The Facts and Fantasies of Angular Momentum

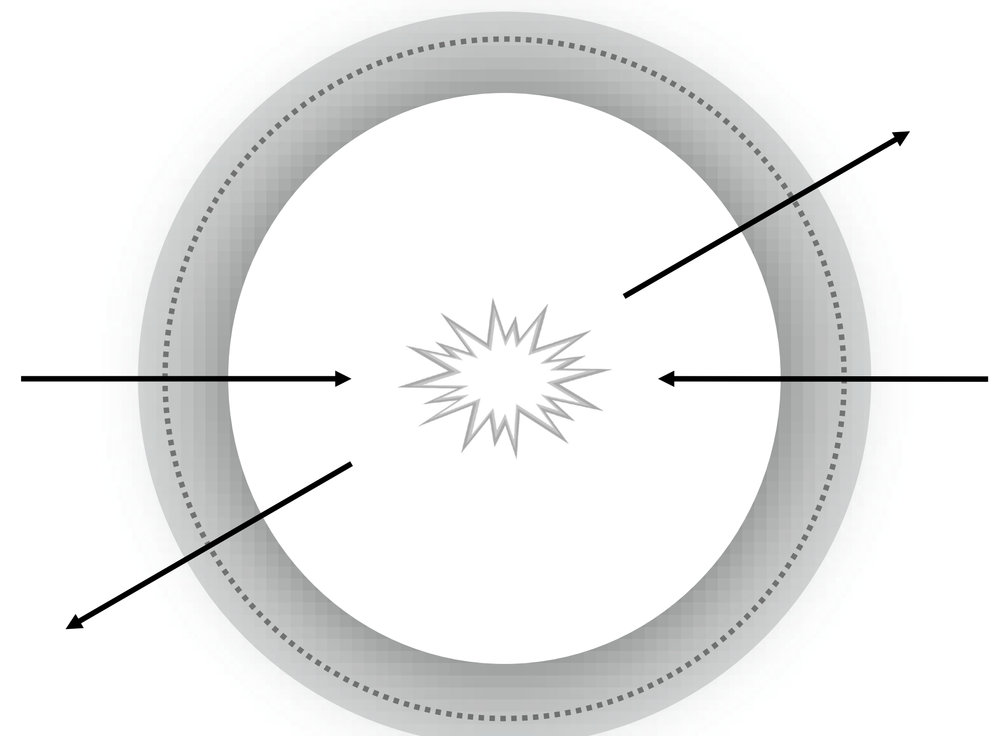
Chia-Hsien Shen 沈家賢 (UC San Diego) @ NCTS

PRL [2203.04283] w/ Manohar and Ridgway

What is the radiated angular momentum after scattering?

— a basic but mysterious quantity

radiation: $E,\ J^{\mu\nu}$



What is the radiated angular momentum after scattering?

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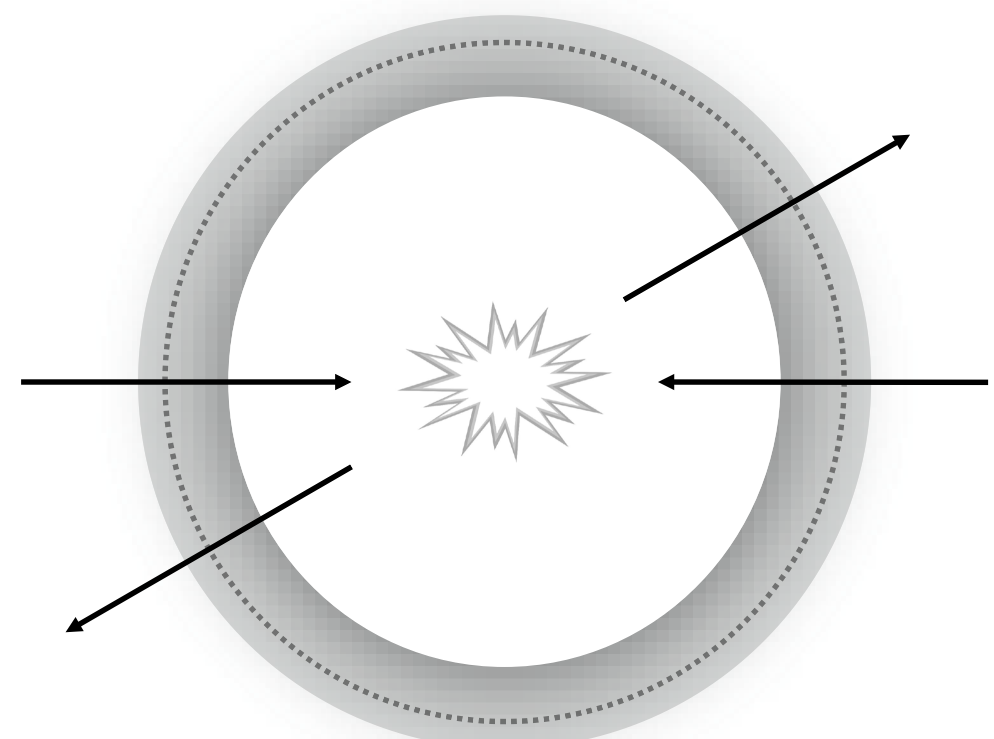
1. $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$
2. If $E = 0$, then $J^{\mu\nu} = 0$
3. Heavy particle decouples

radiation: $E,\ J^{\mu\nu}$



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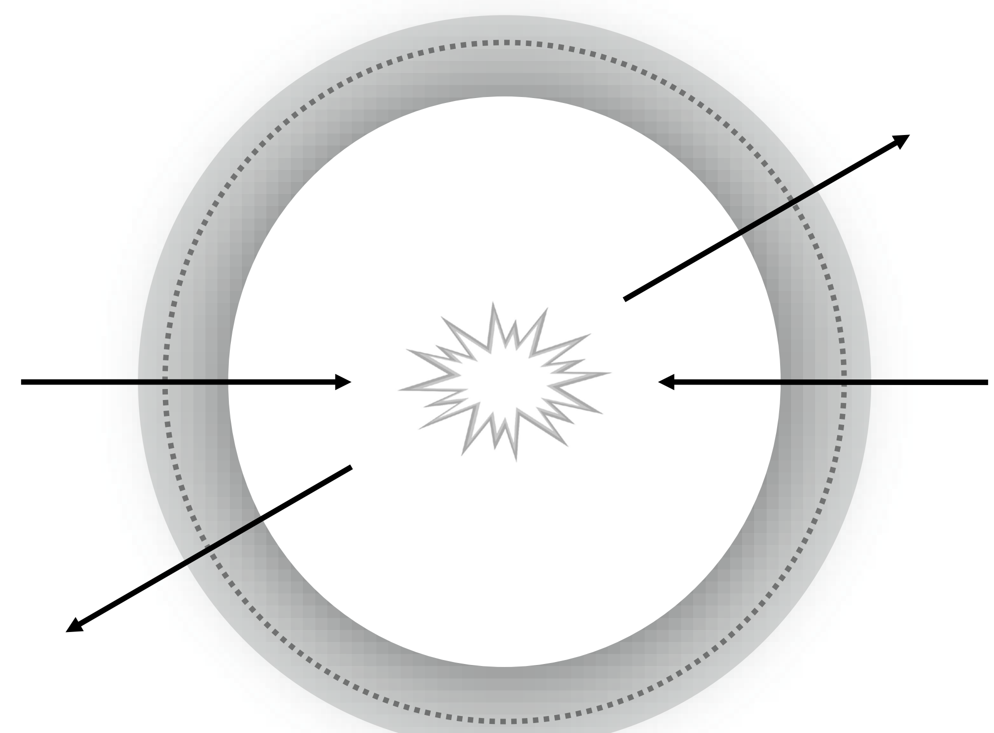

center of mass $\quad v \to 0 \quad$ rest frame

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The G(1) Problem: Fact and Fantasy on the Spin of the Proton #1

R.L. Jaffe (MIT, LNS), Aneesh Manohar (MIT, LNS) (Apr, 1989)

Published in: *Nucl.Phys.B* 337 (1990) 509-546

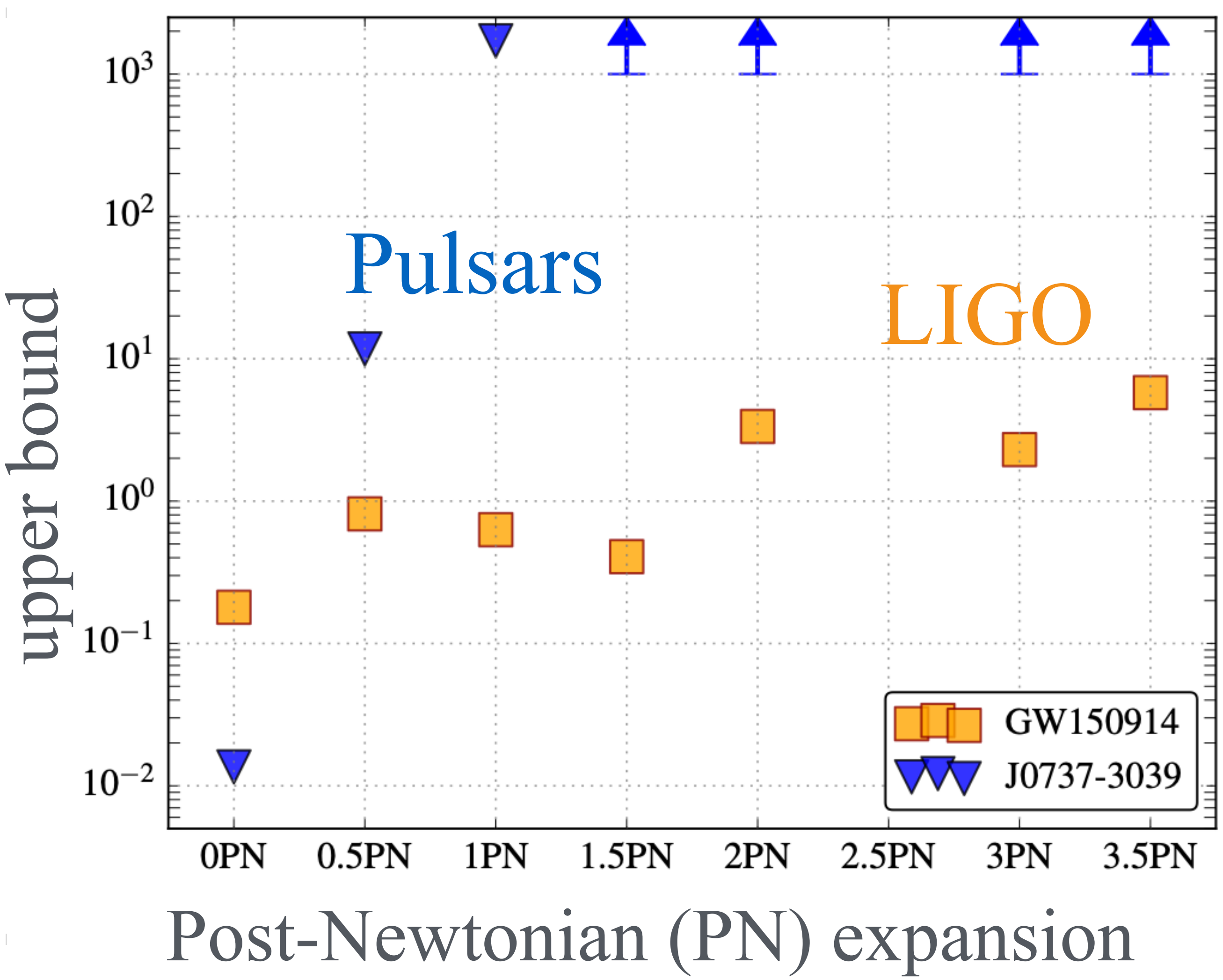
pdf DOI cite 914 citations

Facts or Fantasies?

