Dispersive analysis of neutral meson mixing

Hsiang-nan Li

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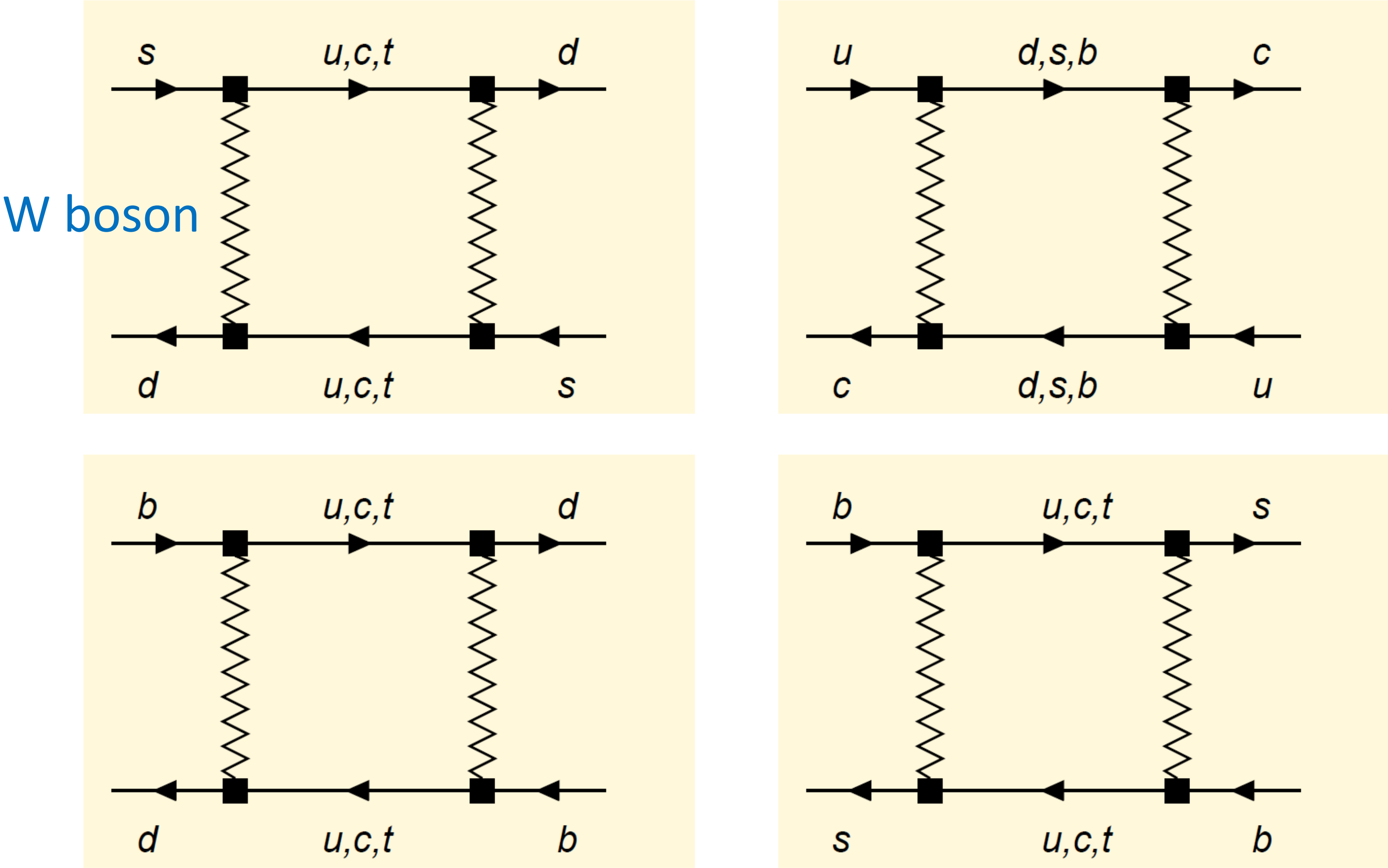
Neutral meson mixing

Only K^0, D^0, B_d^0, and B_s^0 mesons mix with their antiparticles:



Exceptional D mixing

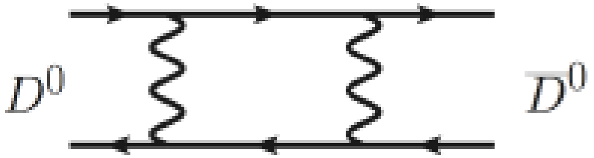
- Perturbation theory explains K, Bd, Bs mixing satisfactorily
- Heavy quark expansion (HQE) works well

Beneke, Buchalla, Dunietz, 1996;
Ciuchini, 2003; Lenz, Nierste 2007

- Short-distance contributions amount up to 89% of measured mass difference of two mass eigenstates for kaon mixing Brod, Gorbahn, 2012
- HQE results lower than observed D mixing by 4 orders of magnitude (10^-7 vs 10^-3)

Golowich, Petrov 2005

$D^0 - \overline{D}^0$ Mixing



- The time evolution

2X2 matrices

$$i\frac{\partial}{\partial t}\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix} = \left(\mathbf{M} - \frac{i}{2}\mathbf{\Gamma}\right)\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix}$$

virtual real contribution

- Mass eigenstates in terms of weak eigenstates

$$|D_{1,2}\rangle = p|D^0\rangle \pm q|\overline{D}^0\rangle$$

- Mass difference and Width difference

$$\boxed{x \equiv \frac{\Delta m}{\Gamma} = \frac{m_1 - m_2}{\Gamma}}$$

$$\boxed{y \equiv \frac{\Delta\Gamma}{2\Gamma} = \frac{\Gamma_1 - \Gamma_2}{2\Gamma}}$$

OPE result for short-distance

D meson

Golowich and Petrov 2005

SM $$\begin{cases} x_D \simeq 6 \times \boxed{10^{-7}} \\ y_D \simeq 6 \times \boxed{10^{-7}} \end{cases}$$

much below data

Suppressed by GIM

HFLAV at CHARM18

Exp. $$\begin{cases} x_D = 3.6^{+2.1}_{-1.6} \times 10^{-3} \\ y_D = 6.7^{+0.6}_{-1.3} \times 10^{-3} \end{cases}$$

