Quenching dynamics across quantum critical points: unusual power laws

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Dziarmaga, Advances in Physics 59 (2010) 1063
Polkovnikov, Sengupta, Silva and Vengalattore, Rev. Mod. Phys. 83 (2011) 863
Dutta, Divakaran, Sen, Chakrabarti, Rosenbaum and Aeppli, arXiv:1012.0653

Outline

- Quantum critical point and critical exponents
- Slow quench across a quantum critical point and Kibble-Zurek scaling of defect density
- Slow quench along a quantum critical line and non-linear quenching
- Effect of spatial periodicity: variable critical exponent ν
- Slow quenching in a Tomonaga-Luttinger liquid

Quantum critical point (QCP)

The ground state of a quantum system may undergo a continuous phase transition as some parameter γ in the Hamiltonian is varied

Example: one-dimensional Ising model in a transverse magnetic field

$$H = - \sum_n [\, \sigma_n^z \sigma_{n+1}^z + \gamma\, \sigma_n^x \,], \quad \text{where} \quad \sigma_n^a \quad \text{are Pauli matrices}$$

There is a QCP at $\gamma_c = 1$

The two-spin correlation function $\langle \sigma_0^z \sigma_n^z \rangle - \langle \sigma_0^z \rangle^2$ goes to zero exponentially with n with a correlation length ξ which diverges near the QCP as $|\, \gamma - \gamma_c \,|^{-\nu}$

Consider the low-lying excitation spectrum $\omega(k)$. At the QCP $\gamma = \gamma_c$, $\omega(k)$ vanishes at some momentum k_c as $|\, k - k_c \,|^z$

Near the QCP, the gap $\Delta E = \omega(k_c)$ between the ground state and the first excited state vanishes as $\Delta E \sim |\, \gamma - \gamma_c \,|^{z\nu}$

These relations define two critical exponents ν and z

For the transverse Ising model,

$$\omega(k) = 2\sqrt{(\gamma - 1)^2 + 4\gamma \cos^2(k/2)}$$

so that $\gamma_c = 1$ and $k_c = \pi$

ω goes to zero linearly as $|\gamma - \gamma_c|$ for $k = k_c$ and as $|k - k_c|$ for $\gamma = \gamma_c$

So the critical exponents are $z = \nu = 1$

Transverse Ising model

$$H = -\sum_{n} [\, \sigma_n^z \sigma_{n+1}^z + \gamma\, \sigma_n^x \,]$$

For $\gamma \to \infty,$ the ground state is

$$\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow$$

The lowest excited state has a spin pointing in the wrong direction

$$\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\leftarrow \;\rightarrow \;\rightarrow \;\rightarrow \;\rightarrow$$

For $\gamma \to 0,$ the ground states are

$$\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow$$

and

$$\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow$$

The lowest excited state is a domain wall

$$\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\uparrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow$$

Quenching in transverse Ising model

What happens if we change γ from ∞ to 0 in a time τ ?

For $\gamma \to 0$, the ground states are

$$\uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow$$

and

$$\downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow$$

But due to quenching at a finite rate, the state reached as $\gamma \to 0$ will have some defects

$$\uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \downarrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \uparrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow$$

How does the defect density depend on the quenching time τ ?

Consider a linear quench with $\gamma = -t/\tau$, where $-\infty < t < 0$

Main result: For the transverse Ising model, if τ is much larger than the inverse of the band width of the low-energy excitations, then the density of defects n scales as $1/\sqrt{\tau}$

Reason: On quenching across a QCP, there are necessarily a number of low-energy modes for which the quenching is not adiabatic. These modes give rise to defects

Zurek, Dorner and Zoller, Phys. Rev. Lett. 95 (2005) 105701
Polkovnikov and Gritsev, Nature Physics 4 (2008) 477

Jordan-Wigner transformation

The model can be solved exactly by mapping spin-1/2's to fermions using the Jordan-Wigner transformation