Motivation:

Precision Frontier of Gravitational Waves

The Need of Precision Gravity



[LIGO/Virgo: 1602.03841]

Tidal Effect: 5PN for BH

Einstein/LISA: 6PN+

LISA
Einstein
Cosmic Explorer

Tian-Qin

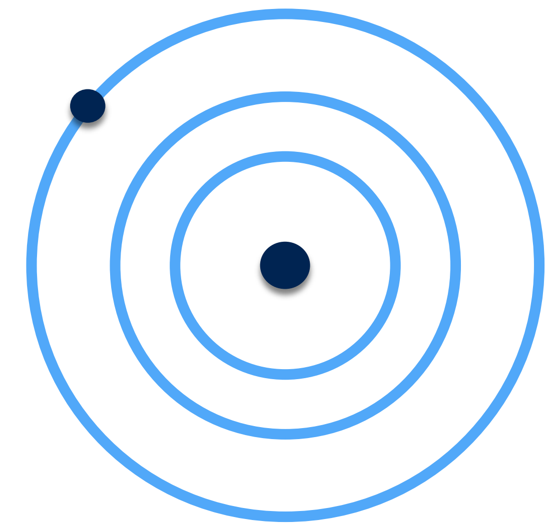
LIGO-India

KAGRA

LIGO/Virgo

2020 2025 2030

Symmetry $\longrightarrow$ Dynamics



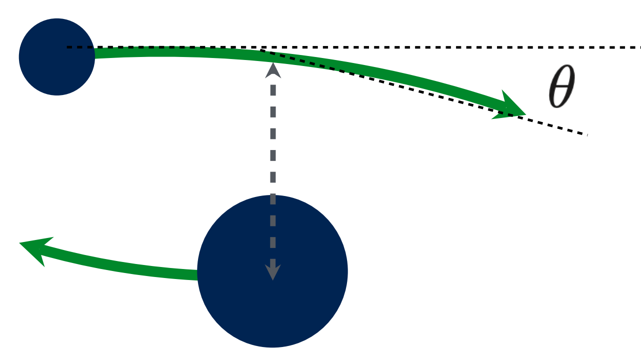
$$V(r, \vec{p}^2, \vec{p} \cdot \hat{r})$$

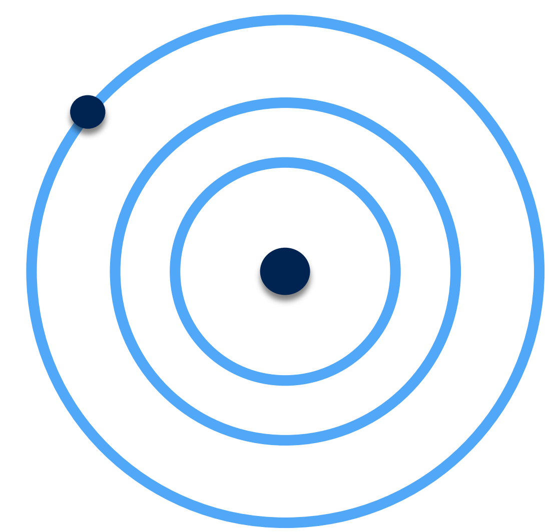
Effective Field Theory

[Goldberger, Rothstein '04]

Symmetry $\longrightarrow$ Conservative Dynamics

Lorentz invariance





$\theta(J, E)$

$V(r, \vec{p}^{\,2}, \cancel{\vec{p}\cdot\hat{r}})$

Effective Field Theory

[Goldberger, Rothstein ’04]

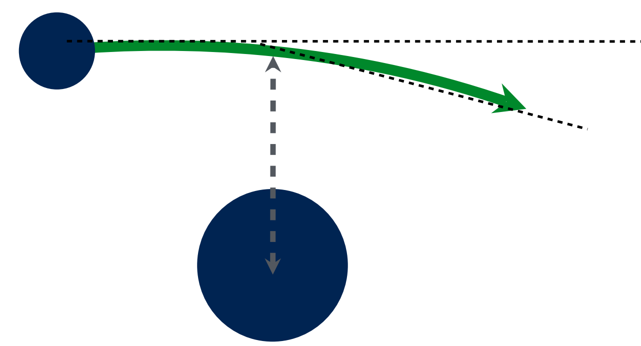
[Buonanno, Damour] [Damour]

[Neill, Rothstein] [Cheung, Rothstein, Solon]

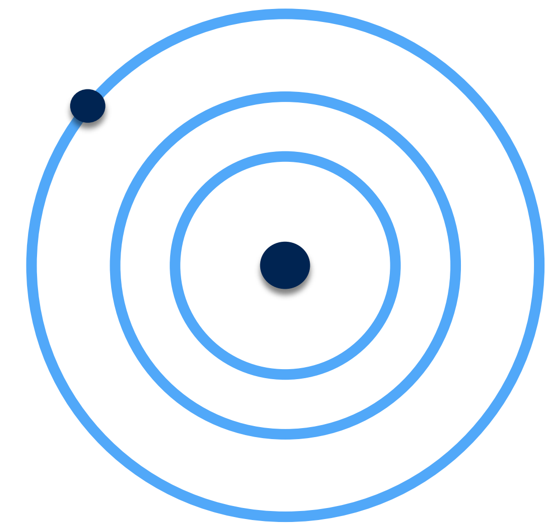
Symmetry $\longrightarrow$ **Conservative Dynamics**

Lorentz invariance

$$\gamma = \sigma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}$$



$$\theta(J, E)$$



$$V(r, \vec{p}^{\,2})$$

[Buonanno, Damour] [Damour]
[Neill, Rothstein] [Cheung, Rothstein, Solon]

Symmetry $\longrightarrow$ **Conservative Dynamics**

$$\gamma = \sigma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}$$

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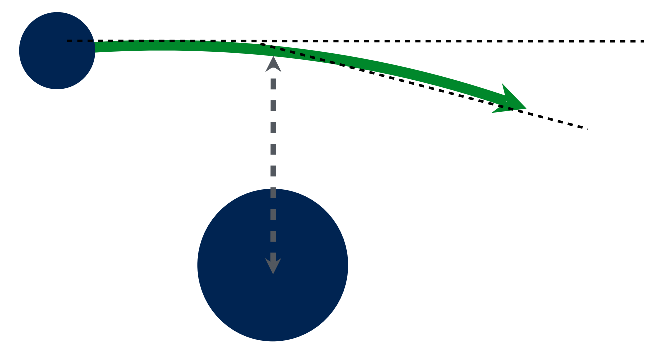

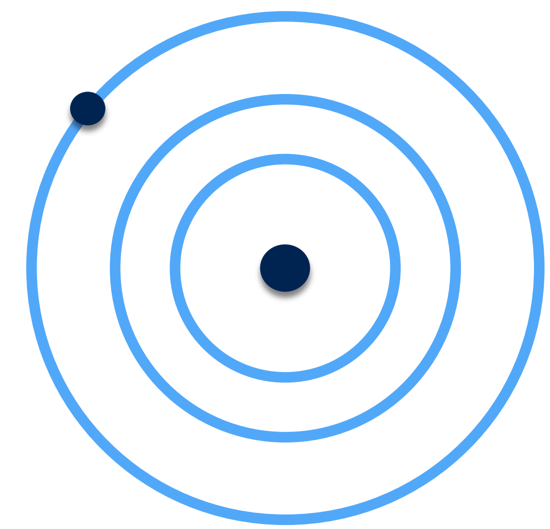

$$\theta(J, E)$$

$$V(r, \vec{p}^{\,2})$$

Scattering Amplitudes

Generalized unitarity [Bern, Dixon, Dunbar, Kosower]…
GR=YM^2 [Kawai, Lewellen, Tye][Bern, Carrasco, Johansson]…

Effective Field Theory

[Goldberger, Rothstein ’04]

See Snowmass white paper [Buonanno, Khalil, O’Connell, Roiban, Solon, Zeng, 2204.05194]

Conservative Dynamics

- Impressive progress from both traditional and new methods

- Higher order potential
  [Bern, Cheung, Parra-Martinez, Roiban, Ruf, **CHS**, Solon, Zeng]
  [Bini, Damour, Geralico] [Blumlein, Maier, Marquard, Schafer] [Dlapa, Kalin, Liu, Porto]
  [Bjerrum-Bohr, Cristofoli, Damgaard, Festuccia, Plante, Vanhove] [di Vecchia, Heissenberg, Russo, Veneziano]
  [Kosower, Maybee, O’Connell] [Damgaard, Haddard, Helset] [Jakobsen, Mogull, Plefka, Steinhoff]
  [Brandhuber, Chen, Travaglini, Wen] [Kol, O’Connell, Telem]….
- Spin
  [Vaidya] [Vines] [Guevara, Ochirov, Vines] [Chung, Huang, Kim, Lee] [Aoude, Haddard, Helset]
  [Bern, Luna, Roiban, **CHS**, Zeng][Bern, Kosmopoulos, Luna, Roiban, Teng]
  [Steinhoff, Levi] [Levi, Von Hippel, McLeod] [Liu, Porto, Yang]
  [Maybee, O’Connell, Vines] [Jakobsen, Mogull, Plefka, Steinhoff] [Chiodaroli, Johansson, Pichini]…

- Tidal effects
  [Bini, Damour][Cheung, Solon][Kalin, Liu, Porto][Aoude, Haddard, Helset]
  [Bern, Parra-Martinez, Roiban, **CHS**, Sawyer] [Cheung, Shah, Solon]…

See Snowmass white paper [Buonanno, Khalil, O'Connell, Roiban, Solon, Zeng, 2204.05194]

Conservative Dynamics

- Impressive progress from both traditional and new methods

- Higher order potential
  [Bern, Cheung, Parra-Martinez, Roiban, Ruf, **CHS**, Solon, Zeng]
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See Snowmass white paper [Buonanno, Khalil, O'Connell, Roiban, Solon, Zeng, 2204.05194]

Conservative Dynamics

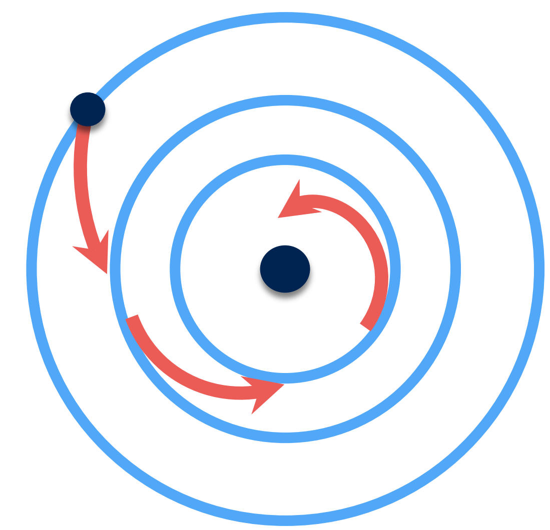
- Impressive progress from both traditional and new methods
- Higher order potential, spin, tidal effects

[Bini, Damour, Geralco]
[Blumlein, Maier, Marquard, Schafer]
[Foffa, Strurani, +Mastrolia, Strum, Torres]

	Newton	*EIH*	*1974/81*	*2000*	*2014*					
	0PN	**1PN**	**2PN**	**3PN**	**4PN**	**5PN**	**6PN**	**7PN**		
	$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ v^{14}$	$+\ \ldots)$	G^1
		$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ \ldots)$	G^2
[Bern, Cheung, Roiban, **CHS**, Solon, Zeng '19]			$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ \ldots)$	G^3
[Bern, Parra-Martinez, Roiban, Ruf, **CHS**, Solon, Zeng '21] [Dlapa, Kalin, Liu, Porto, '21]				$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ \ldots)$	G^4
					$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ \ldots)$	G^5
						$(1$	$+\ v^2$	$+\ v^4$	$+\ \ldots)$	G^6

All orders in velocity at G^3 and G^4

Symmetry $\longrightarrow$ Dissipative Dynamics

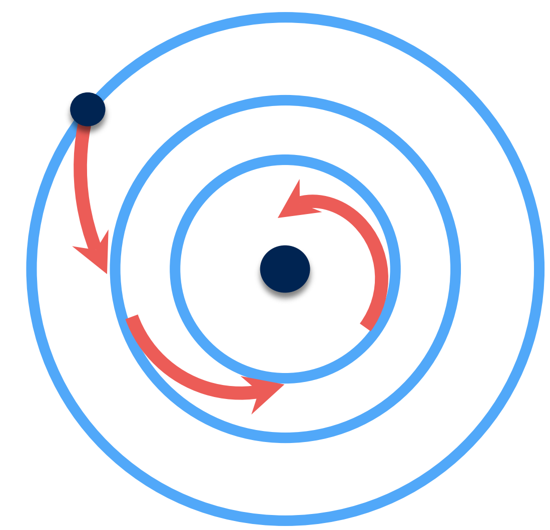


Starts at 2.5PN!!

$$F_{\mathrm{RR}}(r, \vec{p}^{\,2}, \vec{p}\cdot\hat{r})$$

[Burke, Thorne '69]

Symmetry $\longrightarrow$ Dissipative Dynamics



Starts at 2.5PN!!