B_s meson

Artuso, Borissov and Lenz, 2016

SM $$\begin{cases} \Delta M_s = (18.3 \pm 2.7)\mathrm{ps}^{-1} \\ \Delta\Gamma_s = (0.088 \pm 0.020)\mathrm{ps}^{-1} \end{cases}$$

HFAG

Exp. $$\begin{cases} \Delta M_s = (17.757 \pm 0.021)\ \mathrm{ps}^{-1} \\ \Delta\Gamma_s = (0.082 \pm 0.006)\ \mathrm{ps}^{-1} \end{cases}$$

B_d meson

Artuso, Borissov and Lenz, 2016

SM $$\begin{cases} \Delta M_d = (0.528 \pm 0.078)\ \mathrm{ps}^{-1} \\ \Delta\Gamma_d = (2.61 \pm 0.59) \cdot 10^{-3}\ \mathrm{ps}^{-1} \end{cases}$$

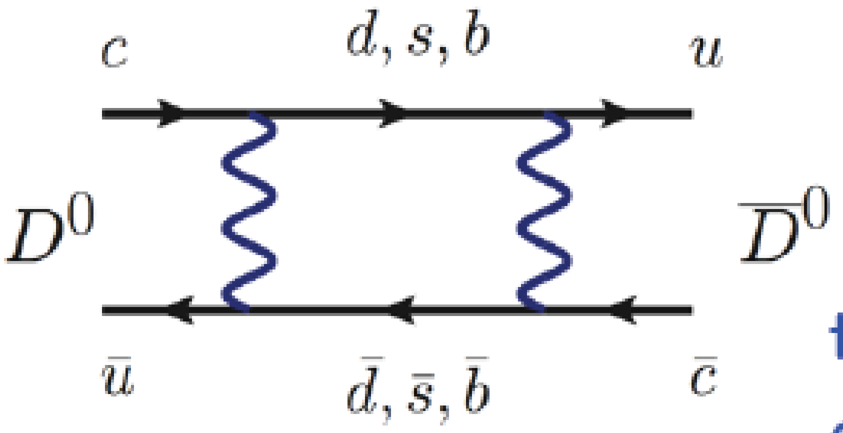
HFAG

Exp. $$\begin{cases} \Delta M_d = (0.5055 \pm 0.0020)\ \mathrm{ps}^{-1} \\ \Delta\Gamma_d = 0.66(1 \pm 10) \cdot 10^{-3}\ \mathrm{ps}^{-1} \end{cases}$$

- Order of magnitude is not reproduced for D meson.
- The SM is in agreement with data for B(d, s) meson.

SU(3) breaking

HQE performed at quark level, quark-hadron duality



third generation can be neglected

$$V_{cd}^* V_{ud} + V_{cs}^* V_{us} + \cancel{V_{cb}^* V_{ub}} = 0$$

$-\lambda * 1$

$1 * \lambda$

$\lambda^2 * \lambda^3$

$\lambda \sim 0.2$

Wolfenstein Parameter; Cabibbo angle

GIM: x~y~0 in the SU(3) limit

Non-zero mixing from SU(3) breaking effects

Main ideas

- Solve dispersion relation obeyed by x(s) and y(s) for a fictitious D meson with arbitrary mass squared s from HQE inputs at high s
- x: real part; y: imaginary part
- CKM factors are independent mathematically
- Get solution for each channel with quark pair dd, ds, ss in loop
- It is likely that sum of three solutions escapes GIM mechanism

Dispersion relation

- Consider “fictitious D meson” of mass squared s

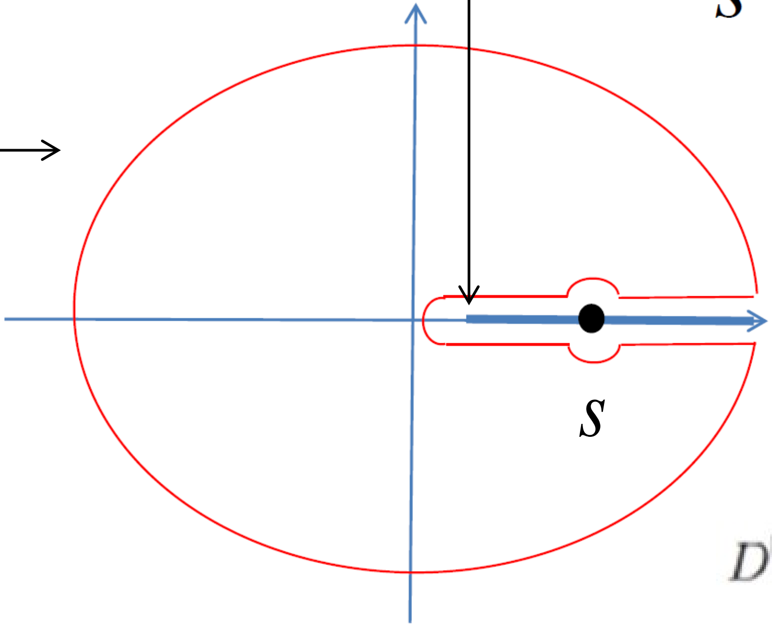
real part $$M_{12}(s) = \frac{P}{2\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Gamma_{12}(s')}{s-s'}$$ imaginary part

$$M_{12}(s) - \frac{i}{2}\Gamma_{12}(s) = \langle D^0|\mathcal{H}|\bar{D}^0\rangle$$

