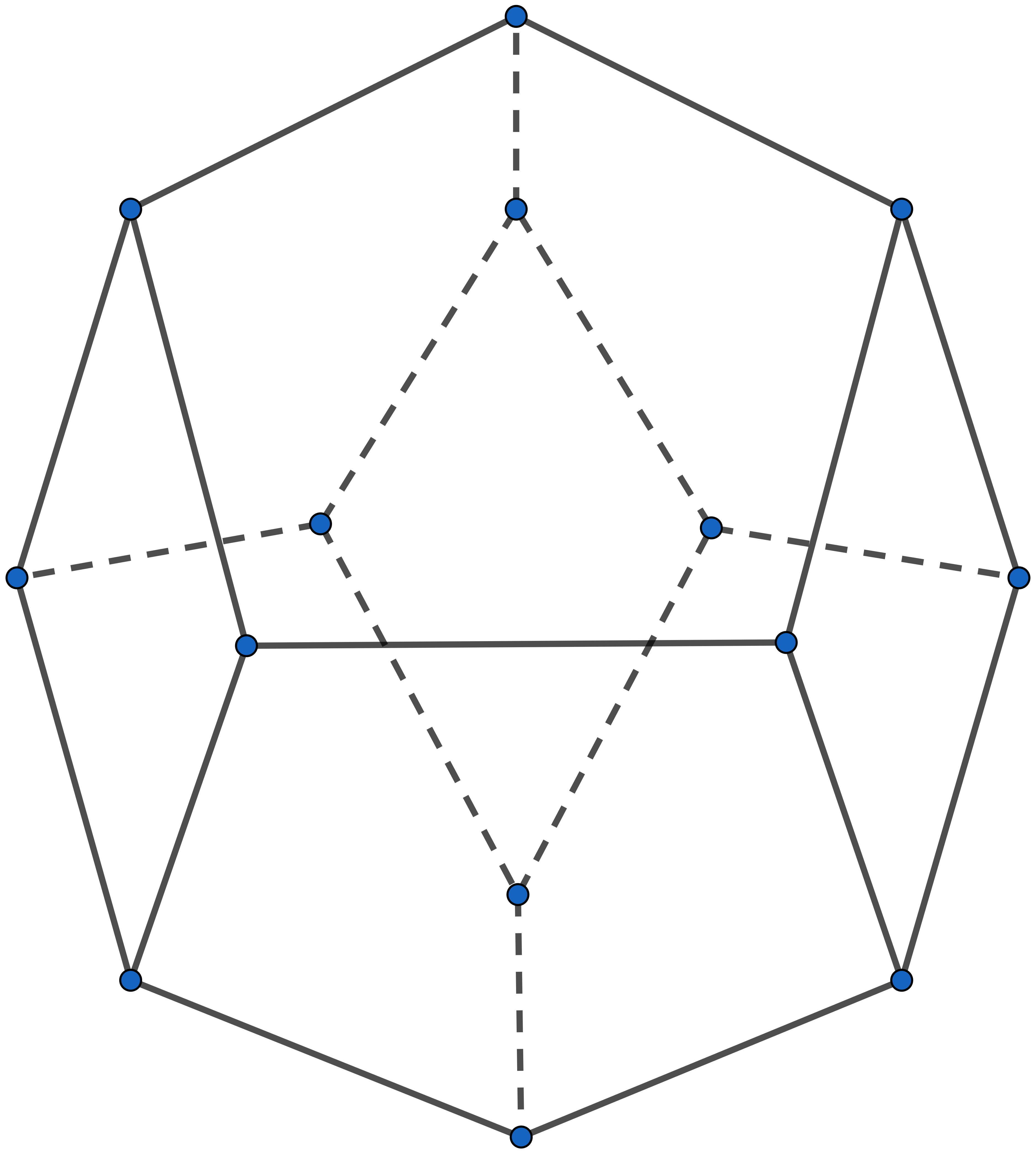
Antipodal Symmetry & Dualities

Andy Liu



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Summary

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- New type of symmetry & dualities

  in planar N = 4 super-Yang-Mills theory

  (toy model for QCD)

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  [AL 2207.11815]

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  [AL 2207.11815]

- *Antipodal* duality - *all* loops, 6-particle / 3-particle (form factor)

  [Dixon, Gürdogan, McLeod, Wilhelm 2112.06243]

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  descends from

- *Antipodal* self-duality - 2 (probably all?) loops, 4-particle (form factor)