Lieb, Schultz and Mattis, Ann. Phys. 16 (1961) 407

$$a_n = \left[\prod_{m=-\infty}^{n-1} \sigma_m^z\right] \sigma_n^+$$
$$a_n^\dagger = \left[\prod_{m=-\infty}^{n-1} \sigma_m^z\right] \sigma_n^-$$

where $\sigma_n^\pm = (1/2)\,(\sigma_n^x \pm i\sigma_n^y)$

Then $\{a_m, a_n\} = 0$ and $\{a_m, a_n^\dagger\} = \delta_{mn}$

These operators create and annihilate spinless fermions

In terms of the fermion operators, the Hamiltonian

$$H = - \sum_n [\, \sigma_n^x \sigma_{n+1}^x + \gamma\, \sigma_n^z \,]$$

becomes

$$H = - \sum_n [\, (a_n^\dagger - a_n)\, (a_{n+1}^\dagger + a_{n+1}) + 2\gamma a_n^\dagger a_n \,]$$

Define

$$a_k = \frac{1}{\sqrt{N}} \sum_n a_n\, e^{-ikn} \qquad \text{and} \qquad a_n = \frac{1}{\sqrt{N}} \sum_{-\pi<k<\pi} a_k\, e^{ikn}$$

where N is the number of sites. Then we get

$$H = 2 \sum_{0<k<\pi} [-(\gamma + \cos k)\, (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) - i \sin k\, (a_k a_{-k} - a_{-k}^\dagger a_k^\dagger)]$$

Energy spectrum

$$H = 2 \sum_{0<k<\pi} [-(\gamma + \cos k)\,(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) - i \sin k\,(a_k a_{-k} - a_{-k}^\dagger a_k^\dagger)]$$

This is a system of non-interacting fermions.
For each pair of momenta $\pm k,$ there are four states:
$|\,\phi>$ (empty state), $|\,k>,\ |\,-k>$ (one-particle states),
and $|\,k,\ -k>$ (two-particle state)

The states $|\,\phi\rangle$ and $|\,k,\ -k\rangle$ are governed by the Hamiltonian

$$h_k = \begin{pmatrix} -4(\gamma + \cos k) & i\,2\sin k \\ -i\,2\sin k & 0 \end{pmatrix}$$

The ground state lies in this two-dimensional subspace, and the energy spectrum is $\omega(k) = 2\sqrt{(\gamma-1)^2 + 4\gamma\cos^2(k/2)}$

As γ varies with time, only these two states mix with each other

We only have to deal with a two-level system for each value of $\pm k$

Damski, Phys. Rev. Lett. 95 (2005) 035701

For $\gamma = -t/\tau$, the Hamiltonian is

$$h_k = \begin{pmatrix} 2(t/\tau - \cos k) & i\,2\sin k \\ -i\,2\sin k & -2(t/\tau - \cos k) \end{pmatrix}$$

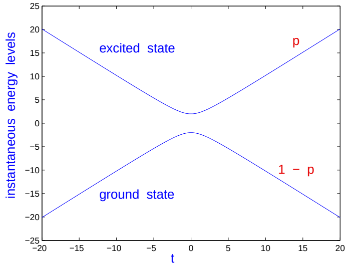
If we start in the ground state of this system at $t = -\infty$, which state do we reach at $t = 0$?

This is the Landau-Zener problem

Zener, Proc. R. Soc. London Ser A 137 (1932) 696
Landau and Lifshitz, Quantum Mechanics: Non-relativistic Theory (Pergamon, Oxford, 1965)

Consider a two-level system with a time-dependent Hamiltonian

$$H = \begin{pmatrix} t/\tau & b \\ b & -t/\tau \end{pmatrix}$$



If we start in the ground state at $t \to -\infty$, the probability of ending in the excited state at $t \to \infty$ is given by $p = \exp\left[-\pi b^2 \tau\right]$

Scaling argument for p

The probability of ending in the excited state is $p = \exp\left[-\pi b^2 \tau\right]$

Note that $p \to 0$ or 1 as $\tau \to \infty$ (adiabatic) or 0 (sudden quench)

A simple scaling argument shows that p must be a function of $b\sqrt{\tau}$.
The Schrödinger equation is

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} t/\tau & b \\ b & -t/\tau \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Multiply throughout by $\sqrt{\tau}$ and re-defining $t' = t/\sqrt{\tau}$ to get

$$i\frac{\partial}{\partial t'}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} t' & b\sqrt{\tau} \\ b\sqrt{\tau} & -t' \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Hence, if we start with $\psi_1(t' = -\infty) = 1$, then $p = |\psi_1(t' = \infty)|^2$ must be a function of the single parameter $b\sqrt{\tau}$, and the function must $\to 0$ or 1 as $b\sqrt{\tau} \to \infty$ or 0