State of the art: (partial) 4.5PN

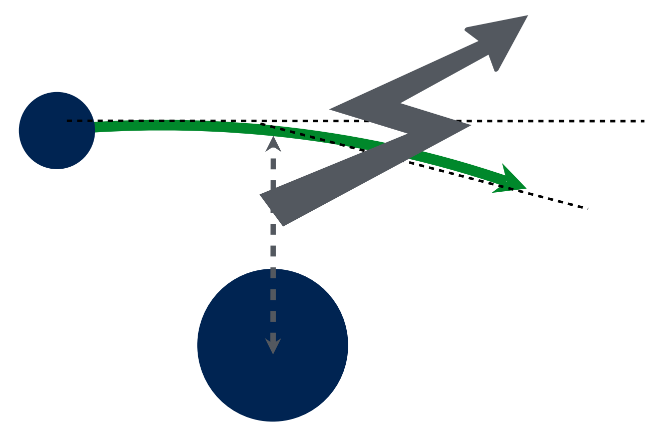
2.5PN **3.5PN** **4.5PN**

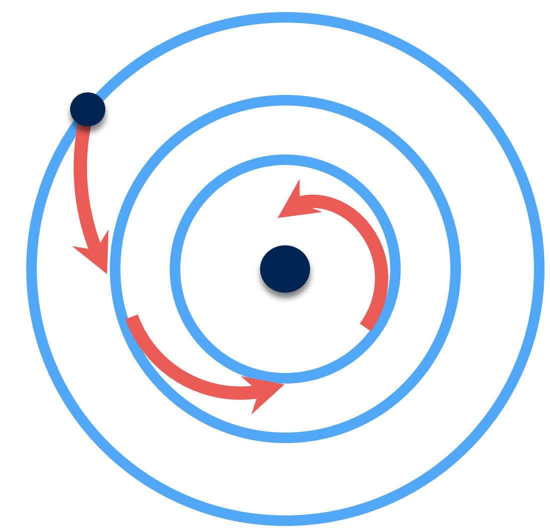
$$
\begin{array}{ccccccccccc}
(v^3 & + & v^5 & + & v^7 & + & v^9 & + & \ldots) & G^2 \\
(\;v & + & v^3 & + & v^5 & + & v^7 & + & \ldots) & G^3 \\
 & & (\;v & + & v^3 & + & v^5 & + & \ldots) & G^4
\end{array}
$$

$$F_{\rm RR}(r, \vec{p}^{\,2}, \vec{p}\cdot\hat{r})$$

[Burke, Thorne '69]

Symmetry → Dissipative Dynamics





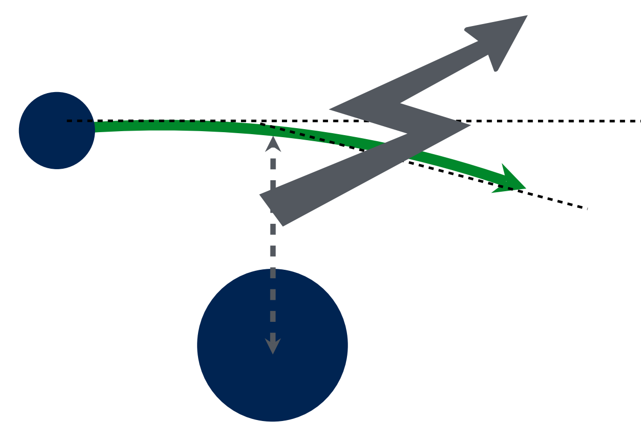
[Kovacs, Thorne ’77]

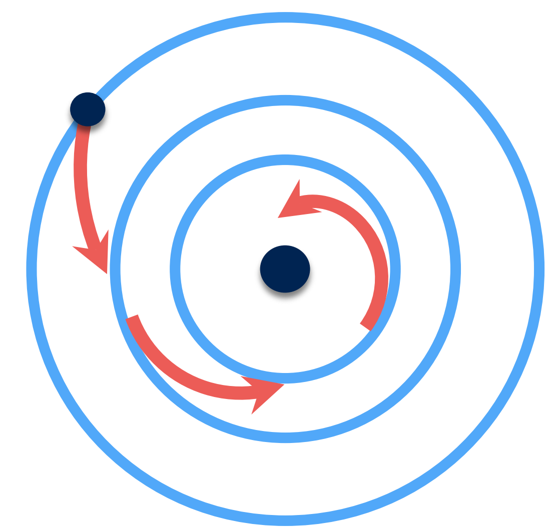
[Goldberger, Ridgway ’16]…

Pioneering idea: [Iyer, Will]

Symmetry $\longrightarrow$ **Dissipative Dynamics**

Poincare invariance





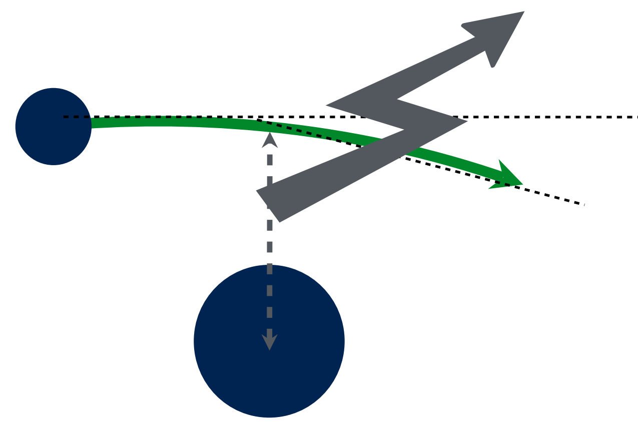
E, J

$F_{\rm RR}(r, \vec{p}^{\,2})$

[Manohar, Ridgway, **CHS**, 2203.04283]

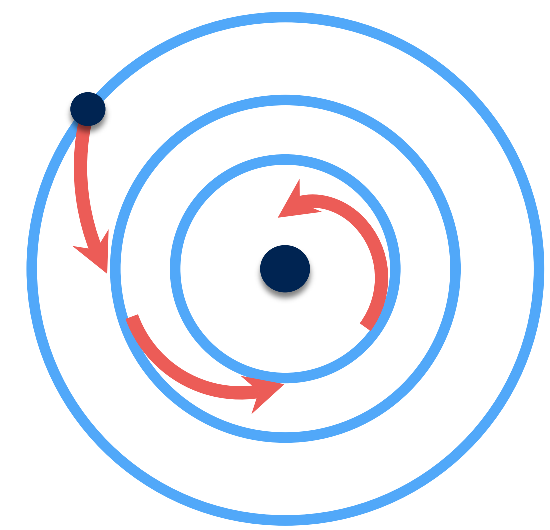
Symmetry $\longrightarrow$ Dissipative Dynamics

Poincare invariance



E, J

[Herrmann, Parra-Martinez, Ruf, Zeng] [This talk]



$F_{\mathrm{RR}}(r, \vec{p}^{\,2})$

2.5PN		3.5PN		4.5PN					
$(v^3$	$+$	v^5	$+$	v^7	$+$	v^9	$+$	$\ldots)$	G^2
$(v$	$+$	v^3	$+$	v^5	$+$	v^7	$+$	$\ldots)$	G^3
		$(v$	$+$	v^3	$+$	v^5	$+$	$\ldots)$	G^4

How to calculate radiated angular momentum?

*We do not intend to resolve the BMS subtlety.

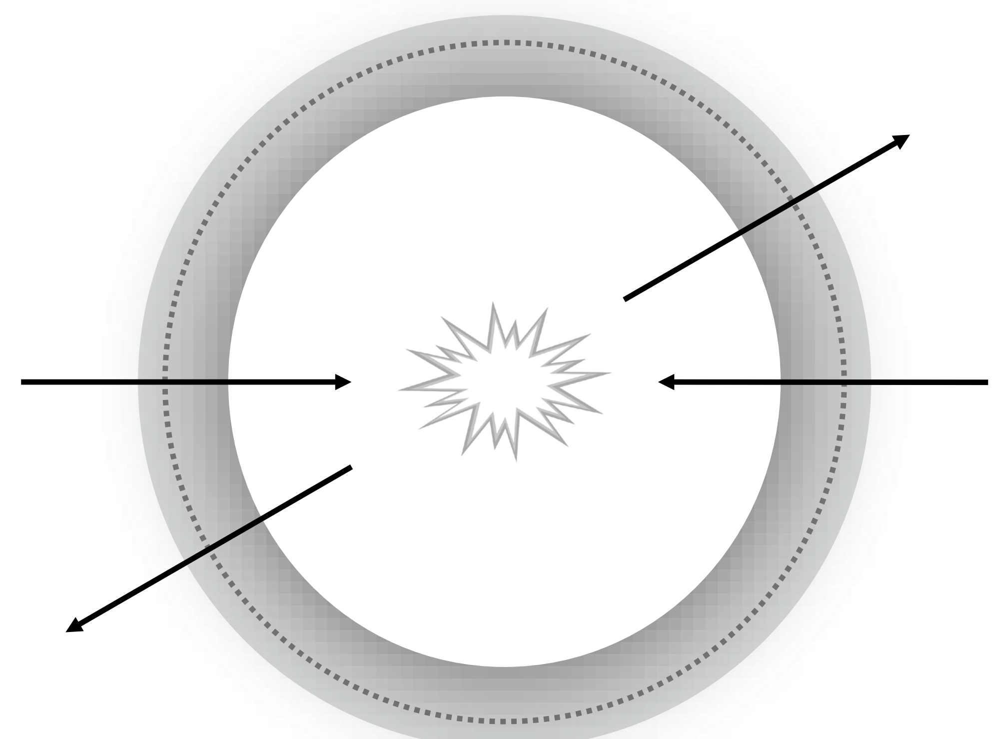
Radiated Poincare Charges

- Consider the final state of scattering.

