Parallel syndrome extraction with shared flag qubits for CSS codes of distance three

arXiv:2208.00581

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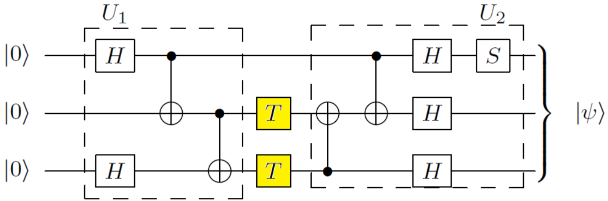
Outline

1. Flagged fault-tolerant quantum computation
2. Parallel syndrome extraction with shared flag qubits for general CSS codes of distance three.
3. Numerical results
4. Conclusion

Flagged fault-tolerant quantum computation

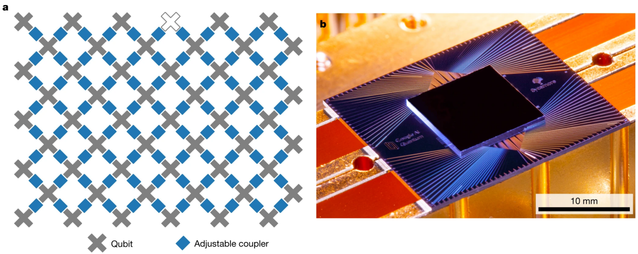
Postulate The evolution of a quantum state in a closed system is described by a unitary operator.

- A quantum algorithm can be implemented in a quantum circuit consisting of quantum wires, single-qubit gates, and two-qubit gates.



- Any quantum computation can be implemented by a universal set of elementary gates with arbitrary accuracy. Ex: $\{H, S, CNOT, T\}$.

Google's Quantum Chip: Sycamore



F. Arute et al., "Quantum supremacy using a programmable superconducting processor," **Nature** 574, 505–510 (2019)

Pauli and measurement errors

Average error	Isolated	Simultaneous
Single-qubit (e_1)	0.15%	0.16%
Two-qubit (e_2)	0.36%	0.62%
Two-qubit, cycle (e_{2c})	0.65%	0.93%
Readout (e_r)	3.1%	3.8%

Physical error rate is about $10^{-2} \sim 10^{-3}$.

Discretization of the Errors

- The Pauli matrices

$$\{I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iXZ\}$$

  form a basis for the space of linear operators on a single-qubit $\mathcal{L}(\mathbb{C}^2)$.

Bit flip	$X\|0\rangle = \|1\rangle,\ X\|1\rangle = \|0\rangle$
Phase flip	$Z\|0\rangle = \|0\rangle,\ Z\|1\rangle = -\|1\rangle$

- (independent) Depolarizing channel with parameter ϵ:
  - no error (I) with probability $1 - \epsilon$
  - X with probability $\epsilon/3$
  - Y with probability $\epsilon/3$
  - Z with probability $\epsilon/3$

n-fold Pauli operator

- n-fold Pauli operators $\{M_1 \otimes M_2 \otimes \cdots \otimes M_n : M_i \in \{I, X, Y, Z\}\}$.
  - $X \otimes X \otimes Y \otimes Z \otimes I \otimes Z = XXYZIZ = X_1 X_2 Y_3 Z_4 Z_6$.
- Every n-fold Pauli operator has eigenvalue ± 1.
- Two Pauli operators either commute or anticommute with each other.

Quantum Stabilizer Codes

- $\mathcal{S} = \langle g_1, g_2, \cdots, g_m \rangle$: an Abelian subgroup of $\{I, X, Y, Z\}^n$ and $-I \notin \mathcal{S}$.

$$\langle g_i, g_j \rangle = 0.$$

- An $[[n, k, d]]$ quantum stabilizer code $\mathcal{C}(\mathcal{S})$ defined by stabilizer group $\mathcal{S}$ is the 2^k-dimensional subspace of the n-qubit state space $\mathbb{C}^{2^n}$ fixed by $\mathcal{S}$ so that any error $E \in \{I, X, Y, Z\}^{\otimes n}$ of $\mathrm{wt}(E) \leq d - 1$ is detectable.

$$\mathcal{C}(\mathcal{S}) = \{|\psi\rangle \in \mathbb{C}^{2^n} : \boxed{g|\psi\rangle = |\psi\rangle}, \forall g \in \mathcal{S}\}.$$