Defect scaling law

Returning to the Hamiltonian for the transverse Ising model

$$H = 2 \sum_{0<k<\pi} [-(\gamma + \cos k)\, (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) \;+\; \sin k\, (a_k a_{-k} + a_{-k}^\dagger a_k^\dagger)]$$

the total defect density is

$$n \;=\; \int_0^\pi \frac{dk}{2\pi}\, p_k \;=\; \int_0^\pi \frac{dk}{2\pi}\, \exp\,[\,-2\pi\tau \sin^2 k\,]$$

For large τ, the integral is dominated by contributions from $k = 0$ and π. Hence $n \;\sim\; \int_0^\infty dk\, \exp\,[\,-2\pi\tau k^2\,] \;\sim\; 1/\sqrt{\tau}$

The power law arises because the quench takes the system across a QCP where the energy vanishes at some values of k. No matter how slowly we quench across this point, there are modes with energies $\lesssim 1/\sqrt{\tau}$ for which the quenching is not adiabatic

Kibble-Zurek scaling at a QCP

General result: For a translation invariant system, if γ is taken across a QCP at a rate $1/\tau$, the density of defects scales as

$$n \sim \frac{1}{\tau^{d\nu/(z\nu+1)}}$$

Polkovnikov, Phys. Rev. B 72 (2005) 161201(R)

A 'hand waving' argument: the defects are produced by a region in momentum space with volume k^d, at a time $t \sim (\gamma - \gamma_c)\tau \sim 1/\omega$, where $k \sim (\gamma - \gamma_c)^\nu$ and $\omega \sim k^z$. Then the density of defects is $n \sim k^d \sim 1/\tau^{d\nu/(z\nu+1)}$

For $d \geq 2(z + 1/\nu)$, we get $n \sim 1/\tau^2$ due to contributions from all momenta, not just the critical momenta

Generalizations of Kibble-Zurek scaling

The Kibble-Zurek scaling relation is

$$n \sim \frac{1}{\tau^{d\nu/(z\nu+1)}}$$

We now discuss some generalizations of this relation:

(i) quenching across a gapless surface in momentum space

(ii) quenching along a critical line in parameter space

(iii) non-linear quenching

Each of these modifies the power law

Gapless surface in momentum space

Suppose that at $\gamma = \gamma_c$, the energy vanishes on a surface of $d - m$ dimensions in momentum space, rather than at a single point

Then the momentum integration appearing in the expression for the defect density will be over m dimensions instead of d dimensions

Hence, we will get

$$n \sim \int_0^\infty d^m k \; p_k(k\tau^{\nu/(z\nu+1)}) \sim \frac{1}{\tau^{m\nu/(z\nu+1)}}$$

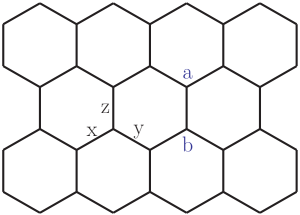
This happens in the Kitaev model which has $d = 2,\ m = 1,$ and $\nu = z = 1.$ Thus $n \sim 1/\sqrt{\tau}$ instead of $1/\tau$ as it normally would have been for a two-dimensional model with $\nu = z = 1$

Kitaev, Ann. Phys. 321 (2006) 2

Spin-1/2 model on a honeycomb lattice, with the Hamiltonian

$$H = \sum_{j+l=\text{even}} \left(J_1\, \sigma^x_{j,l}\sigma^x_{j+1,l} + J_2\, \sigma^y_{j-1,l}\sigma^y_{j,l} + J_3\, \sigma^z_{j,l}\sigma^z_{j,l+1} \right)$$

Can assume that all couplings $J_i \geq 0$



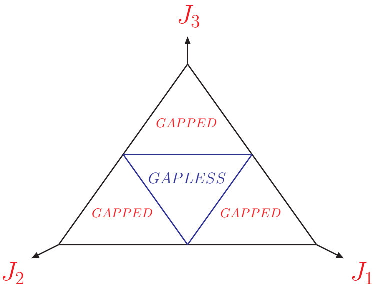
The model can be solved exactly by mapping it to Majorana fermions by a Jordan-Wigner transformation

Phase diagram of Kitaev model

If $J_1 < J_2 + J_3, \quad J_2 < J_3 + J_1$ and $J_3 < J_1 + J_2,$ the system is gapless along some lines in the Brillouin zone