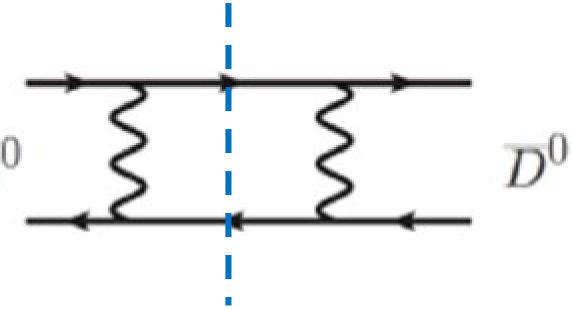
x y

large circle contribution suppressed

$\Gamma_{12} \sim 1/s^2$



branch cut caused by intermediate states



key formulas

- Decomposition $$\Gamma_{12}(s) = \sum_{i,j=d,s,b} \lambda_i \lambda_j \Gamma_{ij}(s) \qquad \lambda_k \equiv V_{ck} V^*_{uk},\ k = d, s, b,$$
- Dispersion relation $$M_{ij}(s) = \frac{1}{2\pi} \int_{m_{IJ}}^{R} ds' \frac{\Gamma_{ij}(s')}{s - s'} + \frac{1}{2\pi i} \int_{C_R} ds' \frac{\Pi^{\rm box}_{ij}(s')}{s - s'}$$

$$m_{\pi\pi} = 4m_\pi^2 \qquad m_{\pi K} = (m_\pi + m_K)^2,\ m_{KK} = 4m_K^2$$

physical threshold

SU(3) breaking

- Box contribution $$M^{\rm box}_{ij}(s) = \frac{1}{2\pi} \int_{m_{ij}}^{R} ds' \frac{\Gamma^{\rm box}_{ij}(s')}{s - s'} + \frac{1}{2\pi i} \int_{C_R} ds' \frac{\Pi^{\rm box}_{ij}(s')}{s - s'}$$

quark-level threshold ~ 0

- Equality of M(s) at high s $$\int_{m_{IJ}}^{R} ds' \frac{\Gamma_{ij}(s')}{s - s'} = \int_{m_{ij}}^{R} ds' \frac{\Gamma^{\rm box}_{ij}(s')}{s - s'}$$
- UV regularization $$\Delta\Gamma_{ij}(s, \Lambda) = \Gamma_{ij}(s) - \Gamma^{\rm box}_{ij}(s)\{1 - \exp[-(s - m_{IJ})^2/\Lambda^2]\}$$

$$\int_{m_{IJ}}^{\infty} ds' \frac{\Delta\Gamma_{ij}(s', \Lambda)}{s - s'} = \int_{m_{IJ}}^{\infty} ds' \frac{\Gamma^{\rm box}_{ij}(s') \exp[-(s' - m_{IJ})^2/\Lambda^2]}{s - s'} + \int_{m_{ij}}^{m_{IJ}} ds' \frac{\Gamma^{\rm box}_{ij}(s')}{s - s'}$$

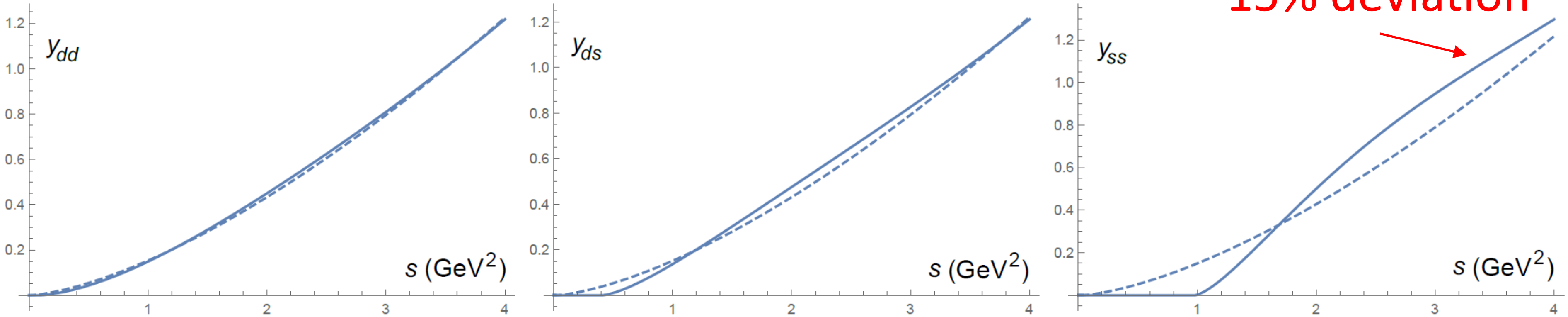
T

Solutions

$$\Lambda = 5\ \mathrm{GeV}^2$$

$$y_{ij}(s) \equiv \Gamma_{ij}(s)/\Gamma \longleftarrow$$ total decay width

15% deviation



dashed: box inputs; solid: solutions

similarity of three inputs yields strong GIM suppression

deviation of solutions from inputs enhanced by thresholds

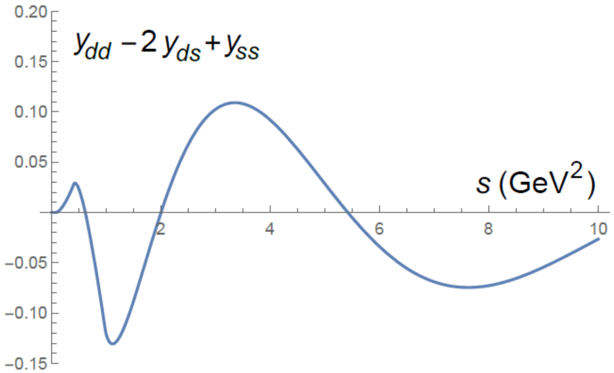
not significant, quark-hadron duality holds reasonably

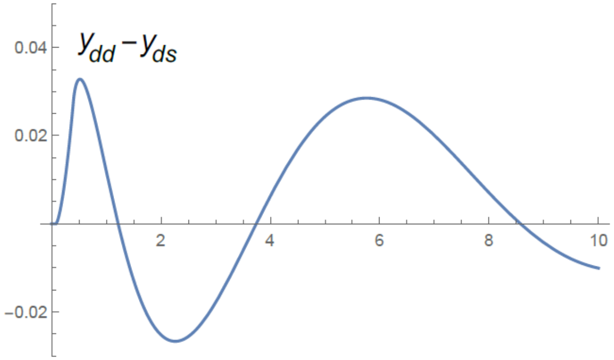
sum over three solutions avoids GIM mechanism