- An error E can be detected if it anticommutes with some $g_j \in \mathcal{S}$:

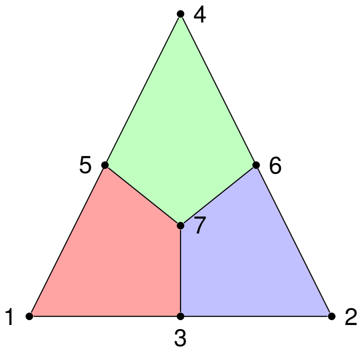
$$g_j(E|\psi\rangle) = -Eg_j|\psi\rangle = \boxed{-}(E|\psi\rangle).$$

- The error syndrome of E is a binary $(n-k)$-tuple of the measurement outcome of $g_1, \ldots, g_m$.

The 7-qubit Steane ode of distance three

- The $[7, 4, 3]$ Hamming code is dual-containing and its parity check matrix is

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$



- Rows of the stabilizer check matrix:

$$\begin{aligned} g_1 &= XIXIXIX & \quad g_4 &= ZIZIZIZ \\ g_2 &= IXXIIXX & \quad g_5 &= IZZIIZZ \\ g_3 &= IIIXXXX & \quad g_6 &= IIIZZZZ \end{aligned}$$

(Calderbank-Shor-Steane (CSS) Code)

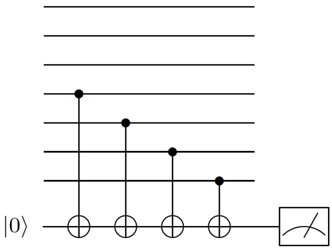
- Any single-qubit Pauli error has a unique error syndrome and can be corrected.

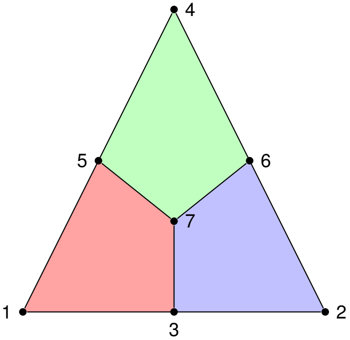
	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
g_1	1	0	1	0	1	0	1
g_2	0	1	1	0	0	1	1
g_3	0	0	0	1	1	1	1

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
g_4	1	0	1	0	1	0	1
g_5	0	1	1	0	0	1	1
g_6	0	0	0	1	1	1	1

Syndrome Measurement

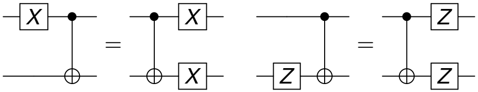
- The circuit for the measurement of *ZZZZ*



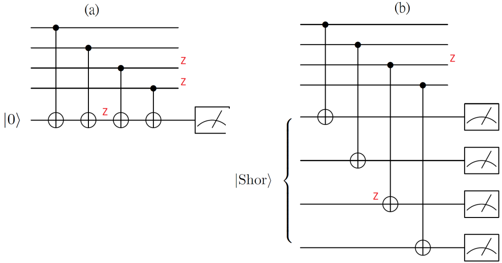


Fault-Tolerant Syndrome Measurement

- Quantum states are vulnerable and quantum gates are faulty. Location failures may occur.
- Errors propagate through CNOT gates.

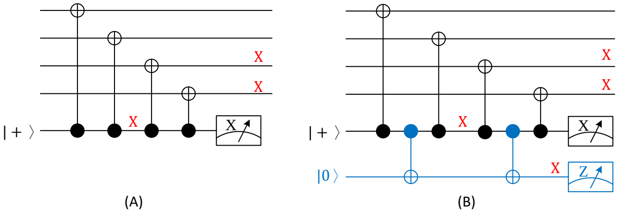


- A location failure in a procedure is said to be a bad location failure if one of its location failures may evolve into an uncorrectable error.
- A procedure is fault-tolerant if it has no bad location failures.
- To measure a stabilizer of weight-w, one may use the Shor syndrome extraction with a weight-w cat state.