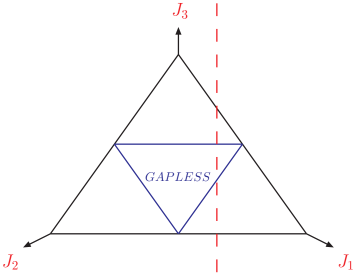
For all other values of $(J_1, J_2, J_3),$ the system is gapped

The phase diagram can be shown in terms of points in an equilateral triangle satisfying $J_1 + J_2 + J_3 = 1$
(the value of J_i is the distance from the opposite side)



Quenching in the Kitaev model

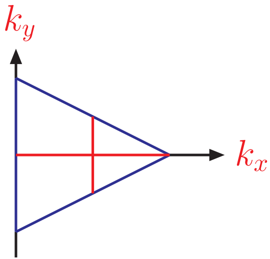
We hold J_1, J_2 fixed, and vary J_3 in time as $Jt/\tau,$ from $t = -\infty$ to $t = \infty$ (as shown by the red dotted line). Then the system will pass through the gapless region for some time



Sengupta, Sen and Mondal, Phys. Rev. Lett. 100 (2008) 077204
Mondal, Sen and Sengupta, Phys. Rev. B 78 (2008) 045101

Scaling of defect density

In the gapless region, the energy of the low-lying excitations vanishes on some lines in the Brillouin zone as indicated in red below



Thus the Kitaev model has $d = 2$ but $m = 1$. Also, $\nu = z = 1$

Hence the defect density scales as $n \sim 1/\sqrt{\tau}$ instead of $1/\tau$

$$n = \frac{3\sqrt{3}}{4\pi^2} \int\int d^2\vec{k} \; p_{\vec{k}} \,,$$

$$p_{\vec{k}} = e^{-2\pi\tau \; [J_1 \sin(\vec{k}\cdot\vec{M}_1) - J_2 \sin(\vec{k}\cdot\vec{M}_2)]^2/J}$$

Quenching along a critical line

A different scaling occurs if one quenches along a critical line in parameter space. In terms of a two-level system, suppose that the Hamiltonian for the modes with momenta $\pm k$ is

$$H = \begin{pmatrix} |k|^a \, t/\tau & |k|^z \\ |k|^z & -|k|^a \, t/\tau \end{pmatrix}$$

A scaling argument then shows that the defect density goes as $n \sim 1/\tau^{d/(2z-a)}$ for a system in d dimensions

Example: the spin-1/2 XY chain with a transverse magnetic field

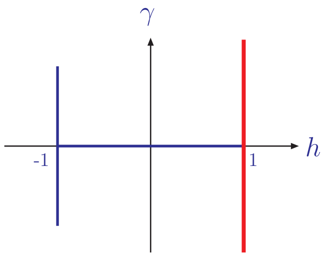
$$H = - \sum_n [\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \gamma(\sigma_n^x \sigma_{n+1}^x - \sigma_n^y \sigma_{n+1}^y) + h \sigma_n^z]$$

Mukherjee *et al.*, Phys. Rev. B 76 (2007) 174303

Divakaran, Dutta and Sen, Phys. Rev. B 78 (2008) 144301

Quenching along a critical line · · ·

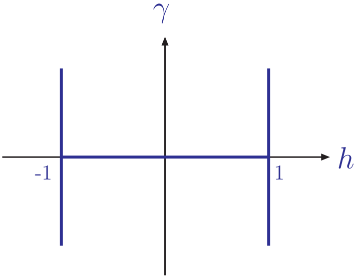
$$H = -\sum_n [\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \gamma(\sigma_n^x \sigma_{n+1}^x - \sigma_n^y \sigma_{n+1}^y) + h\sigma_n^z]$$



The critical lines are $h = -1$, $h = 1$ and $-1 \leq h \leq 1$, $\gamma = 0$

If we quench along the red line $h = 1$, we get $d = 1$, $z = 2$, $a = 1$

Hence the defect density scales as $n \sim 1/\tau^{d/(2z-a)} \sim 1/\tau^{1/3}$



Quenching along one of the vertical lines $h = \pm 1$ gives $n \sim 1/\tau^{1/3}$