  The radiated linear and angular momentum are

$$P^\mu = \int \mathrm{d}^3x \, T^{\mu 0}$$

$$J^{\mu\nu} = \int \mathrm{d}^3x \, x^{[\mu} T^{\nu]0}$$



$u = t - r \sim 0$

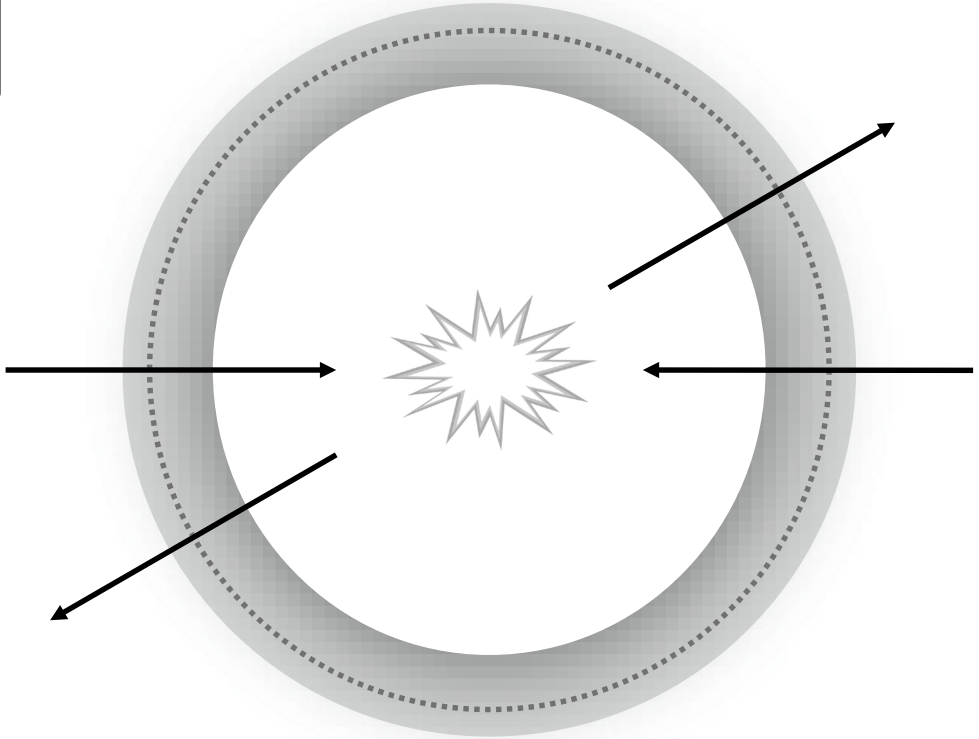
Radiated Poincare Charges

- Textbook formula for angular momentum

$$J^i = \frac{c^2}{32\pi G}\int d^3x \left[-\epsilon^{ikl}\dot{h}^{\rm TT}_{ab} x^k \partial^l h^{\rm TT}_{ab} + 2\epsilon^{ikl} h^{\rm TT}_{ak}\dot{h}^{\rm TT}_{al}\right] . \qquad (2.51)$$

How to see gauge invariance?

Why not covariant?



$u = t - r \sim 0$

Radiated Poincare Charges

- Consider the final state of scattering.
  The radiated linear and angular momentum are

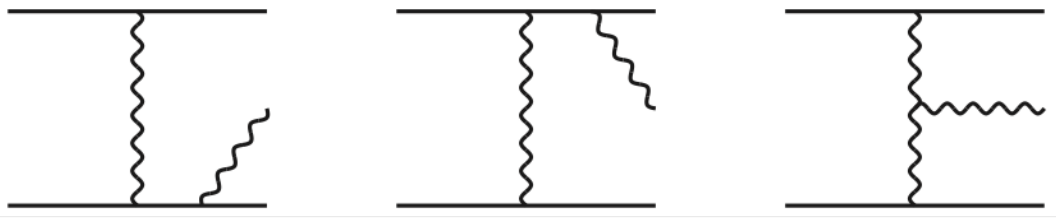
$$P^\mu = \int \mathrm{d}^3x \, T^{\mu 0}$$

$$J^{\mu\nu} = \int \mathrm{d}^3x \, \underline{x^{[\mu}} T^{\nu]0}$$

Sources $\longrightarrow$ Fields $\longrightarrow$ $T^{\mu\nu}$ $\longrightarrow$ linear/angular momentum

Sources

- EM: currents $\mathcal{J}^{\mu}$
  - 3 degrees of freedom under conservation $k_\nu \mathcal{J}^\nu(k) = 0 \quad \partial_\mu \mathcal{J}^\mu(x) = 0$
  - "on-shell" part can be projected to transverse mode
    $$\mathcal{J}^\nu(k) \to \mathcal{J}^\nu(k) + \alpha k^\nu$$



- Gravity: stress-energy pseudotensor $\mathcal{T}^{\mu\nu}$
  - 6 degrees of freedom under conservation $k_\mu \mathcal{T}^{\mu\nu}(k) = 0 \quad \partial_\mu \mathcal{T}^{\mu\nu}(x) = 0$
  - "on-shell" part can be projected to traceless and transverse mode
    $$\mathcal{T}^{\mu\nu}(k) \to \mathcal{T}^{\mu\nu}(k) + k^\mu \epsilon^\nu(k) + k^\nu \epsilon^\mu(k)$$

Fields

- Solved by fixing a gauge, e.g., under Lorentz gauge $\Box A_\mu = 4\pi J_\mu$
- No ambiguity after gauge fixing, even for static sources
- Position space:

$$A^\mu(x) = \int d^4y D_{\rm ret}(x,y) J^\mu(y)$$

Needed for angular momentum

$$A^\mu(x) = \int d\omega e^{-i\omega u} \left(\frac{1}{r} \mathcal{J}^\mu(k = \omega(1,\hat{r})) + \frac{1}{r^2} \mathcal{O}(\partial_k J) \right)$$

residual gauge redundancy
→ **2 degrees** of freedom
→ amplitudes!

No gauge redundancy
→ **≥2 degrees** of freedom

$$\vec{E} = \frac{Q\hat{r}}{r^2}$$

Fields

- Solved by fixing a gauge, e.g., under Lorentz gauge $\Box A_\mu = 4\pi J_\mu$

- No ambiguity after gauge fixing, even for static sources

- Momentum space:

$$A_\mu(x) = \int \underline{\widetilde{dk}}\, (\underline{P_{\mu\nu}}\, \underline{\mathcal{J}^\nu(k)}\, e^{-ik\cdot x} + \text{c.c.})\,,$$

$$h_{\mu\nu}(x) = \sqrt{8\pi G} \int \underline{\widetilde{\mathrm{d}k}}\, (\underline{P_{\mu\nu\rho\sigma}}\, \underline{\mathcal{T}^{\rho\sigma}(k)}\, e^{-ik\cdot x} + \text{c.c.})$$

Lorentz-invariant phase space

sources (current or stress-energy pseudotensor)

gauge-dependent tensor

Stress-energy Tensor

- EM: gauge-invariant

$$T^{\mu\nu} = -F^{\mu}{}_{\rho}F^{\nu\rho} + \eta^{\mu\nu}\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

- Gravity: *not* gauge invariant

$$T_{\mu\nu} = -\frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$$

- But global charges are gauge invariant

- No separation of “orbital” and “spin” parts of $T^{\mu\nu}$ [Jaffe, Manohar]

Radiated Poincare Charges

- Radiated linear and angular momentum are

$$P^\mu = \int \mathrm{d}^3x\, T^{\mu 0}$$

$$J^{\mu\nu} = \int \mathrm{d}^3x\, x^{[\mu} T^{\nu]0}$$

- Need to include the $1/r^2$ of the field,
  so cannot project to transverse parts

Sources ⟶ Fields ⟶ $T^{\mu\nu}$ ⟶ linear/angular momentum

Radiated Linear Momentum

- Linear momentum:
  same as cross section weighted by momentum

- EM: $$P^\mu = \int \underbrace{\widetilde{dk}}_{\text{Phase space integral}} \underbrace{k^\mu}_{\text{momentum}} \underbrace{\left(-\mathcal{J}^{*\rho}(k)\mathcal{J}_\rho(k)\right)}_{\text{polarization sum}}$$

  Phase space integral   momentum   polarization sum

- GR: $$P^\mu = 8\pi G \int \widetilde{\mathrm{d}k}\, k^\mu \left(\mathcal{T}^{*\rho\sigma}(k)\mathcal{T}_{\rho\sigma}(k) - \frac{\mathcal{T}_\rho^{*\rho}(k)\mathcal{T}_\sigma^\sigma(k)}{D-2}\right)$$

- Covariance and gauge invariance are obvious

- Directly related to on-shell amplitudes [Kosower, Maybee, O'Connell]

Radiated Angular Momentum

- New formulae for radiated angular momentum

- EM

$$J^{\mu\nu} = \int \widetilde{dk} \left(- \mathcal{J}^{*\rho}(k) \mathcal{L}^{\mu\nu} \mathcal{J}_\rho(k) - i \mathcal{J}^{*[\mu}(k) \mathcal{J}^{\nu]}(k) \right). \quad \mathcal{L}^{\mu\nu} = i k^\mu \frac{\partial}{\partial k_\nu} - i k^\nu \frac{\partial}{\partial k_\mu}$$

- GR

$$J^{\mu\nu} = 8\pi G \int \widetilde{dk} \left(\mathcal{T}^{*\rho\sigma}(k) \mathcal{L}^{\mu\nu} \mathcal{T}_{\rho\sigma}(k) - \frac{\mathcal{T}^{*\rho}_{\rho}(k) \mathcal{L}^{\mu\nu} \mathcal{T}^{\sigma}_{\sigma}(k)}{D-2} + 2i \, \mathcal{T}^{*\rho[\mu}(k) \mathcal{T}^{\nu]}{}_{\rho}(k) \right)$$

[Manohar, Ridgway, **CHS**]

[Manohar, Ridgway, **CHS**]

Radiated Poincare Charges

- EM

$$P^\mu = \int \widetilde{dk}\, k^\mu \left(-\mathcal{J}^{*\rho}(k)\mathcal{J}_\rho(k)\right),$$

$$\mathcal{L}^{\mu\nu} = ik^\mu \frac{\partial}{\partial k_\nu} - ik^\nu \frac{\partial}{\partial k_\mu}$$

$$J^{\mu\nu} = \int \widetilde{dk}\left(-\mathcal{J}^{*\rho}(k)\,\mathcal{L}^{\mu\nu}\mathcal{J}_\rho(k) - i\mathcal{J}^{*[\mu}(k)\mathcal{J}^{\nu]}(k)\right).$$

- Poincare algebra:

$$x^\mu \to x^\mu + a^\mu$$

$$\mathcal{J}^\mu(k) \to \mathcal{J}^\mu(k)e^{ik\cdot a}$$

$\longrightarrow$

$$P^\mu \to P^\mu$$

$$J^{\mu\nu} \to J^{\mu\nu} + a^{[\mu}P^{\nu]}$$

- Gauge invariance: no physical separation into orbital and spin

[Jaffe, Manohar]

$$\mathcal{J}^\nu(k) \to \mathcal{J}^\nu(k) + \alpha k^\nu$$

[Manohar, Ridgway, **CHS**]

Radiated Poincare Charges

- GR

$$P^\mu = 8\pi G \int \widetilde{dk}\, k^\mu \left(\mathcal{T}^{*\rho\sigma}(k)\mathcal{T}_{\rho\sigma}(k) - \frac{\mathcal{T}_\rho^{*\rho}(k)\mathcal{T}_\sigma^{\sigma}(k)}{D-2} \right), \qquad \mathcal{L}^{\mu\nu} = ik^\mu \frac{\partial}{\partial k_\nu} - ik^\nu \frac{\partial}{\partial k_\mu}$$

$$J^{\mu\nu} = 8\pi G \int \widetilde{dk} \left(\mathcal{T}^{*\rho\sigma}(k)\mathcal{L}^{\mu\nu}\mathcal{T}_{\rho\sigma}(k) - \frac{\mathcal{T}_\rho^{*\rho}(k)\mathcal{L}^{\mu\nu}\mathcal{T}_\sigma^{\sigma}(k)}{D-2} + 2i\, \mathcal{T}^{*\rho[\mu}(k)\mathcal{T}^{\nu]}{}_\rho(k) \right)$$

- Poincare algebra:

$$x^\mu \to x^\mu + a^\mu \qquad\qquad P^\mu \to P^\mu$$

$$\mathcal{T}^{\mu\nu}(k) \to \mathcal{T}^{\mu\nu}(k) e^{ik\cdot a} \quad \longrightarrow \quad J^{\mu\nu} \to J^{\mu\nu} + a^{[\mu}P^{\nu]}$$