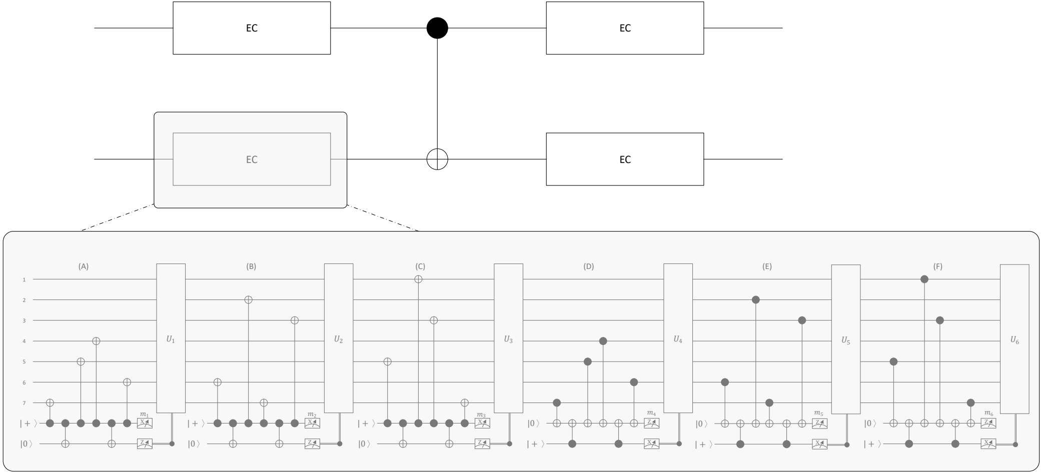


Fault-tolerant quantum computation with flag qubits

Flagged syndrome extraction



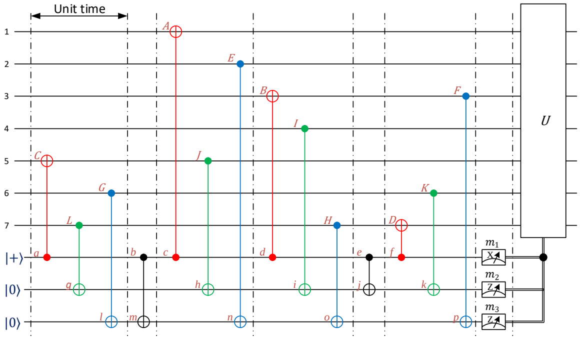
- Circuits for measuring the stabilizer $XXXX$. (A) Unflagged syndrome measurement. (B) Flagged syndrome measurement.
  - R. Chao and B. W. Reichardt, "Quantum error correction with only two extra qubits," Phys. Rev. Lett. 121, 050502 (2018).
  - R. Chao and B. W. Reichardt, "Fault-tolerant quantum computation with few qubits," npj Quantum Inf. 4, 42 (2018).
  - Only one additional ancilla qubit (called flag) is required for fault-tolerant syndrome extraction.
- When a flag qubit rises, we know that there is some high-weight error in the codeword. Then we perform a complete unflagged syndrome extraction and choose the most likely error accordingly. (Usually a lookup table is used.)



- However, the circuit depth is increased.

Parallel syndrome extraction for the Steane code with shared flag qubits

- B. W. Reichardt, “Fault-tolerant quantum error correction for Steane’s seven-qubit color code with few or no extra qubits," Quantum Sci. Tech. 6, 015007 (2020)



– Parallel syndrome extraction circuits for the [[7, 1, 3]] code: $X_1X_3X_5X_7$ (red), $Z_2Z_3Z_6Z_7$ (blue), and $Z_4Z_5Z_6Z_7$ (green).
– A total of three ancilla qubits are used in the measurement of three stabilizers. No additional ancillas are required.