The quenching procedure discussed earlier was to keep γ fixed at a non-zero value and cross one of the lines $h = \pm 1$.
This gives $n \sim 1/\tau^{1/2}$

Finally, quenching through one of the multicritical points at $h = \pm 1, \ \gamma = 0$ gives $d = 1, \ z = 3, \ a = 0.$ Hence
$n \sim 1/\tau^{d/(2z-a)} \sim 1/\tau^{1/6}$

Non-linear quenching

We can change the quenching parameter γ through a QCP in a non-linear way. A 'hand waving' way of studying this is to take

$$H = \begin{pmatrix} \Delta E \; sign(t) & |k|^z \\ |k|^z & -\Delta E \; sign(t) \end{pmatrix}$$

where $\Delta E \sim |\gamma - \gamma_c|^{z\nu}$ and we set $|\gamma - \gamma_c| = |t/\tau|^\alpha$

Then a scaling argument will show, for a system in d dimensions, that the defect density goes as $n \sim 1/\tau^{d\nu\alpha/(z\nu\alpha+1)}$

This is like the power law for linear quenching but with $\nu \rightarrow \nu\alpha$

For $d = \nu = z = 1$, we obtain $n \sim 1/\tau^{\alpha/(\alpha+1)}$

Sen, Sengupta and Mondal, Phys. Rev. Lett. 101 (2008) 016806
Mondal, Sengupta and Sen, Phys. Rev. B 79 (2009) 045128

Variation of power law with periodicity

For quenching across a QCP, the power law for the defect density can depend on the periodicity of the term whose coefficient is varied in time. Consider a tight-binding model in one dimension in which the chemical potential is periodic in space

$$H = -2 \sum_{n=-\infty}^{\infty} [\, c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n + h(t) \cos(\frac{\pi n}{q} + \phi)\, c_n^\dagger c_n \,],$$

$$h(t) = \frac{t}{\tau}$$

where $q = 1, 2, 3, \cdots$. The chemical potential has period $2q$

The ground state at $t = -\infty$ is half-filled, and the Fermi momenta lie at $k = \pm\pi/2$. Only states near these momenta contribute to the defect density. So we find the effective Hamiltonian governing pairs of states at $-\pi/2 + k$ and $\pi/2 + k$, where $|k| \ll \pi/2$

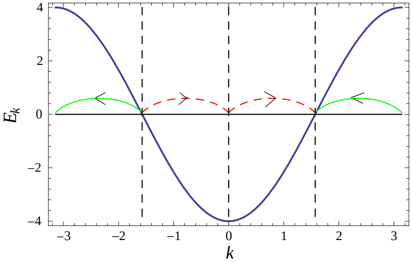
Sen and Vishveshwara, EPL 91 (2010) 66009

Thakurathi *et al.*, Phys. Rev. B 85 (2012) 165425

Effective Hamiltonian

$$H = -\sum_{k=-\pi}^{\pi} 4\cos k \; c_k^\dagger c_k - \sum_{n=-\infty}^{\infty} 2h \; \cos(\frac{\pi n}{q} + \phi) \; c_n^\dagger c_n$$

The low-energy states near $\pm \pi/2$ differ by a momentum of π. They are connected by matrix elements coming from the $\cos(\pi n/q + \phi)$ term. Since this has Fourier components at $\pm\pi/q$, we have to go to the q-th order in perturbation theory; this involves going through $q-1$ intermediate states along one of two possible paths, shown by red and green for $q = 2$



The matrix element between the states near $\pm\pi/2$ is given by

$$
\begin{aligned}
|\Delta| &= \frac{h^q}{4^{q-1}} \frac{2|\cos(q\phi)|}{\prod_{s=1}^{q-1} \sin(\pi s/q)} \quad \text{if } q \text{ is odd} \\
&= \frac{h^q}{4^{q-1}} \frac{2|\sin(q\phi)|}{\prod_{s=1}^{q-1} \sin(\pi s/q)} \quad \text{if } q \text{ is even}
\end{aligned}
$$

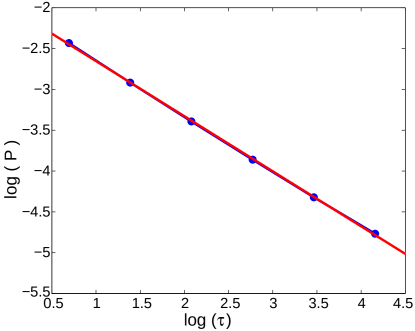
for small h. The $\cos(q\phi)$ or $\sin(q\phi)$ comes from the relative phase between the two possible paths