- Gauge invariance: no physical separation into orbital and spin

[Jaffe, Manohar]

$$\mathcal{T}^{\mu\nu}(k) \to \mathcal{T}^{\mu\nu}(k) + k^\mu\epsilon^\nu(k) + k^\nu\epsilon^\mu(k)$$

Leading Order: Zero-Frequency Limit

(Weinberg soft theorem, memory, …)

Stress-energy Pseudotensor

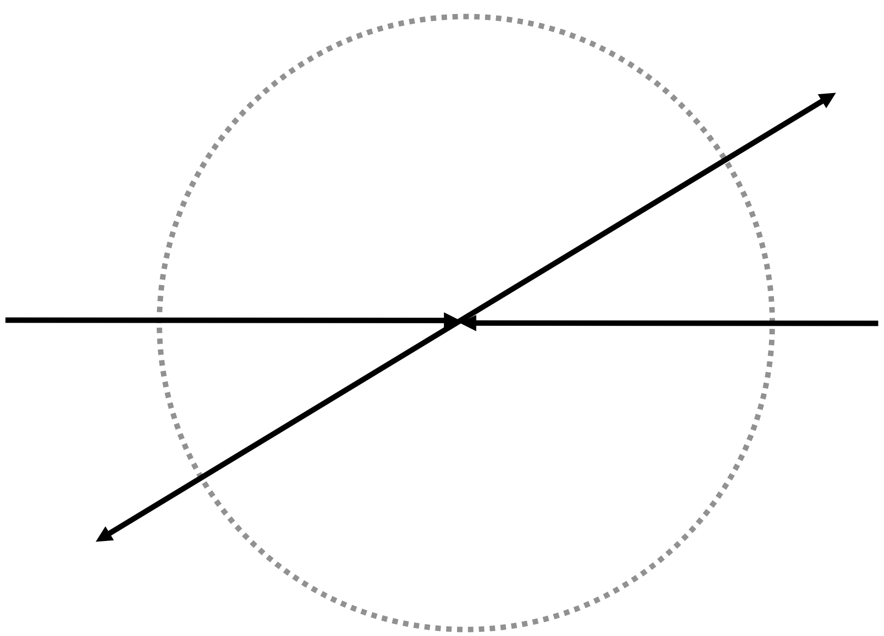
- Zoom out the time scale so collision occurs at t=0 (zero-frequency limit)

$$x^\mu(\tau) = \underline{v_i^\mu \tau} + \underline{(v_f^\mu - v_i^\mu)\theta(\tau)\tau}$$

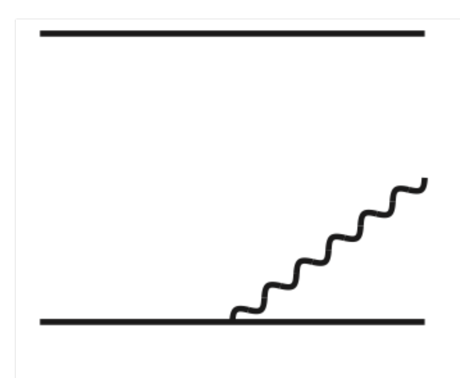
$$\mathcal{T}^{\mu\nu}(k)|_{\omega\to 0^+} = \underline{-i\pi\delta(\omega)\sum_a \frac{p_a^\mu p_a^\nu}{E_a - \hat{\mathbf{k}}\cdot\mathbf{p}_a}} + \underline{\frac{1}{\omega + i0}\sum_a \left(\frac{p_a^\mu p_a^\nu}{E_a - \hat{\mathbf{k}}\cdot\mathbf{p}_a}\right)\Big|_i^f}$$

Free particles
(Coulomb mode)

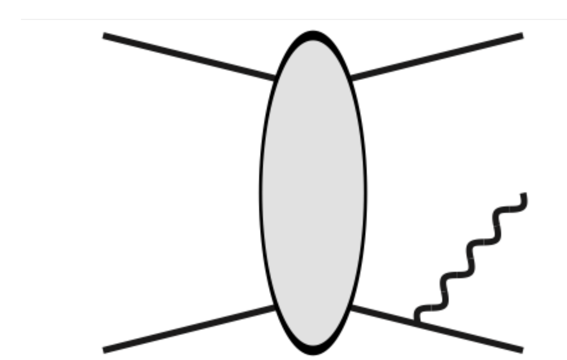
deflection turned on at t=0
valid to all orders



$$\delta(u = t - r)$$



$$\delta(p_a \cdot k) = \frac{\delta(\omega)}{E_a - \hat{k}\cdot\vec{p}_a}$$

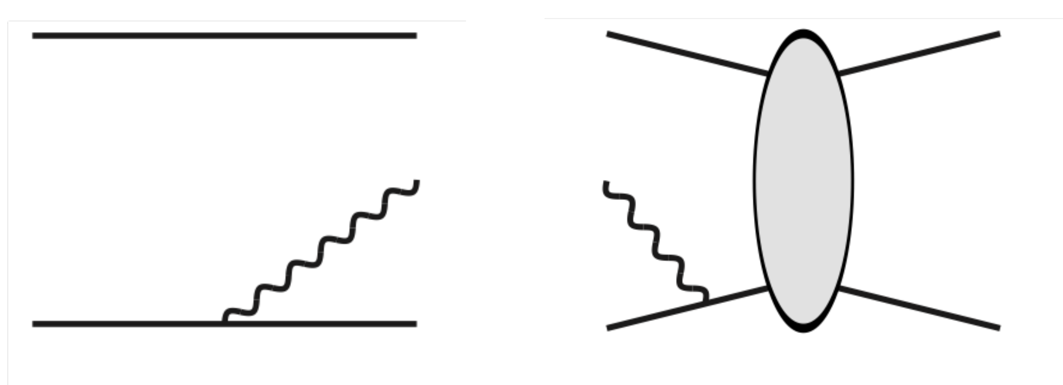


Radiated angular momentum

- Zero-frequency limit

$$P^\mu = 0,$$

$$J^{\mu\nu} = 8\pi G \int \underbrace{\widetilde{dk}}_{d\omega\,\omega} \Big(\underbrace{\mathcal{T}^{*\rho\sigma}(k)}_{\delta(\omega)} \mathcal{L}^{\mu\nu} \underbrace{\mathcal{T}_{\rho\sigma}(k)}_{\frac{1}{\omega}} - \frac{\mathcal{T}_\rho^{*\rho}(k)\mathcal{L}^{\mu\nu}\mathcal{T}_\sigma^{\sigma}(k)}{D-2} + 2i\,\mathcal{T}^{*\rho[\mu}(k)\mathcal{T}^{\nu]}{}_\rho(k) \Big)$$



Radiated angular momentum

- Zero-frequency limit

$$P^\mu = 0,$$

$$J^{\mu\nu} = 8\pi G \int \widetilde{dk} \left(\mathcal{T}^{*\rho\sigma}(k) \mathcal{L}^{\mu\nu} \mathcal{T}_{\rho\sigma}(k) - \frac{\mathcal{T}^{*\rho}_{\rho}(k) \mathcal{L}^{\mu\nu} \mathcal{T}^{\sigma}_{\sigma}(k)}{D-2} + 2i\, \mathcal{T}^{*\rho[\mu}(k) \mathcal{T}^{\nu]}{}_{\rho}(k) \right)$$

- There is nothing wrong with zero energy but non-zero angular momentum

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} \sim \int d\Omega r^2\, \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$$

$$\frac{1}{r^2} \qquad \frac{1}{r}$$

- Energy is infinitesimal but still finite angular momentum.
  A common feature in scalar, EM, and gravity

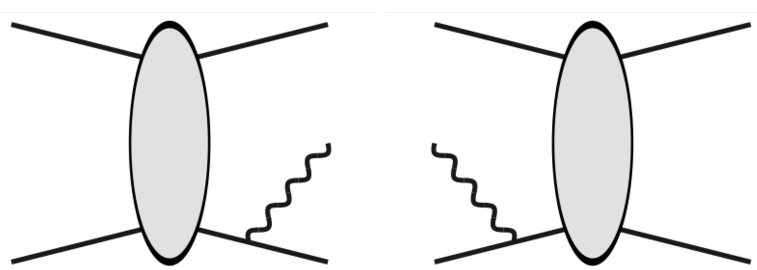
Soft Finiteness to all orders

- Log divergence from phase space integration

$$P^\mu = 0,$$

$$J^{\mu\nu} = 8\pi G \int \widetilde{dk} \left(\mathcal{T}^{*\rho\sigma}(k) \mathcal{L}^{\mu\nu} \mathcal{T}_{\rho\sigma}(k) - \frac{\mathcal{T}_\rho^{*\rho}(k) \mathcal{L}^{\mu\nu} \mathcal{T}_\sigma^{\sigma}(k)}{D-2} + 2i\, \mathcal{T}^{*\rho[\mu}(k) \mathcal{T}^{\nu]}{}_\rho(k) \right)$$

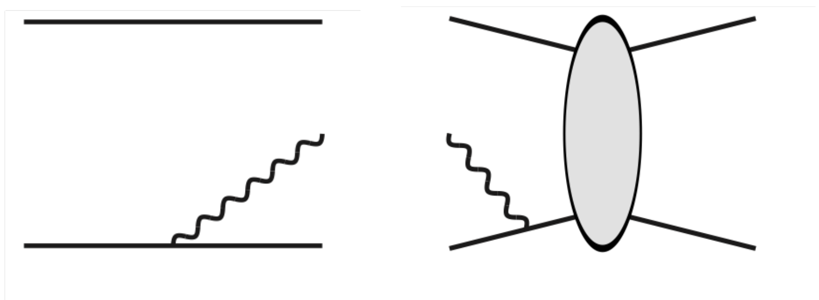
$$\int d\Omega \;\; d\omega\, \omega \quad \frac{1}{\omega} \qquad \frac{1}{\omega}$$



- Soft divergence cancels after integrating the full solid angle

Radiated angular momentum

- Leading order in deflection angle θ



$$P^{\mu} = 0$$

$$\frac{J^{12}_{\mathrm{CM},2}}{\mathrm{J_{CM}}} = 2 \times \frac{J^{12}_{\mathrm{rest},2}}{\mathrm{J_{rest}}} = 2 m_1 m_2 \mathcal{I}(\sigma)\, \theta$$

- Model independent (GR, with spin, dilaton gravity, supergravity, etc)

- Radiated angular momentum ***is positive*** when scattering is ***attractive***

Comparison

- J^{12} at G^2 agrees with [Damour]
- Fully agrees arbitrary deflection [Di Vecchia, Heissenberg, Russo]
- Fully agrees with matter system [Bini, Damour]
- Disagree with the standard formula in the rest frame by x2
  [Jakobsen, Mogull, Plefka, Steinhoff][Mougiakakos, Riva, Vernizzi]

Standard v.s. Our formula

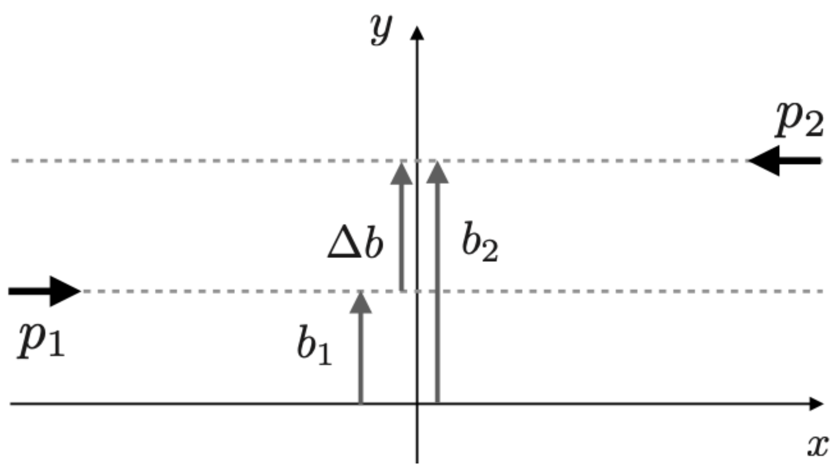
Standard formula	Our formula
TT part of metric	*stress-energy pseudotensor*
$\dfrac{J^{12}_{\mathrm{rest},2}}{\mathsf{J}_{\mathrm{rest}}} = \dfrac{J^{12}_{\mathrm{CM},2}}{\mathsf{J}_{\mathrm{CM}}}$	$\dfrac{J^{12}_{\mathrm{rest},2}}{\mathsf{J}_{\mathrm{rest}}} = \dfrac{1}{2} \times \dfrac{J^{12}_{\mathrm{CM},2}}{\mathsf{J}_{\mathrm{CM}}}$