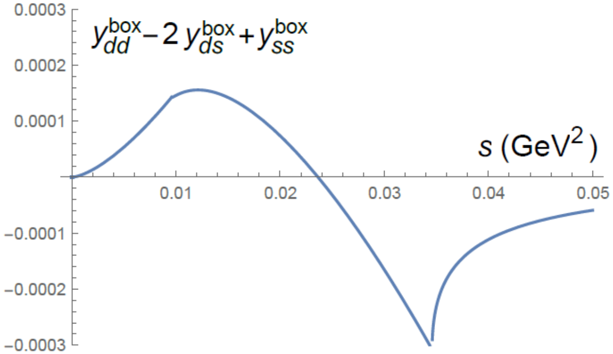
Combinations

$$\Gamma_{12}(m_D^2) = \lambda_s^2[\Gamma_{dd}(m_D^2) - 2\Gamma_{ds}(m_D^2) + \Gamma_{ss}(m_D^2)] \\ + 2\lambda_s\lambda_b[\Gamma_{dd}(m_D^2) - \Gamma_{ds}(m_D^2)] + \lambda_b^2\Gamma_{dd}(m_D^2)$$

$\Lambda = 5\ \mathrm{GeV}^2$





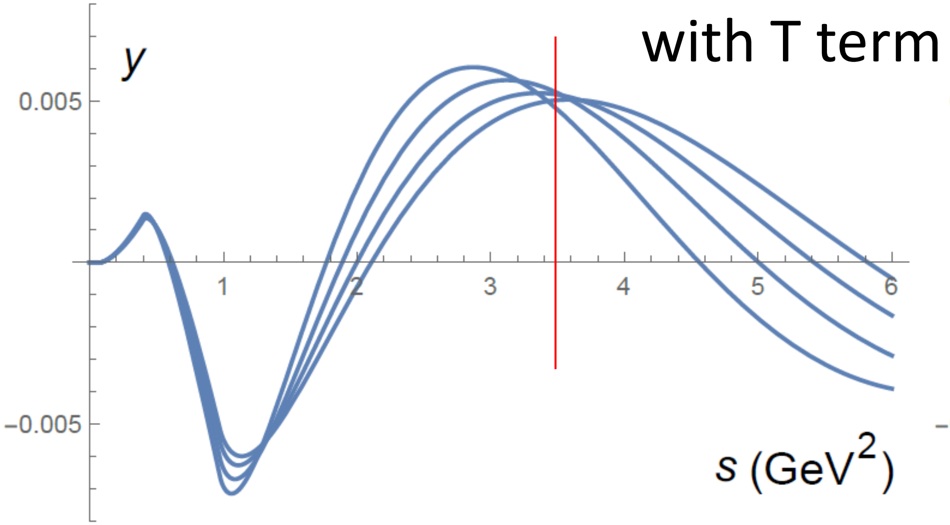


GIM mechanism avoided
small duality violation
enough for explaining data

4 orders of
magnitude smaller

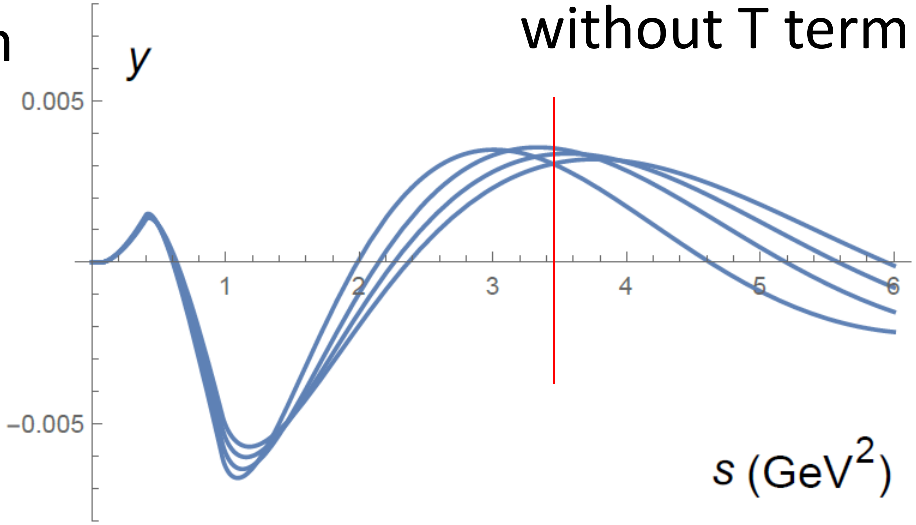
Parameter y

$\Lambda = 4.0\ \mathrm{GeV}^2,\ 4.5\ \mathrm{GeV}^2,\ 5.0\ \mathrm{GeV}^2$ and $5.5\ \mathrm{GeV}^2$ from left to right