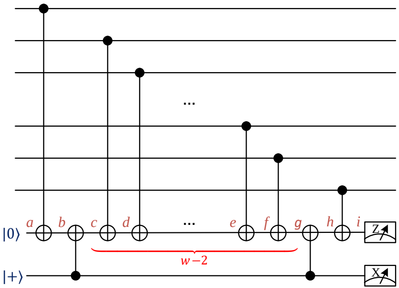
Failure error	Data error	$m_{1,2,3}$	m'_{123456}	m_{456}	m''_{123456}
X_A	X_1	000	-	100	000100
X_B	X_3	001	001100	-	-
X_C	X_5	010	010100	-	-
X_D	X_7	000	-	100	011100
X_E	X_2	001	001000	-	-
X_F	X_3	001	001100	-	-
X_G	X_6	011	011000	-	-
X_H	X_7	001	011100	-	-
X_I	X_4	010	010000	-	-
X_J	X_5	010	010100	-	-
X_K	X_6	010	011000	-	-
X_L	X_7	011	011100	-	-
Z_A	Z_1	100	100000	-	-
Z_B	Z_3	100	100001	-	-
Z_C	Z_5	100	100010	-	-
Z_D	Z_7	100	100011	-	-
Z_E	Z_2	000	-	001	000001
Z_F	Z_3	000	-	001	100001
Z_G	Z_6	000	-	011	000011
Z_H	Z_7	100	100011	-	-
Z_I	Z_4	000	-	010	000010
Z_J	Z_5	000	-	010	100010
Z_K	Z_6	000	-	011	000011
Z_L	Z_7	100	100011	-	-

X_a	$X_{1,3,5,7}$	000	-	-	-
X_b	$X_{1,3,7}$	010	010100	-	-
X_c	$X_{1,3,7}$	011	010100	-	-
X_d	$X_{3,7}$	011	010000	-	-
X_e	X_7	010	011100	-	-
X_f	X_7	000	-	100	011100
X_g	No error	010	000000	-	-
X_h	No error	010	000000	-	-
X_i	No error	010	000000	-	-
X_j	No error	010	000000	-	-
X_k	No error	010	000000	-	-
X_l	No error	001	000000	-	-
X_m	No error	001	000000	-	-
X_n	No error	001	000000	-	-
X_o	No error	001	000000	-	-
X_p	No error	001	000000	-	-
Z_a	No error	100	000000	-	-
Z_b	No error	100	000000	-	-
Z_c	No error	100	000000	-	-
Z_d	No error	100	000000	-	-
Z_e	No error	100	000000	-	-
Z_f	No error	100	000000	-	-
Z_g	$Z_{4,5,6,7}$	000	-	-	-
Z_h	$Z_{4,5,6}$	100	100011	-	-
Z_i	$Z_{4,6}$	100	000001	-	-
Z_j	Z_6	100	000011	-	-
Z_k	Z_6	000	-	011	000011
Z_l	$Z_{2,3,6,7}$	000	-	-	-
Z_m	$Z_{2,3,7}$	000	-	011	000011
Z_n	$Z_{2,3,7}$	100	000011	-	-
Z_o	$Z_{3,7}$	100	000010	-	-
Z_p	Z_7	000	-	011	100011

Our procedure for general CSS codes of distance three.

Flagged syndrome extraction with shared flags

- standard flagged syndrome extraction circuit



  – Let $C(g_i)$ denote the syndrome extraction circuit for g_i.

  – Let $B(C(g_i))$ be the number of locations that will trigger the flag qubit.

Lemma 1 In $C(g_i)$, each bad location failures leads to residue errors of the same type as g_i, up to one Pauli error of weight one.

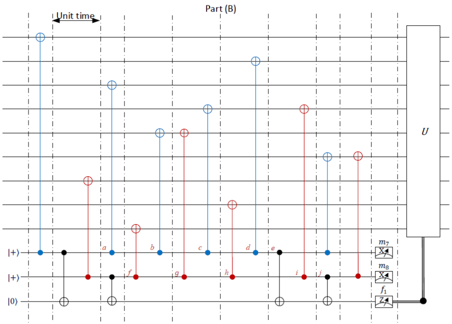
Lemma 2 Suppose that g is an X- or Z-stabilizer of weight w. Then

$$B(C(g_i)) \leq w - 1 .$$

- Suppose that g_1 and g_2 are stabilizers of the same type (X or Z). Let $C(g_1) \cup C(g_2)$ be the joint circuit of $C(g_1)$ and $C(g_2)$ with one shared flag qubit and the CNOTs connecting the ancillas and the flag qubit are placed in sequence.

Lemma 3

$$B(C(g_1) \cup C(g_2)) = B(C(g_1)) + B(C(g_2)).$$



– For Shor's [[9,1,3]] code, the syndrome bits for $X_1X_2X_3X_4X_5X_6$ (blue) and $X_4X_5X_6X_7X_8X_9$ (red) are extracted with one shared flag qubit.