- Both agree in the CM frame
- Independent checks: general covariance and known RR force

[Jaranowski, Schafer; Nissanke, Blanchet]
[Bini, Damour]

General Structure

- Form factors parametrization:



$$P^\mu = F_1 p_1^\mu + F_2 p_2^\mu + F_3 \Delta b^\mu,$$

$$J^{\mu\nu} = \bar{b}^{[\mu} \left(F_1 p_1^{\nu]} + F_2 p_2^{\nu]} + F_3 \Delta b^{\nu]} \right) + \Delta b^{[\mu} \left(G_1 p_1^{\nu]} - G_2 p_2^{\nu]} \right) + H_{12}\, p_2^{[\mu} p_1^{\nu]}$$

$$F_1 \overset{m_1 \leftrightarrow m_2}{=} F_2, \qquad G_1 \overset{m_1 \leftrightarrow m_2}{=} G_2,$$

$$F_3 \overset{m_1 \leftrightarrow m_2}{=} -F_3, \qquad H_{12} \overset{m_1 \leftrightarrow m_2}{=} -H_{12},$$

- Only assume Lorentz covariance, Poincare algebra, and $1 \leftrightarrow 2$ symmetry
- Form factors are functions of $m_{1,2}, |\Delta b|, \sigma$

[Manohar, Ridgway, **CHS**]

Nonperturbative result

CM Frame

$$\left.\frac{J^{12}_{\rm CM}}{\mathsf{J}_{\rm CM}}\right|_{\omega=0} = G_1 + G_2$$

Rest Frame

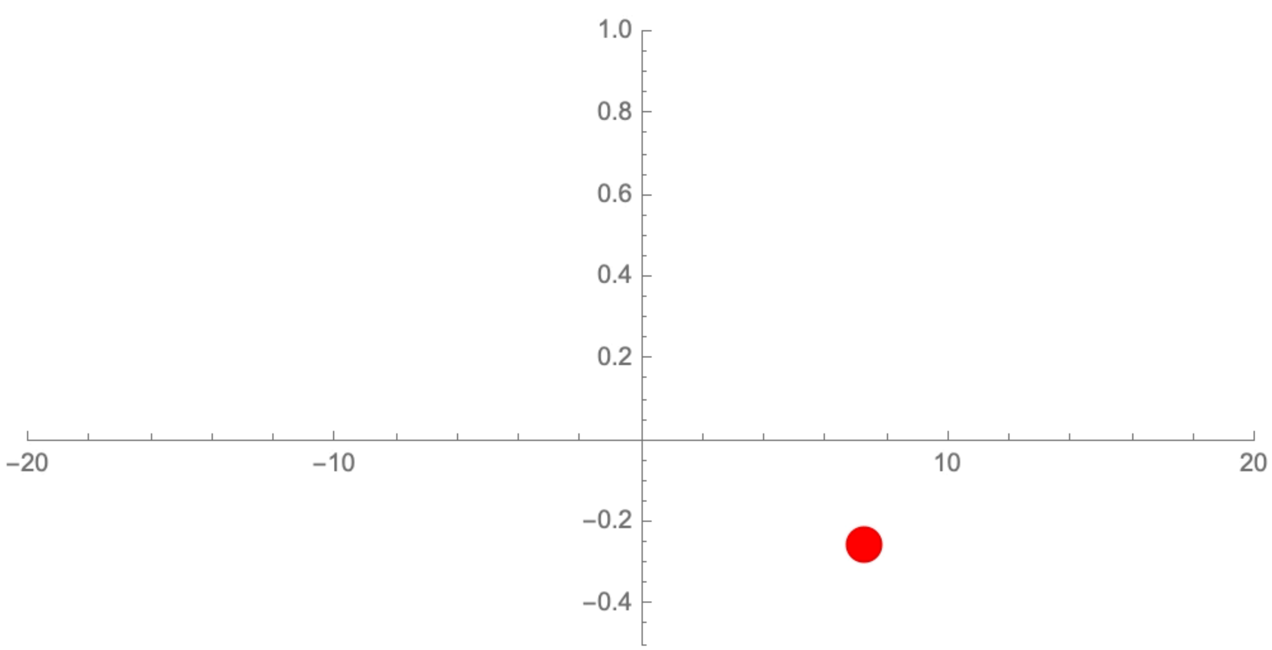
$$\left.\frac{J^{12}_{\rm rest}}{\mathsf{J}_{\rm rest}}\right|_{\omega=0} = G_2$$

- This is an exact relation
- Since $G_1 = G_2$ at this order, our answer agrees with this general prediction

Crosscheck with Burke-Throne

- Burke-Thorne force at G^2:

$$\mathbf{a_1} = -\mathbf{a_2} = \frac{4G^2 m_1 m_2}{5r^3}\left(3v^2 v_r \hat{\mathbf{r}} - v^2 \mathbf{v}\right)$$

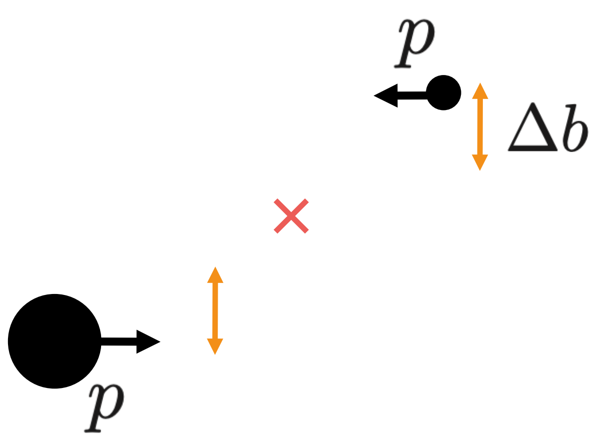


- Final energy is the same as initial
- Impact parameter shrinks equally
- Non-decoupling of heavy particle

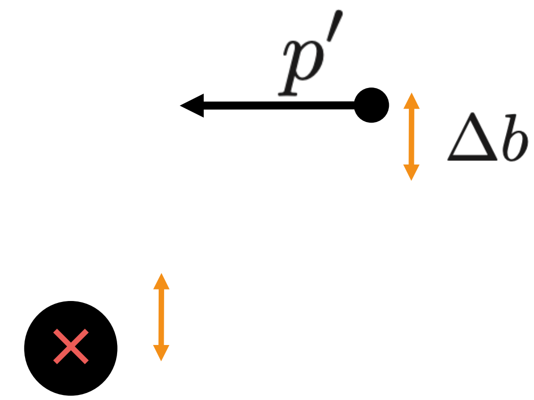
Radiation Reaction Force

- Burke-Thorne force at G^2:

CM Frame:



Rest Frame:



$$\frac{J^{12}_{\text{CM},2}}{\mathrm{J}_{\text{CM}}} = \frac{2p\Delta b}{pb} = 2 \times \frac{\Delta b}{b}$$

$$\frac{J^{12}_{\text{rest},2}}{\mathrm{J}_{\text{rest}}} = \frac{p'\Delta b}{p'b} = \frac{\Delta b}{b}$$

Back reaction is important!

$$J^{\mu\nu} = 8\pi G \int \widetilde{dk} \left(\mathcal{T}^{*\rho\sigma}(k) \mathcal{L}^{\mu\nu} \mathcal{T}_{\rho\sigma}(k) - \frac{\mathcal{T}_\rho^{*\rho}(k) \mathcal{L}^{\mu\nu} \mathcal{T}_\sigma^\sigma(k)}{D-2} + 2i\, \mathcal{T}^{*\rho[\mu}(k) \mathcal{T}^{\nu]}{}_\rho(k) \right)$$

Our formula agrees with covariance and Burke-Thorne force

Standard formula is incomplete for scattering

because of the presence of zero-frequency mode

Facts and Fantasies of Angular momentum

1. $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$

   ✗ Not a physical separation

2. If $E = 0$, then $J^{\mu\nu} = 0$

   ✗ Zero mode ($1/r^2$) contributes

3. Heavy particle decouples

   ✗ Back reaction on heavy particle is relevant

Precision Frontier:

Radiated angular momentum at G^3

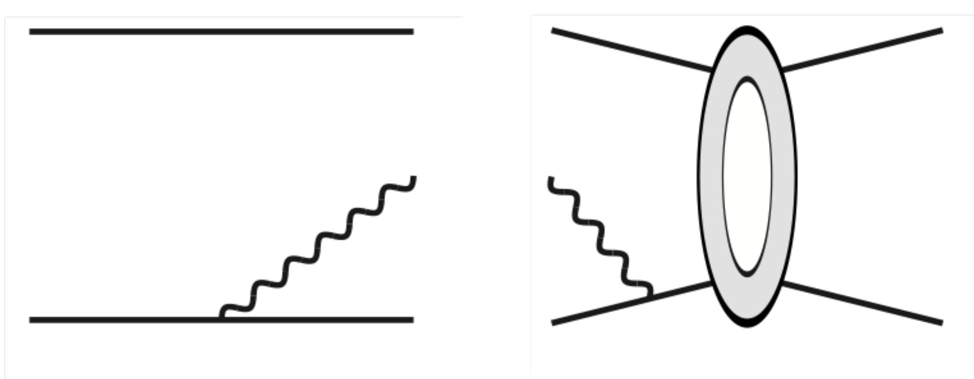
Radiated Poincare Charges

- State of the art precision at G^3

$P^{\mu} \rightarrow$ Known [Herrmann, Parra-Martinez, Ruf, Zeng]

$J^{\mu\nu} \rightarrow$ Both zero and finite frequency contributions

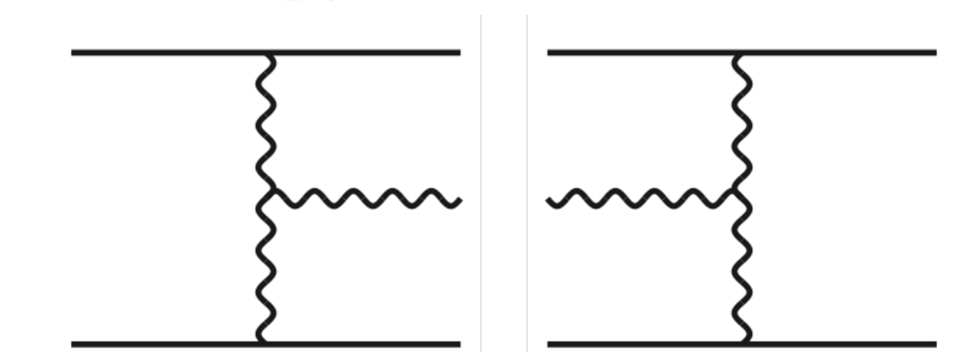
Soft Theorem



- Same as before, just use G^2 impulses

[Westpfahl 80's]

Double Copy & Generalized Unitarity



- Waveform from 2-to-3 amplitude
- Resum velocity expansion from $O(v^{60})$ series
- Verified by modern integration methods

[Di Vecchia, Heisenberg, Russo, Veneziano;
Herrmann, Moss, Parra-Martinez, Ruf]