The effective Hamiltonian for the states $\pm\pi/2 + k$ is

$$
H_k = \begin{pmatrix} -4k & \Delta \\ \Delta^* & 4k \end{pmatrix}
$$

for $|k| \ll \pi/2$. Since $\Delta \sim h^q$, the QCP at $h = 0$ has $\nu = q$ and $z = 1$. So we expect the excitation probability p_k to be a function of $k\tau^{q/(q+1)}$ and the defect density to scale as $1/\tau^{q/(q+1)}$

Defect density for $q = 2$ and $\phi = \pi/4$

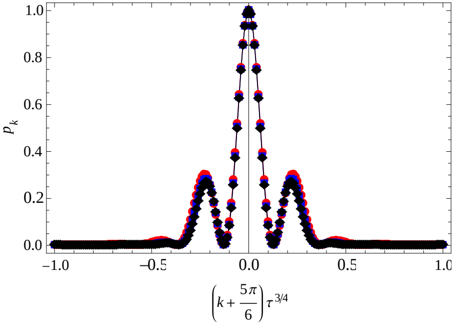


Log - log plot of defect density versus τ

A linear fit gives $P \sim 1/\tau^{0.67}$ which is close to a $-2/3$ power law

Sen and Vishveshwara, EPL 91 (2010) 66009

p_k for $q = 3$ and $\phi = 0$



Plot of excitation probability p_k versus scaled variable $(k + 5\pi/6)\,\tau^{2/3}$

for τ = 2 (red), 4 (blue) and 8 (black)

Thakurathi *et al.*, Phys. Rev. B 85 (2012) 165425

Effect of interactions

Consider a nearest-neighbor interaction between particles

$$V = \sum_{n=-\infty}^{\infty} V_0 \, c_n^\dagger c_n \, c_{n+1}^\dagger c_{n+1}$$

This gives a Tomonaga-Luttinger liquid which is characterized by a Luttinger parameter K and a velocity v. $K = 1$ for $V_0 = 0$ and < 1 for $V_0 > 0$ (repulsive interactions)

In terms of a bosonic field ϕ, the action is of the sine-Gordon form

$$S = \frac{1}{2} \int \int dx dt \left[\frac{1}{v} \left(\frac{\partial \phi}{\partial t} \right)^2 - v \left(\frac{\partial \phi}{\partial x} \right)^2 + h^q \cos(2\sqrt{\pi K} \phi) \right]$$

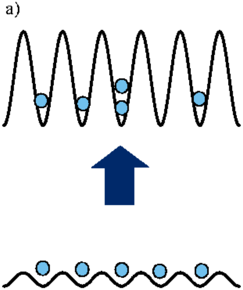
The 'cosine' term has scaling dimension K. So it gives rise to a finite correlation length which scales as $\xi \sim h^{-q/(2-K)}$. Hence the correlation length exponent is $\nu = q/(2-K)$

If $h(t)$ is quenched across the QCP at $h = 0$, the defect density will scale as $1/\tau^{\nu/(\nu+1)} = 1/\tau^{q/(q+2-K)}$

Quenching in a Tomonaga-Luttinger liquid

The loading of interacting bosons in a one-dimensional optical lattice gives another example of quenching in a Tomonaga-Luttinger liquid

Suppose that the periodic potential of the optical lattice is changed in time as $V(x,t) = V(t)\cos(2\pi x/a)$, where the lattice spacing a is commensurate with the bosonic density



De Grandi, Barankov and Polkovnikov, Phys. Rev. Lett. 101 (2008) 230402

This is a special case of our model with $q = 1$, i.e.,
there is only one particle in each well of the periodic potential

If the periodic potential is changed slowly as $V(t) = t/\tau,$ the number of defects (potential wells having less than or more than one particle) will scale as

$$n \sim 1/\tau^{q/(q+2-K)} = 1/\tau^{1/(3-K)}$$

This result holds only for $K < 2$

$K = 2$ is the Kosterlitz-Thouless point

For $K > 2,$ the cosine term is irrelevant, and the defects receive contributions from all modes, not just the low-momentum modes.
Then one finds that $n \sim 1/\tau$ for any value of q

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