Lemma 4 Suppose that $g_{j_1}, g_{j_2}, \ldots, g_{j_s}$ are stabilizers of the same type and $C(g_{j_1}), C(g_{j_2}), \ldots, C(g_{j_s})$ can be performed in parallel using one shared flag qubit. Then the following conditions hold.

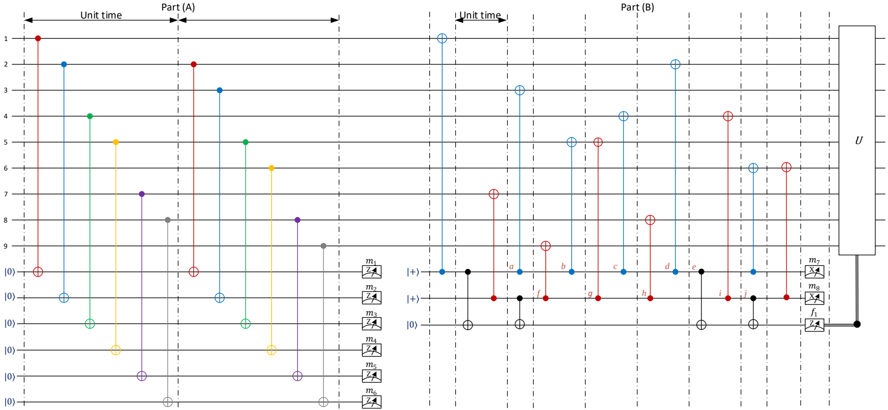
1. Each distinct bad location failure has a unique error syndrome when a flag rises.
2. $\begin{cases} B(\cup_i C(g_{j_i})) = \sum_{i=1}^{s} B(C(g_{j_i})) \leq 2^a, & \text{if } g_{j_1}, \ldots, g_{j_s} \text{ are } X\text{-type;} \\ B(\cup_i C(g_{j_i})) = \sum_{i=1}^{s} B(C(g_{j_i})) \leq 2^b, & \text{if } g_{j_1}, \ldots, g_{j_s} \text{ are } Z\text{-type,} \end{cases}$

   assuming that there are b X-type stabilizers and a Z-type stabilizers.

Example: Shor's [[9, 1, 3]] code

- Shor's [[9,1,3]] code is defined by stabilizers

$$\begin{aligned} g_1 &= Z_1 Z_2,\ g_2 = Z_2 Z_3,\ g_3 = Z_4 Z_5, & g_4 &= Z_5 Z_6,\ g_5 = Z_7 Z_8,\ g_6 = Z_8 Z_9, \\ g_7 &= X_1 X_2 X_3 X_4 X_5 X_6, & g_8 &= X_4 X_5 X_6 X_7 X_8 X_9. \end{aligned} \quad (1)$$



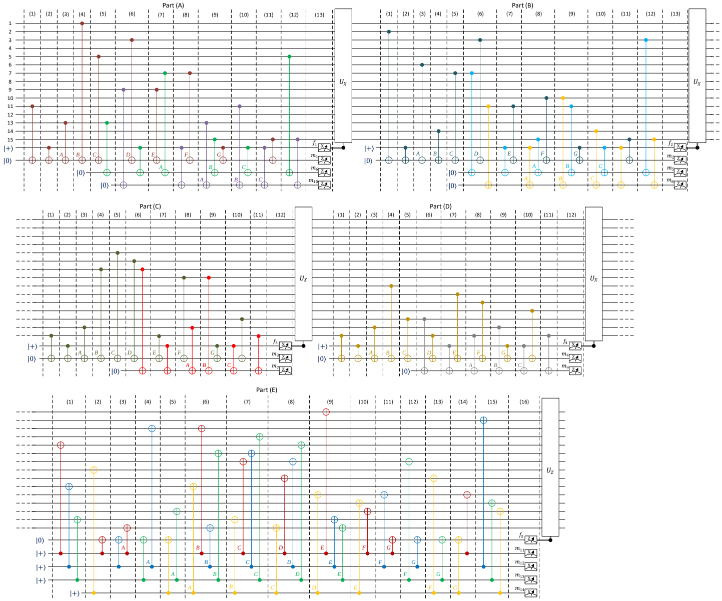
– No flag qubit is required for the measurement of Z-stabilizers.