New Results in General Relativity

- New results for G^3 radiated angular momentum

$$J^{12}_{\text{rest},3} = b m_1 m_2^2 \left(m_1 \mathcal{C}(\sigma) + (m_1 + m_2)\mathcal{D}(\sigma) \right),$$

$$\mathcal{I}(\sigma) = -\frac{16}{3} + \frac{2\sigma^2}{\sigma^2-1} + \frac{(2\sigma^2-3)}{\sigma^2-1}\frac{\sigma \operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}$$

$$\frac{\mathcal{E}(\sigma)}{\pi} = f_1 + f_2 \log\left(\frac{\sigma+1}{2}\right) + f_3 \frac{\sigma \operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}$$

$$\frac{\mathcal{C}(\sigma)}{\pi} = g_1 + g_2 \log\left(\frac{\sigma+1}{2}\right) + g_3 \frac{\sigma \operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}$$

$$\mathcal{D}(\sigma) = \frac{3\pi(5\sigma^2-1)}{8}\mathcal{I}(\sigma)$$

$$f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48(\sigma^2-1)^{3/2}}$$

$$f_2 = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2-1}}$$

$$f_3 = \frac{(2\sigma^2-3)(35\sigma^4 - 30\sigma^2 + 11)}{8(\sigma^2-1)^{3/2}}$$

$$g_1 = \frac{105\sigma^7 - 411\sigma^6 + 240\sigma^5 + 537\sigma^4 - 683\sigma^3 + 111\sigma^2 + 386\sigma - 237}{24(\sigma^2-1)^2}$$

$$g_2 = \frac{35\sigma^5 - 90\sigma^4 - 70\sigma^3 + 16\sigma^2 + 155\sigma - 62}{4(\sigma^2-1)}$$

$$g_3 = \frac{-(2\sigma^2-3)(35\sigma^5 - 60\sigma^4 - 70\sigma^3 + 72\sigma^2 + 19\sigma - 12)}{4(\sigma^2-1)^2}$$

- As the form factors show, radiated energy enters when translating from rest to CM frame

$$\frac{J^{12}_{\text{CM},3}}{\mathrm{J}_{\text{CM}}} = \frac{m_1 m_2 (m_1 + m_2)}{\sqrt{\sigma^2-1}}\left[\mathcal{C}(\sigma) + 2\mathcal{D}(\sigma) - \frac{m_1 m_2 \sqrt{\sigma^2-1}}{E^2}\mathcal{E}(\sigma)\right]$$

- Elucidate the relation originally found by Bini, Damour, Geralico when considering G^4 scattering

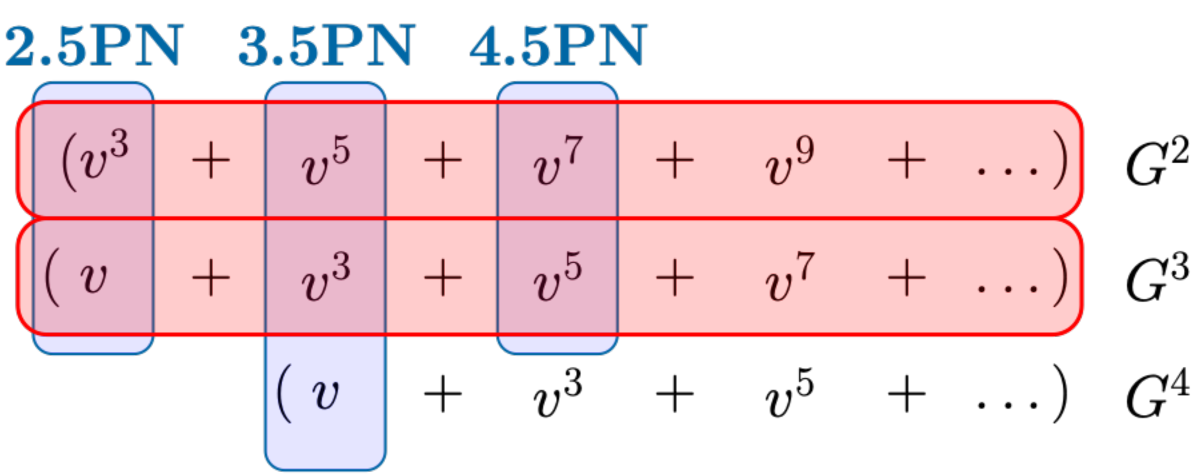
New Results in General Relativity

- New results for G^3 radiated angular momentum

$$\frac{J_3}{\pi} = \frac{28}{5}p_\infty^2 + \left(\frac{739}{84} - \frac{163}{15}\nu\right)p_\infty^4 + \left(-\frac{5777}{2520} - \frac{5339}{420}\nu + \frac{50}{3}\nu^2\right)p_\infty^6$$
$$+ \left(\frac{115769}{126720} + \frac{1469}{504}\nu + \frac{9235}{672}\nu^2 - \frac{553}{24}\nu^3\right)p_\infty^8 + \dots$$

[Bini, Damour, Geralico '21]

[Manohar, Ridgway, **CHS**]

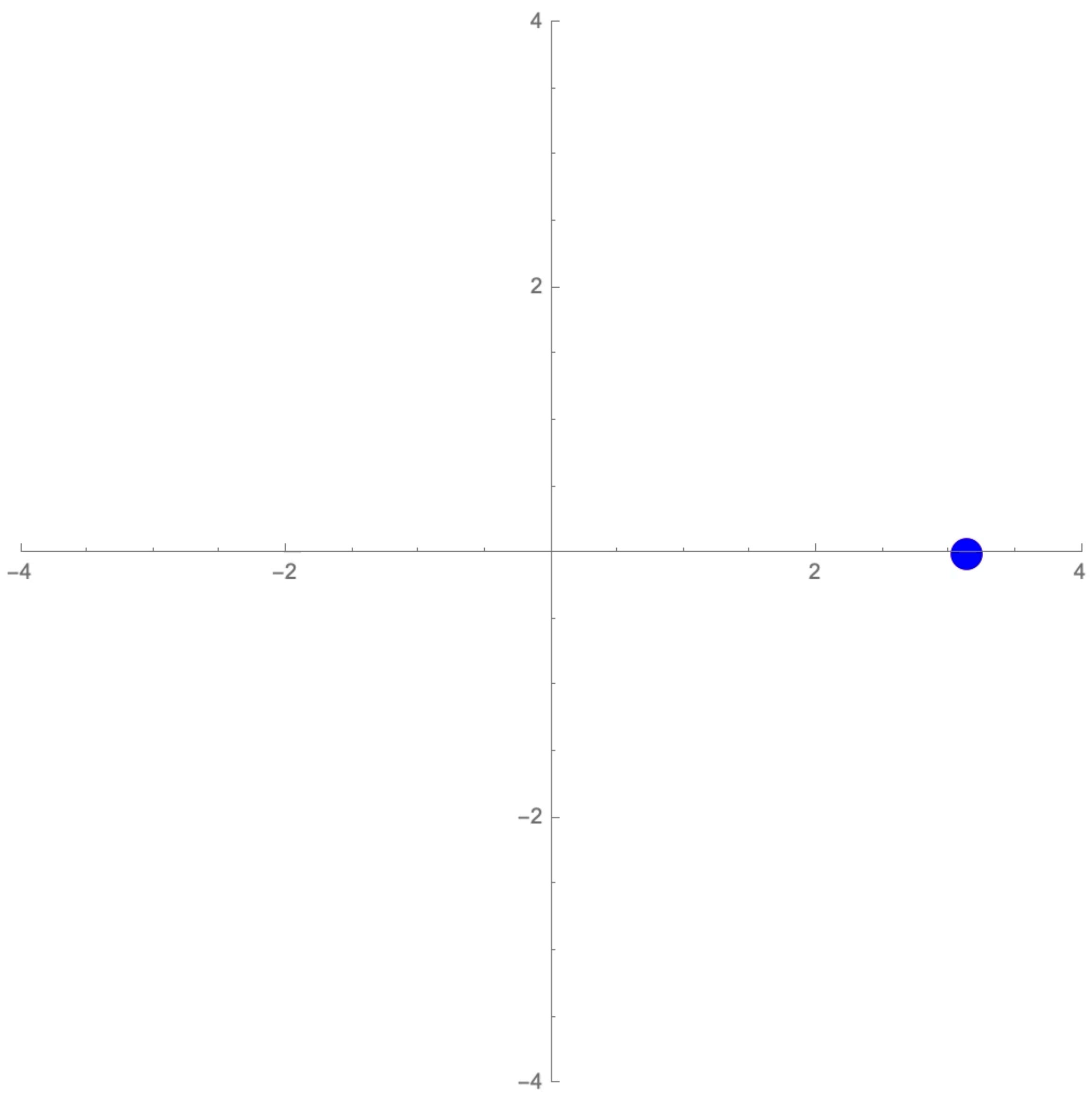


Precision Binary Dynamics

m1=m2, G=0.01, E[0]=–0.0176, J[0]=0.4, v[0]=0.128

t = 0.0

{E,J} = {–0.0176, 0.400}



$$\dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{p}}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{x}} + \boldsymbol{F}_{\mathrm{RR}}$$

Conservative

All orders in v to $\mathbf{G^4}$

via scattering angle

[Bern, Cheung, Roiban, **CHS**, Solon, Zeng, ’19]

[Bern, Parra-Martinez, Roiban, Ruf, **CHS**, Solon, Zeng, ’21]

[Dlapa, Kalin, Liu, Porto ’21]

Dissipative

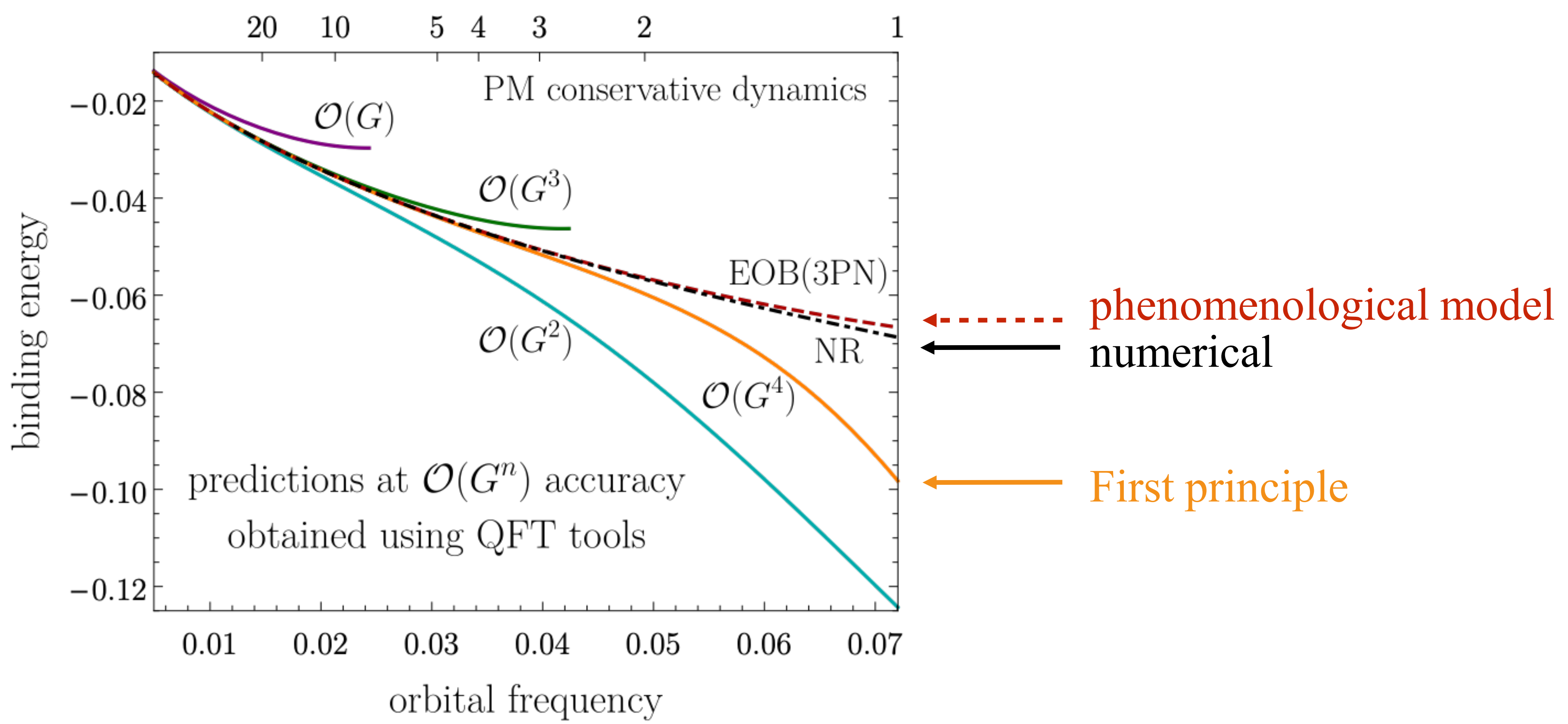
All orders in v to $\mathbf{G^3}$

via radiated E and J

[Manohar, Ridgway, **CHS** ‘22]

Precision Binary Dynamics

Can dissipation bring closer to numerical simulation?