  [Dixon, Gürdogan, AL, McLeod, Wilhelm 2212.02410]

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Summary

- New type of <u>symmetry</u> & <u>dualities</u>

  in planar N = 4 super-Yang-Mills theory

  (toy model for QCD)

All involve an ***Antipode*** map of the underlying math functions

- *Antipodal* symmetry - 2 loops, *all* # of particles

  [AL 2207.11815]

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Collider experiments - Best direct probe of nature

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$\Uparrow$

Cross sections

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Cross sections $\Leftarrow$ **Amplitudes**

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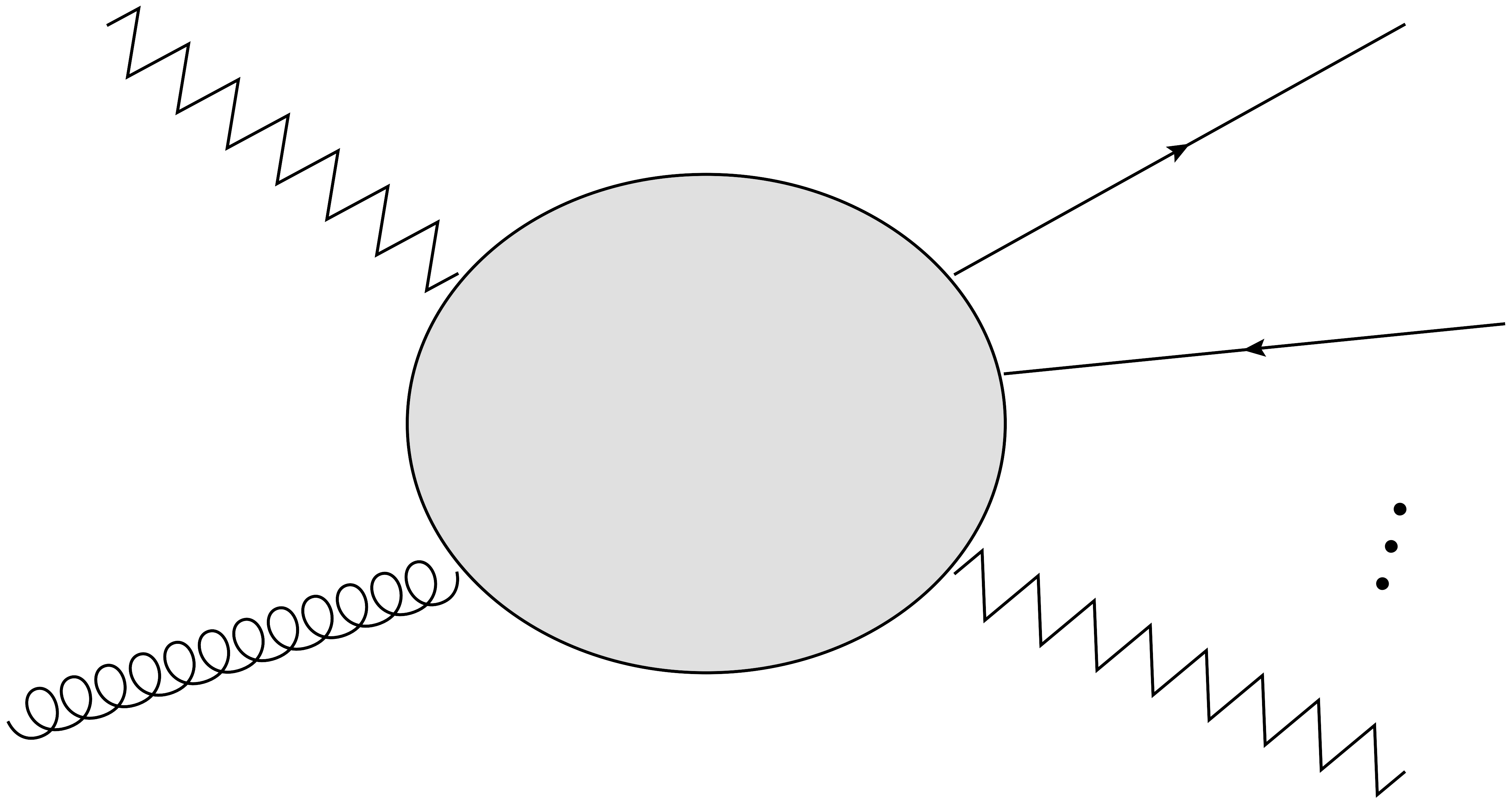
Cross sections $\Leftarrow$ **Amplitudes**