with T term

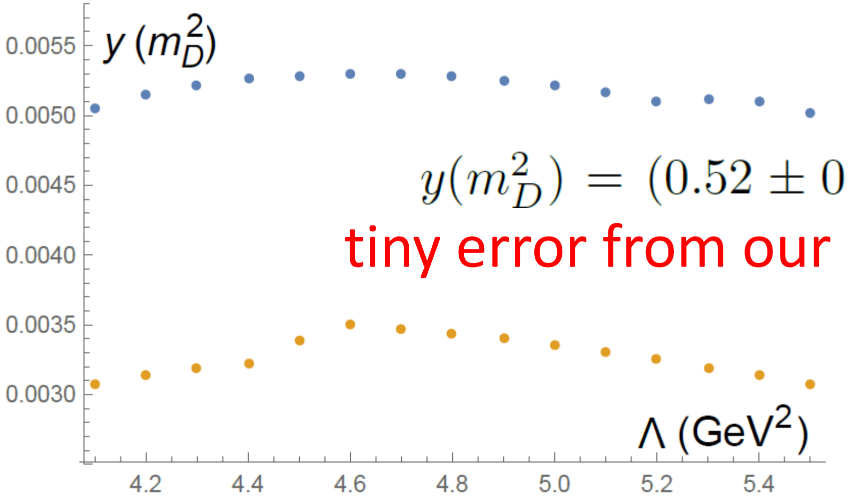
all curves cross at $s \approx m_D^2$



without T term

overlap area spreads

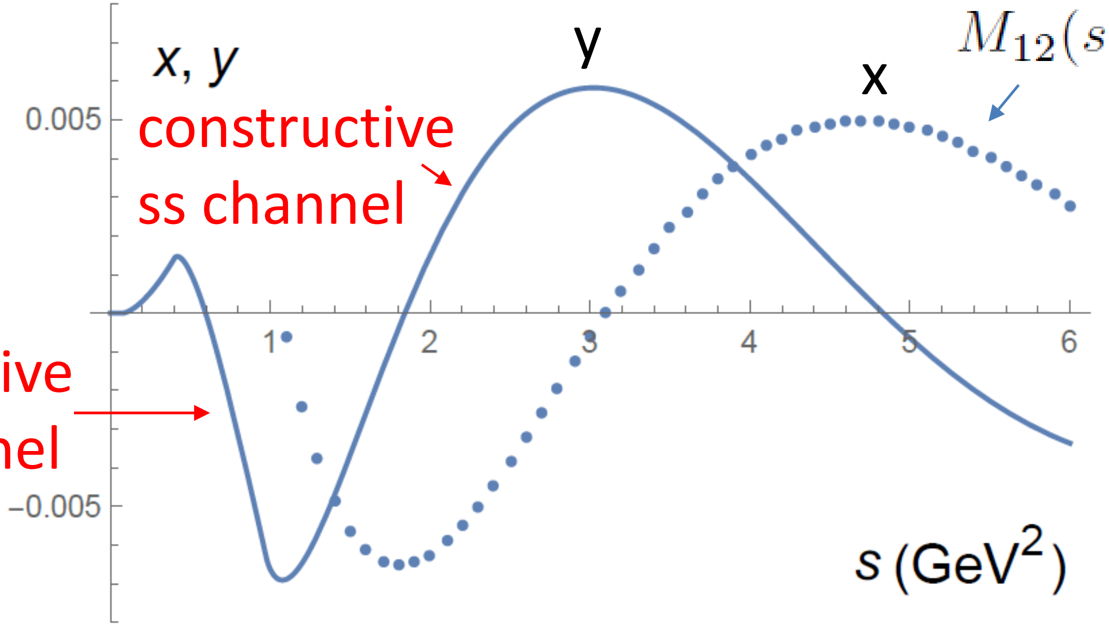
T term stabilizes solutions of $y(m_D^2)$ with respect to variation of Λ



$y(m_D^2) = (0.52 \pm 0.01)\%,$

tiny error from our method

Results of x, y



constructive
ss channel

destructive
ds channel

$$M_{12}(s) = \frac{P}{2\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Gamma_{12}(s')}{s-s'}$$

$$x(m_D^2) = (0.21^{+0.04}_{-0.07})\%, \quad y(m_D^2) = (0.52 \pm 0.03)\%$$

include errors from ms

Data (CP conserving)

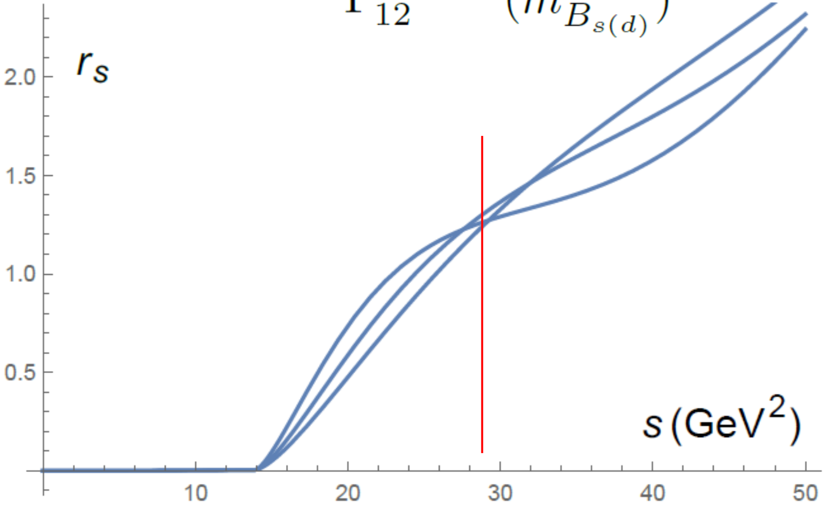
$$x = (0.44^{+0.13}_{-0.15})\%, \quad y = (0.63 \pm 0.07)\%$$

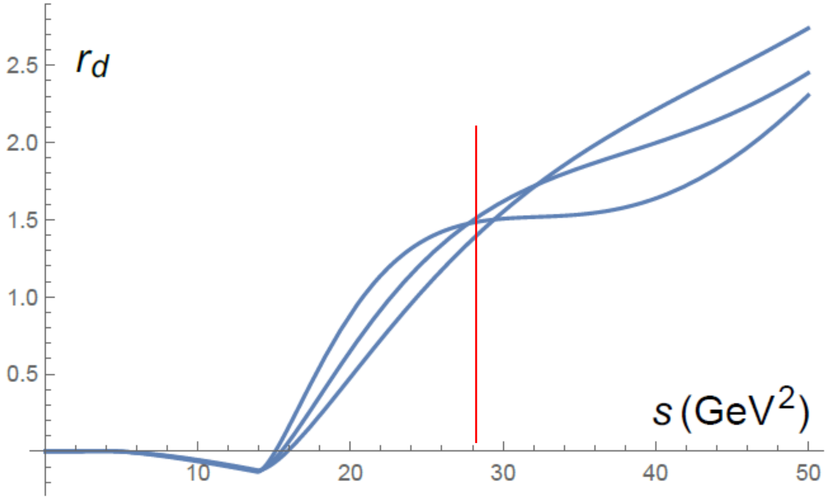
consistency expected to be improved by including higher-order inputs

Bd, Bs mixings

$$r_{s(d)}(s) \equiv \frac{\Gamma_{12}^{s(d)}(s)}{\Gamma_{12}^{s(d)\mathrm{box}}(m_{B_{s(d)}}^2)}$$