Snowmass white paper [Buonanno, Khalil, O'Connell, Roiban, Solon, Zeng, 2204.05194]
[Khalil, Buonanno, Steinhoff, Vines, 2204.05047]

Summary/Outlook

- Exciting era for gravitational waves!
- New formula for radiated angular momentum
- We bootstrap the state-of-the-art dissipative force via Poincare
- Applications of angular momentum to collider physics?
- Toward solving two-body problem in GR

Thank you

Backups

Common concerns

- Can zero-energy radiation carries angular momentum?
- Is radiated angular momentum infrared finite
  (due to 1/r potential in 4D)?
- Are distribution functions (e.g. delta functions) well-defined?
- Is there BMS ambiguity on angular momentum?
  ….[Veneziano, Vilkovisky]

Need to analyze each question by calculations
in scalar, EM, gravity

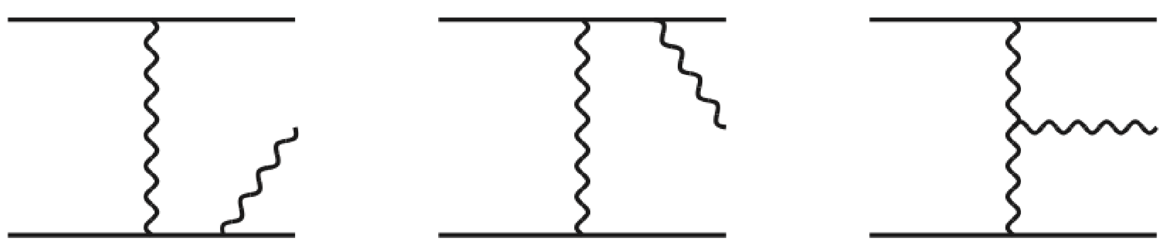
Comparison

	Scalar	EM	GR
Spacetime	Minkowski	Minkowski	Dynamical
Gauge freedom	No	Yes	Yes
zero E but nonzero J	yes	yes	yes
RR force	agree	Abrham-Lorentz-Dirac agree	Burke-Thorne agree
BMS ambiguity	No	No	Maybe?

- Any consideration of BMS cannot change scalar and EM results, and needs to explain the match to radiation reaction force

Sources

- Include both matter and self-interaction



- Imposing off-shell conservation, they can be obtained via field-theory scattering amplitudes

$$\mathcal{T}^{\mu\nu}(k) = i \int \hat{d}\ell \; \hat{\delta}(2p_1 \cdot l)\hat{\delta}(2p_2 \cdot \ell) e^{il\cdot b_1 - i\ell\cdot b_2} \mathcal{M}_5^{\mu\nu}(\ell, k)$$

New Results in General Relativity

- Predict for G^4 odd-in-v impulses via Bini-Damour formula

$$\Delta p_{\perp,4} = \nu M^5 \left(\frac{G}{b}\right)^4 \left(c_{b,4}^{\text{cons}} + c_{b,4}^{\text{rr,even}} + c_{b,4}^{\text{rr,odd}}\right)$$

Conservative; even in v

[Bern, Parra-Martinez, Roiban, Ruf, **CHS**, Solon, Zeng, '21]
[Dlapa, Kalin, Liu, Porto '21]

Dissipative; even in v

Unknown!!

Dissipative; odd in v

[Manohar, Ridgway, **CHS** '22]

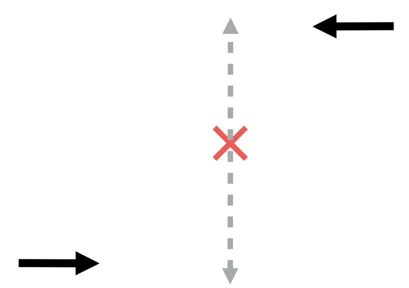
$$c_{b,4}^{\text{rr,odd}} = \nu \left[\frac{\sigma(6\sigma^2-5)}{\sigma^2-1} - \frac{m_1}{M}\frac{2\sigma^2-1}{(\sigma+1)}\right]\frac{\mathcal{E}(\sigma)}{p_\infty} - \frac{\nu(2\sigma^2-1)}{\sigma^2-1}\left[\frac{3\pi(5\sigma^2-1)}{2}\mathcal{I}(\sigma) + \mathcal{C}(\sigma) + 2\mathcal{D}(\sigma)\right] \quad (19)$$

- Using ideas from factorization in EFT,
  simply a G^4 problem into mostly leading order inputs

CM Frame v.s. Rest frame

CM Frame:

$$p_2^\mu = (E_2, -|\mathbf{p}|, 0, 0)$$

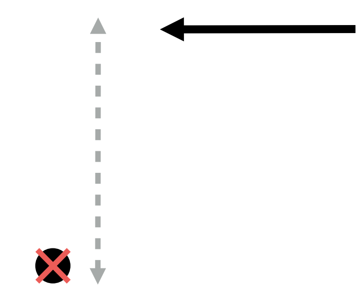


$$p_1^\mu = (E_1, |\mathbf{p}|, 0, 0)$$

$$\mathrm{J}_{\mathrm{CM}} = |\mathbf{p}| b$$

Rest Frame:

$$p_2^\mu = (\sigma m_2, -\sqrt{\sigma^2 - 1} m_2, 0, 0)$$



$$p_1^\mu = (m_1, 0, 0, 0)$$

$$\mathrm{J}_{\mathrm{rest}} = \sqrt{\sigma^2 - 1} m_2 b$$

- They are related by boost and translation

Precision Binary Dynamics

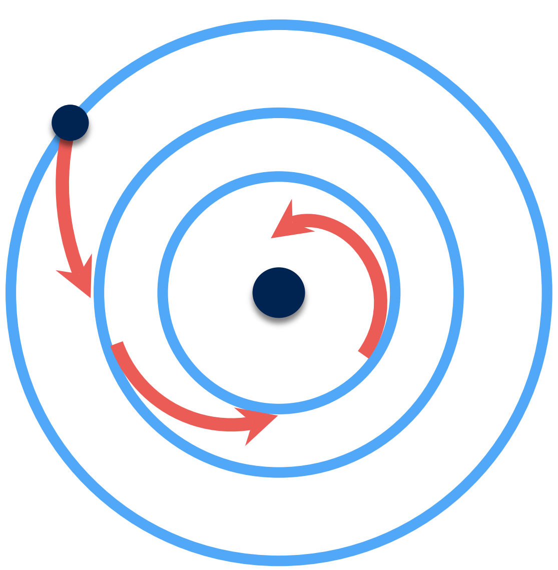
- State-of-the-art EOM all orders in v to G^3

$$H(r, p^2)$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1 - 2\sigma^2\right),$$

$$c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1 - 5\sigma^2\right) - \frac{4\nu\sigma\left(1 - 2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1 - 2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\left[\frac{1}{12}\left(3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3\right) - \frac{4\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right.$$
$$- \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7 - 20\sigma^2\right)}{2\gamma\xi} + \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3}$$
$$\left. - \frac{\nu^2\left(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2\right)\left(1 - 2\sigma^2\right)}{4\gamma^3\xi^2} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\right].$$



$$\mathbf{F}_{\mathrm{RR}} = c_r p_r \hat{\mathbf{r}} + c_p \mathbf{p}$$

$$c_r = \frac{G^2}{r^3} c_{r,2}\left(\mathbf{p^2}\right) + \frac{G^3}{r^4} c_{r,3}\left(\mathbf{p^2}\right) + \ldots,$$

$$c_p = \frac{G^2}{r^3} c_{p,2}\left(\mathbf{p^2}\right) + \frac{G^3}{r^4} c_{p,3}\left(\mathbf{p^2}\right) + \ldots,$$

$$c_{r,2}\left(\mathbf{p^2}\right) = -3 c_{p,2}\left(\mathbf{p^2}\right),$$

$$c_{p,2}\left(\mathbf{p^2}\right) = -\frac{\nu^2 M^4}{E_1 E_2}(2\sigma^2 - 1)\mathcal{I}(\sigma)$$

$$\mathcal{I}(\sigma) = -\frac{16}{3} + \frac{2\sigma^2}{\sigma^2-1} + \frac{(2\sigma^2-3)}{\sigma^2-1}\frac{\sigma \operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}$$

$$\frac{\mathcal{E}(\sigma)}{\pi} = f_1 + f_2 \log\left(\frac{\sigma+1}{2}\right) + f_3 \frac{\sigma \operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}$$

$$\frac{\mathcal{C}(\sigma)}{\pi} = g_1 + g_2 \log\left(\frac{\sigma+1}{2}\right) + g_3 \frac{\sigma \operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}$$

$$\mathcal{D}(\sigma) = \frac{3\pi(5\sigma^2-1)}{8}\mathcal{I}(\sigma)$$

$$f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48(\sigma^2-1)^{3/2}}$$

$$f_2 = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2-1}}$$

$$f_3 = \frac{(2\sigma^2-3)(35\sigma^4 - 30\sigma^2 + 11)}{8(\sigma^2-1)^{3/2}}$$

$$g_1 = \frac{105\sigma^7 - 411\sigma^6 + 240\sigma^5 + 537\sigma^4 - 683\sigma^3 + 111\sigma^2 + 386\sigma - 237}{24(\sigma^2-1)^2}$$

$$g_2 = \frac{35\sigma^5 - 90\sigma^4 - 70\sigma^3 + 16\sigma^2 + 155\sigma - 62}{4(\sigma^2-1)}$$

$$g_3 = \frac{-(2\sigma^2-3)(35\sigma^5 - 60\sigma^4 - 70\sigma^3 + 72\sigma^2 + 19\sigma - 12)}{4(\sigma^2-1)^2}$$

$$c_{p,3}(\mathbf{p^2}) = -\frac{2p_\infty J^{12}_{\mathrm{CM},3}}{\pi\xi E J_0} + \left(2\xi E c'_{p,2}(\mathbf{p^2}) - \left(2 - \frac{p_\infty^2(1-3\xi)}{\xi^2 E^2}\right)\frac{J^{12}_{\mathrm{CM},2}}{2p_\infty J_0}\right) c_{H,1}(\mathbf{p^2}) - p_\infty c'_{H,1}(\mathbf{p^2})\frac{J^{12}_{\mathrm{CM},2}}{J_0}$$

$$c_{r,3}(\mathbf{p^2}) = \frac{8}{\pi p_\infty}\left(\frac{p_\infty^2}{J_0 E\xi} J^{12}_{\mathrm{CM},3} - E_{\mathrm{CM},3}\right) + \left(-6\xi E c'_{p,2}(\mathbf{p^2}) + 2\left(1 + \frac{p_\infty^2(1-3\xi)}{\xi^2 E^2}\right)\frac{J^{12}_{\mathrm{CM},2}}{p_\infty J_0}\right) c_{H,1}(\mathbf{p^2}) + 4p_\infty c'_{H,1}$$