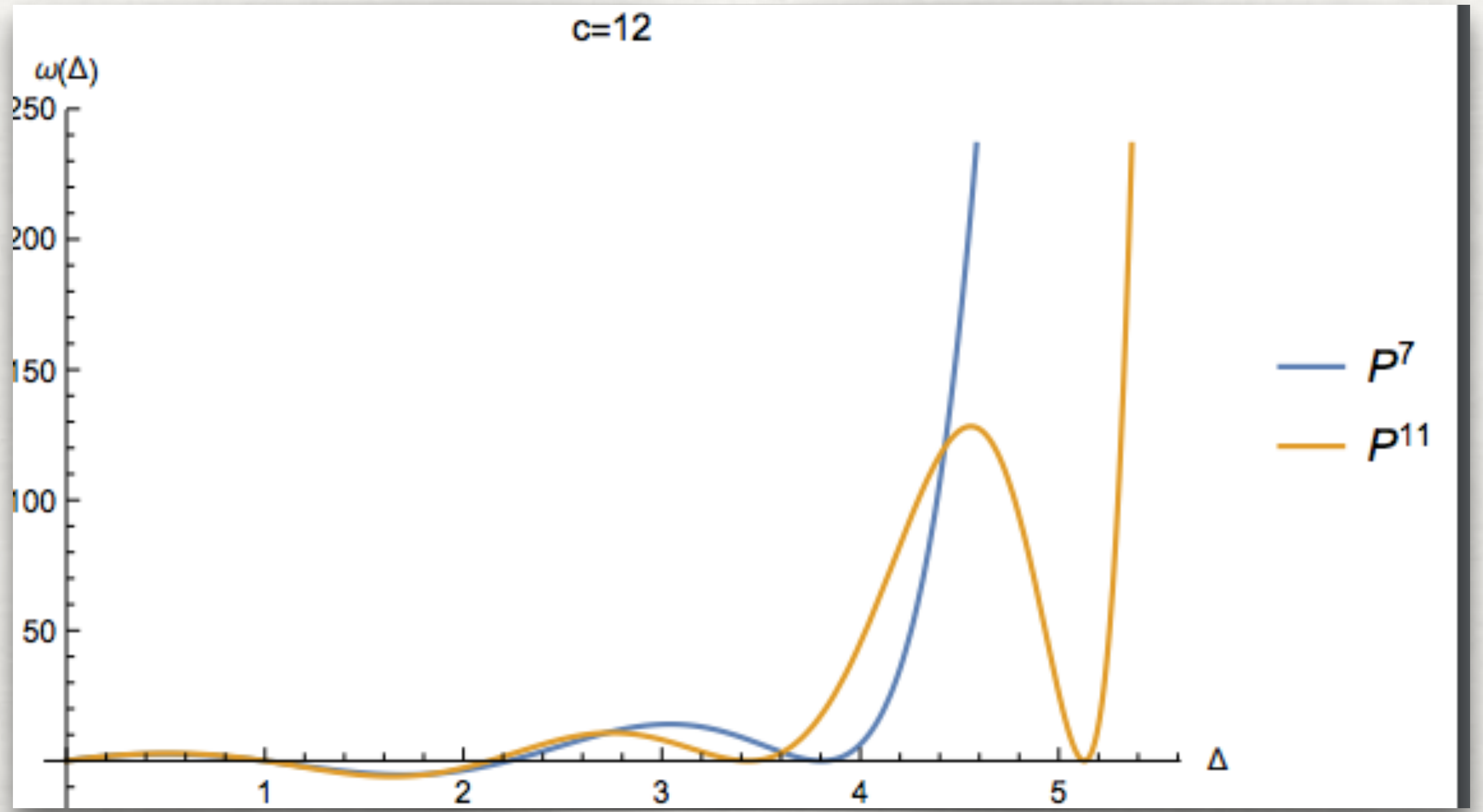
The numerical optimal functional again given by the degenerate intersection conditions



SUMMARY

- We have seen that the geometry behind the bootstrap program can be approached analytically in many instances, 1 D CFT, 2D CFT diagonal limit, Modular bootstrap
- The optimal functionals, which characterize the boundary of the theory space, is given by a degenerate simplex, which is unique. Allows us to analytically solve the functional
- The same approach applies globally for the spectrum, where the optimal functional for each operator is given by intersection condition, and the space is carved out recursively.
- The same geometric interpretation leads to a geometric definition of the kink
- Similarly there is a geometric definition of the bound for OPE coefficients
- Inclusion of spins, and move off-diagonal

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coming to theater soon!