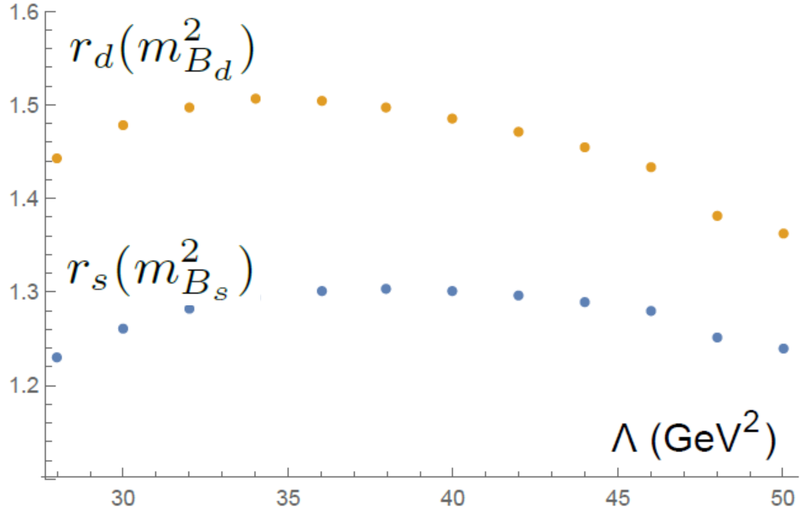
deviation from box inputs





$\Lambda = 30\ \mathrm{GeV}^2,\ 40\ \mathrm{GeV}^2$ and $50\ \mathrm{GeV}^2$

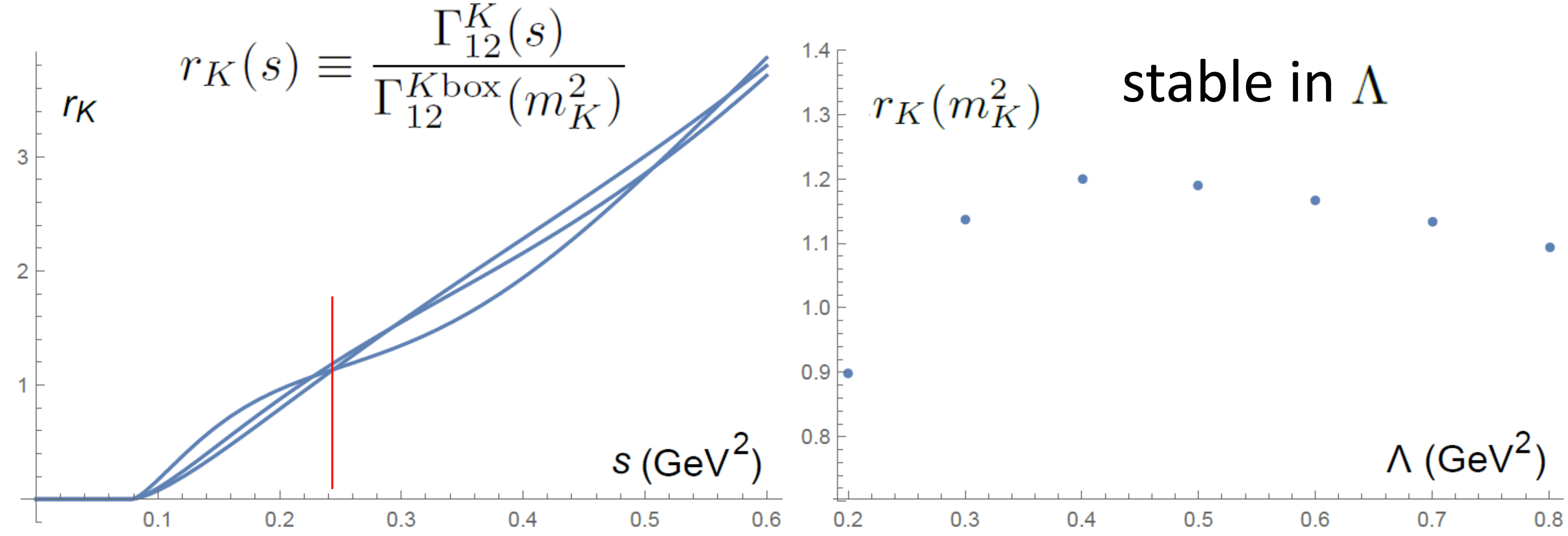
stable in Λ



solutions do not deviate from inputs much

explain why HQE work for Bd, Bs mixing

K mixing



$$r_K(s) \equiv \frac{\Gamma_{12}^{K}(s)}{\Gamma_{12}^{K\,\mathrm{box}}(m_K^2)}$$

stable in Λ

$\Lambda = 0.3\ \mathrm{GeV}^2,\ 0.5\ \mathrm{GeV}^2\ \mathrm{and}\ 0.7\ \mathrm{GeV}^2$

solutions do not deviate from inputs much
explain why pert theory works for K mixing

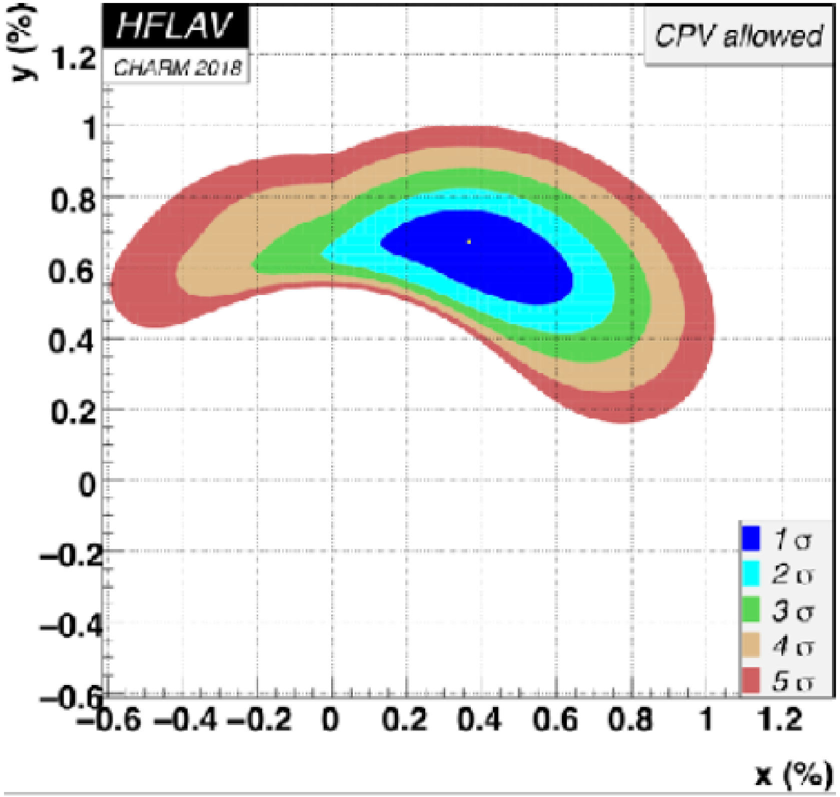
turn out that D has smallest duality violation, 15%
K: 20%, Bd: 50%, Bs: 30%