Example: $[[15,1,3]]$ Reed-Muller code

[15,1,3] Reed-Muller code is defined by stabilizers

$$
\begin{aligned}
g_1 =& Z_{1,3,5,7,9,11,13,15},\ g_6 = Z_{5,7,13,15},\ g_{10} = Z_{9,11,13,15}\\
g_2 =& Z_{2,3,6,7,10,11,14,15},\ g_5 = Z_{3,7,11,15},\ g_8 = Z_{10,11,14,15},\\
g_3 =& Z_{4,5,6,7,12,13,14,15},\ g_7 = Z_{6,7,14,15},\\
g_4 =& Z_{8,9,10,11,12,13,14,15},\ g_9 = Z_{12,13,14,15},\\
g_{11} =& X_{1,3,5,7,9,11,13,15},\ g_{12} = X_{2,3,6,7,10,11,14,15},\\
g_{13} =& X_{4,5,6,7,12,13,14,15},\ g_{14} = X_{8,9,10,11,12,13,14,15}.
\end{aligned}
$$

- 4 X-stabilizers and 10 Z-stabilizers, each of weight 4 or 8
- Distinct syndromes for Z errors: $2^4 = 16$. So at most 2 or 3 Z-stabilizers can be measured in parallel with one shared flag.
- Distinct syndromes for X errors: $2^{10} = 1024$. So the 4 X-stabilizers can be measured in parallel with one shared flag.

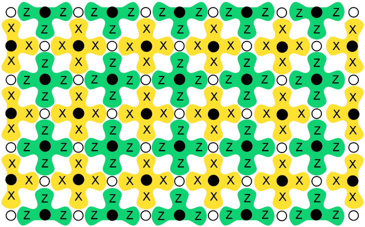

Part (A)
Part (B)
Part (C)
Part (D)
Part (E)

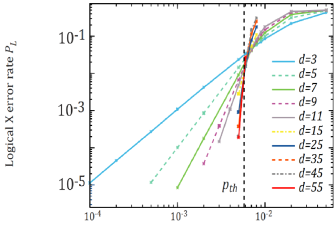
Numerical results

Threshold Theorem

- The error threshold for a procedure is the (physical) error rate below which the logical error rate of the procedure would be lower than the error rate.
- Computation Error threshold: quantum computation can be implemented with arbitrary accuracy provided that the error rate of each physical gate is below a threshold.
- Memory Error threshold: quantum memory can be maintained with an arbitrarily long duration provided that the error rate of each physical gate is below a threshold.

- ▸ Surface codes have the highest known simulated memory error threshold of about 0.1 $\sim$ 0.5%. arXiv:0905.0531, arXiv:1208.0928





  - Qubits are located at the white circles.
  - The stabilizers (black circles) are of low-weight 4 or 3 and have local support.

Comparison of various coding schemes

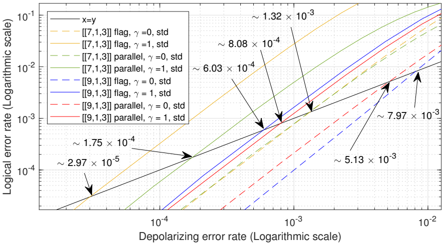
- In $C(g_i)$, each bad location failures leads to residue errors of the same type as g_i, up to one Pauli error of weight one. Therefore, a complete unflagged syndrome extraction circuit is not necessary.

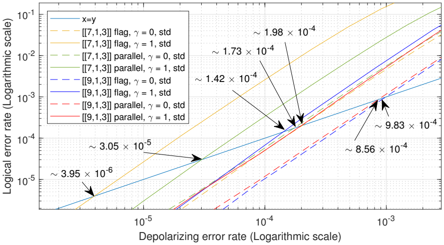
Standard decoder		
	$\gamma = 0$	$\gamma = 1$
Scheme	Memory pseudo-threshold	
[[9, 1, 3]] Parallel (comp.)	4.56×10^{-3}	7.13×10^{-4}
[[9, 1, 3]] Parallel (Alg.)	5.13×10^{-3}	8.08×10^{-4}
Scheme	Computation pseudo-threshold	
[[9, 1, 3]] Parallel (comp.)	6.61×10^{-4}	1.6×10^{-4}
[[9, 1, 3]] Parallel (Alg.)	8.56×10^{-3}	1.98×10^{-4}