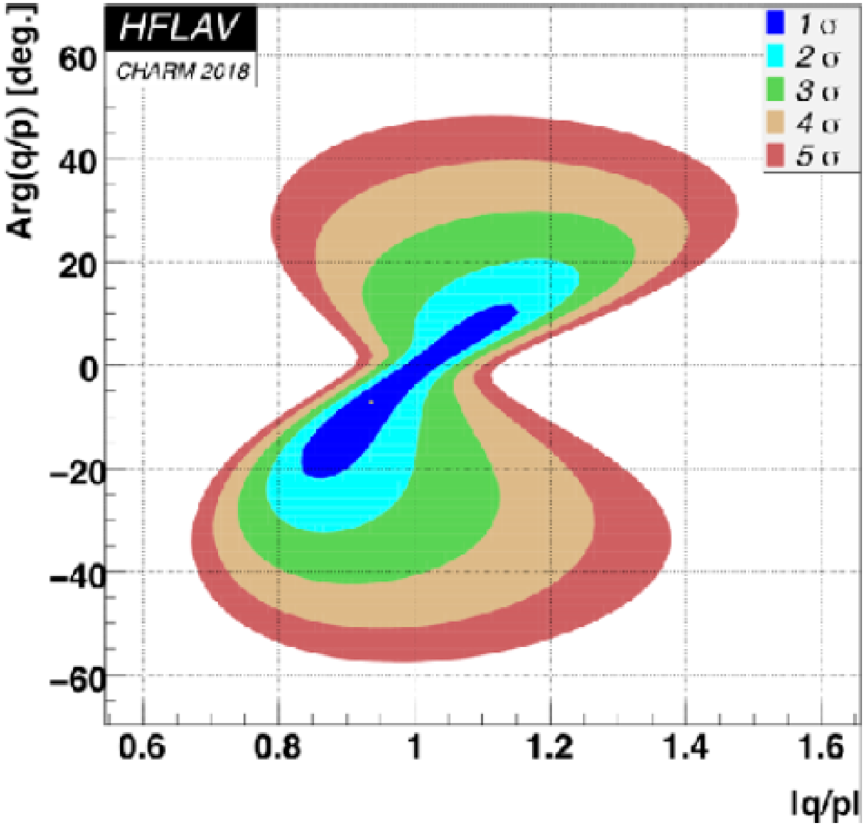
Conclusion

- Quark-hadron duality assumed in HQE still holds for D mixing
- D mixing data puzzling not because they are too large, but because theory too small
- u, d, s quark masses degenerate in SU(3) limit, and CKM factors happen to be the same but opposite in sign for different channels
- SU(3) breaking is key to understand D mixing
- Tiny duality violation enough for explaining data
- LO analysis, not aim at perfect match with data

Back-up slides

$D^0 - \bar{D}^0$ mixing: data





$(x, y) = (0, 0)$ is excluded by 11.5σ

➡ Mixing is confirmed

$(|q/p|, \arg) = (1, 0)$ is allowed

➡ No evidence of CP violation

Parametrization

~10%, convergent

- Propose $$y(s) = \frac{Ns[b_0 + b_1(s-m^2) + b_2(s-m^2)^2]}{[(s-m^2)^2 + d^2]^2}$$

pion mass neglected

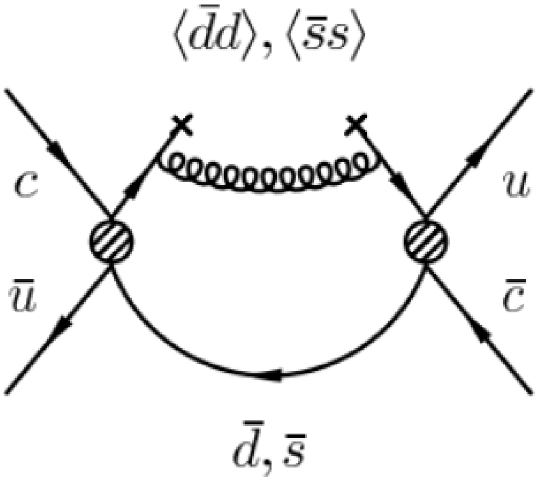
- Obeys boundary condition y(0)=0, and distributes around scale m in narrow width d
- Normalization constant N chosen such that $Ns/[(s-m^2)^2 + d^2]^2 \to \delta(s-m^2)$ as $d \to 0.$
- For given m, d, tune parameters to minimize

$$\sum_{i=1}^{200} \left| \int_0^{\Lambda} ds' \frac{y(s')}{s_i - s'} - \omega(s_i) \right|^2 \qquad 30\ \mathrm{GeV}^2 < s_i < 250\ \mathrm{GeV}^2.$$

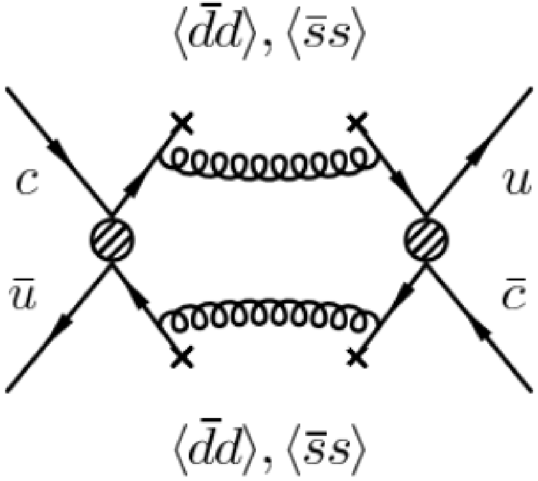
goodness of fit

Higer-dimensional operators

[1002.4794]



D=9



D=12

$$\mathcal{O}(\alpha_s(4\pi)\langle \bar{q}q\rangle/m_c^3)$$

$$\mathcal{O}(\alpha_s^2(4\pi)^2\langle \bar{q}q\rangle^2/m_c^6)$$

y	no GIM	with GIM
$D = 6, 7$	$2 \cdot 10^{-2}$	$5 \cdot 10^{-7}$
$D = 9$	$5 \cdot 10^{-4}$	?
$D = 12$	$2 \cdot 10^{-5}$	?

do not work

Exclusive Approach

phase space factor

$$y \approx \frac{\Gamma_+ - \Gamma_-}{2\Gamma} = \frac{1}{2}\sum_n (Br(D_+ \to n) - Br(D_- \to n))$$

$$= \frac{1}{2\Gamma}\sum_n \rho_n \left(|\langle D_+|H_w|n\rangle|^2 - |\langle D_-|H_w|n\rangle|^2\right),$$

$CP|n\rangle = \eta_{\rm CP}|\bar{n}\rangle$ $\quad \mathcal{A}(\overline{D}^0 \to n) = \mathcal{A}(D^0 \to \bar{n})$ No CPV

$$y = \frac{1}{2\Gamma}\sum_n \rho_n \eta_{\rm CP}(n)(\langle D^0|H_w|n\rangle\langle\bar{n}|H_w|D^0\rangle + \langle D^0|H_w|\bar{n}\rangle\langle n|H_w|D^0\rangle)$$

$$= \sum_n \eta_{\rm CKM}(n)\eta_{\rm CP}(n)\cos\delta_n\sqrt{Br(D^0 \to n)Br(D^0 \to \bar{n})},$$

$(-1)^{n_S}$

number of s quarks

sum up all the intermediate states

$$y_{PP+VP} = (0.36 \pm 0.26)\% \text{ or } (0.24 \pm 0.22)\%$$

[Cheng, Chiang, 2010]

Factorization-assisted topological amplitudes

- With higher precision in global fit

$$y_{PP+PV} = (0.21 \pm 0.07)\%,$$

1705.07335

$$y_{VV} = (0.28 \pm 0.47) \times 10^{-3}$$

- PP, PV, VV (amount up to 50% Br of D decays) cannot explain y
- Other 2-body and multi-body modes relevant
- Not applicable to evaluation of x; exclusive approach not practical

Buras' formulas for B mixing

virtual contribution

$$M_{12}=\frac{G_F^2 f_p^2 m_p M_W^2 B_p}{12\pi^2}\left(\lambda_c^2 U_{cc}^{(d)}+\lambda_t^2 U_{tt}^{(d)}+2\lambda_c\lambda_t U_{ct}^{(d)}\right)$$

real contribution

$$\Gamma_{12}=\frac{G_F^2 f_p^2 m_p M_W^2 B_p}{12\pi^2}\left(\lambda_c^2 U_{cc}^{(a)}+\lambda_t^2 U_{tt}^{(a)}+2\lambda_c\lambda_t U_{ct}^{(a)}\right)$$

$$U_{ij}^{(x)}=A_{uu}^{(x)}+A_{ij}^{(x)}-A_{ui}^{(x)}-A_{uj}^{(x)} \qquad x=\mathrm{a,\,d}$$

$$x_i=\frac{m_i^2}{M_W^2}$$

source of SU(3) breaking

$$A_{ij}^{(a)}=-\frac{\pi}{2x_h^2}\,\frac{1}{(1-x_i)(1-x_j)}\sqrt{(x_i-x_j)^2+x_h^2-2x_h(x_i+x_j)}$$

$$\times\{(1+\tfrac{1}{4}x_i x_j)[3x_h^2-x_h(x_i+x_j)-2(x_i-x_j)^2]+2x_h(x_i+x_j)(x_i+x_j-x_h)\}$$

How to solve dispersion relation?

- Typical Fredholm integral equation

spectral density, unknown

notoriously
difficult to solve

$$\int_0^\infty dy \frac{\rho(y)}{x-y} = \omega(x) \longleftarrow$$

box (HQE) input

- Discretize integral equation usually

$$\sum_i M_{ij}\rho_j = \omega_i \qquad M_{ij} = \begin{cases} 1/(i-j), & i \neq j \\ 0, & i = j \end{cases}$$

unknowns input

- Rows Mij and M(i+1)j become almost identical and matrix M becomes singular quickly for fine meshes, solution diverges

Idea

- Suppose $\rho(y)$ decreases quickly enough
- Expansion into powers of 1/x justified

$$\frac{1}{x-y} = \sum_{m=1}^{N} \frac{y^{m-1}}{x^m} \qquad \omega(x) = \sum_{n=1}^{N} \frac{b_n}{x^n}$$

true for HQE

- Suppose $\omega(x)$ can be expanded
- Decompose

$$\rho(y) = \sum_{n=1}^{N} a_n \underset{\uparrow}{y^\alpha} e^{-y} L_{n-1}^{(\alpha)}(y)$$

generalized Laguerre polynomials

depend on $\rho(y)$ at $y \to 0$

$\alpha = 3/2,$

- Orthogonality

$$\int_0^\infty \underline{y^\alpha e^{-y}} L_m^{(\alpha)}(y) L_n^{(\alpha)}(y) dy = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{mn}$$

Initial condition

- Near threshold, fictitious D meson decay dominated by D -> PP

center-of-mass momentum of P

- Two-body decay width $p_c|\mathcal{M}|^2/s,$

$\mathcal{M} \propto s - m_P^2$ in the naive factorization

$p_c \sim O(m_P)$, $\mathcal{M} \sim O(m_P^2)$ and $\Gamma_{ij} \sim O(m_P^3)$ around the threshold $s \sim O(m_P^2)$

- $\Gamma_{ij}(s) \sim O(m_{IJ}^{3/2})$ $\alpha = 3/2,$ for Laguerre expansion
- Agree with chiral perturbation theory

Inverse matrix method

- Equate coefficients of $1/x^n$ on two sides

$$Ma = b \qquad M_{mn} = \int_0^\infty dy y^{m-1+\alpha} e^{-y} L_{n-1}^{(\alpha)}(y)$$

matrix

input $b = (b_1, b_2, \cdots, b_N)$

unknown $a = (a_1, a_2, \cdots, a_N)$

- Solution $a = M^{-1}b$, easy by using Math
- True solution can be approached by increasing N, before M^{-1} diverges, stability in N

N=15~20 usually

- Additional polynomial gives $1/x^{N+1}$ correction, beyond considered precision

due to orthogonality