Two-step decoder		
	$\gamma = 0$	$\gamma = 1$
Scheme	Memory pseudo-threshold	
[[9, 1, 3]] Parallel (comp.)	7.49×10^{-3}	1.95×10^{-3}
[[9, 1, 3]] Parallel (Alg.)	7.74×10^{-3}	2.01×10^{-3}
Scheme	Computation pseudo-threshold	
[[9, 1, 3]] Parallel (comp.)	1.48×10^{-3}	2.89×10^{-4}
[[9, 1, 3]] Parallel (Alg.)	1.69×10^{-3}	3.02×10^{-4}

Table: Comparisons of the two procedures for unflagged syndrome extraction. (Alg.) denotes the X and Z separated unflagged syndrome extraction, while (comp.) denotes a complete unflagged syndrome extraction.

- γ: the ratio of memory error rate to gate error rate.



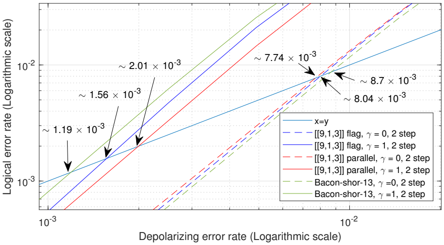


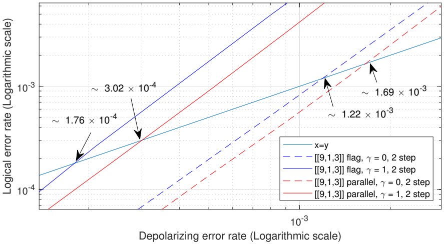
	$\gamma = 0$	$\gamma = 1$
Scheme	Memory pseudo-threshold	
[[7, 1, 3]] Flag	1.32×10^{-3}	2.97×10^{-5}
[[7, 1, 3]] Flag	-	3.39×10^{-5} [CB18]*
[[7, 1, 3]] Parallel [Rei20]	1.29×10^{-3}	1.75×10^{-4}
[[9, 1, 3]] Flag	7.97×10^{-3}	6.03×10^{-4}
[[9, 1, 3]] Parallel	5.13×10^{-3}	8.08×10^{-4}
Scheme	Computation pseudo-threshold	
[[7, 1, 3]] Flag	2.03×10^{-4}	3.95×10^{-6}
[[7, 1, 3]] Parallel [Rei20]	1.73×10^{-4}	3.05×10^{-5}
[[9, 1, 3]] Flag	9.83×10^{-4}	1.42×10^{-4}
[[9, 1, 3]] Parallel	8.56×10^{-4}	1.98×10^{-4}

Table: Memory and computation pseudo-thresholds using the standard decoder.

$*$: A two-round decoder is used.

- γ: the ratio of memory error rate to gate error rate.





	$\gamma = 0$	$\gamma = 1$
Scheme	Memory pseudo-threshold	
[[9, 1, 3]] Flag	8.04×10^{-3}	1.56×10^{-3}
[[9, 1, 3]] Parallel	7.74×10^{-3}	2.01×10^{-3}
Bacon-Shor-13 [LMB18]	8.70×10^{-3}	1.19×10^{-3}
Scheme	Computation pseudo-threshold	
[[9, 1, 3]] Flag	1.22×10^{-3}	1.76×10^{-4}
[[9, 1, 3]] Parallel	1.69×10^{-3}	3.02×10^{-4}

Table: Memory and computation pseudo-thresholds using the two-step decoder.

- M. Li, D. Miller, and K. R. Brown, "Direct measurement of Bacon-Shor code stabilizers," Phys. Rev. A 98, 050301 (2018).

- γ: the ratio of memory error rate to gate error rate.

Future work

- General framework for quantum codes of distance larger than three?
  - R. Chao and B. W. Reichardt, “Flag fault-tolerant error correction for any stabilizer code," PRX Quantum 1, 010302 (2020).
  - C. Chamberland and M. E. Beverland, “Flag fault-tolerant error correction with arbitrary distance codes," Quantum 2, 53 (2018).
- General decoding algorithms for flagged quantum error correction, other than the lookup table.
- Flagged syndrome extraction for subsystem codes.