$$\sigma \sim \left| \quad \right|^2$$



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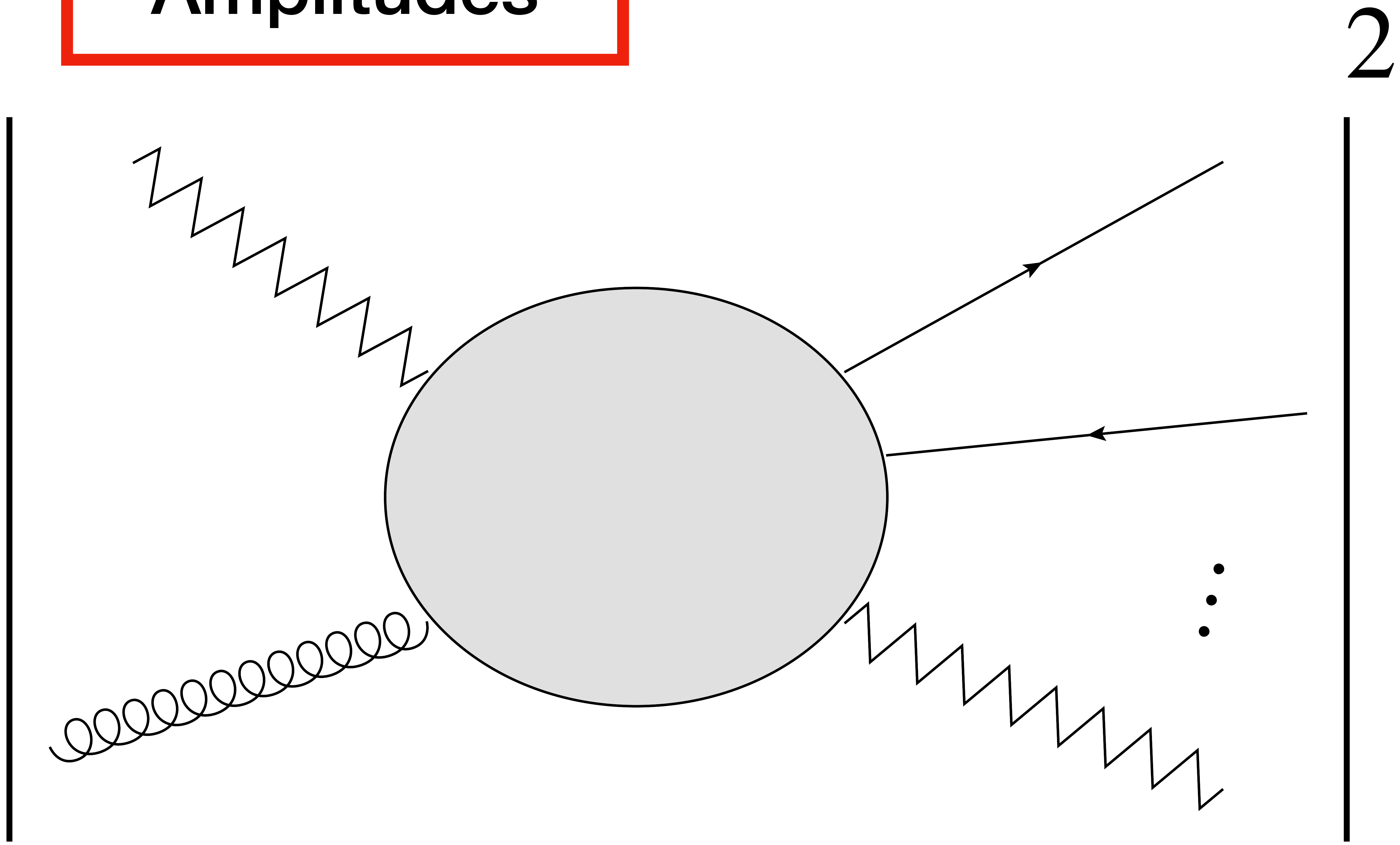

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- $N = 4$ Super-Yang-Mills - toy model for **QCD**

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- Maximal SUSY - "simplest theory"

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- Gluon, Gluinos x 4, Scalars x 6

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- N = 4 Super-Yang-Mills - toy model for **QCD**
- Maximal SUSY - "simplest theory"
- Gluon, Gluinos x 4, Scalars x 6
- AdS/CFT - dual to type IIB string theory on $AdS_5 \times S^5$

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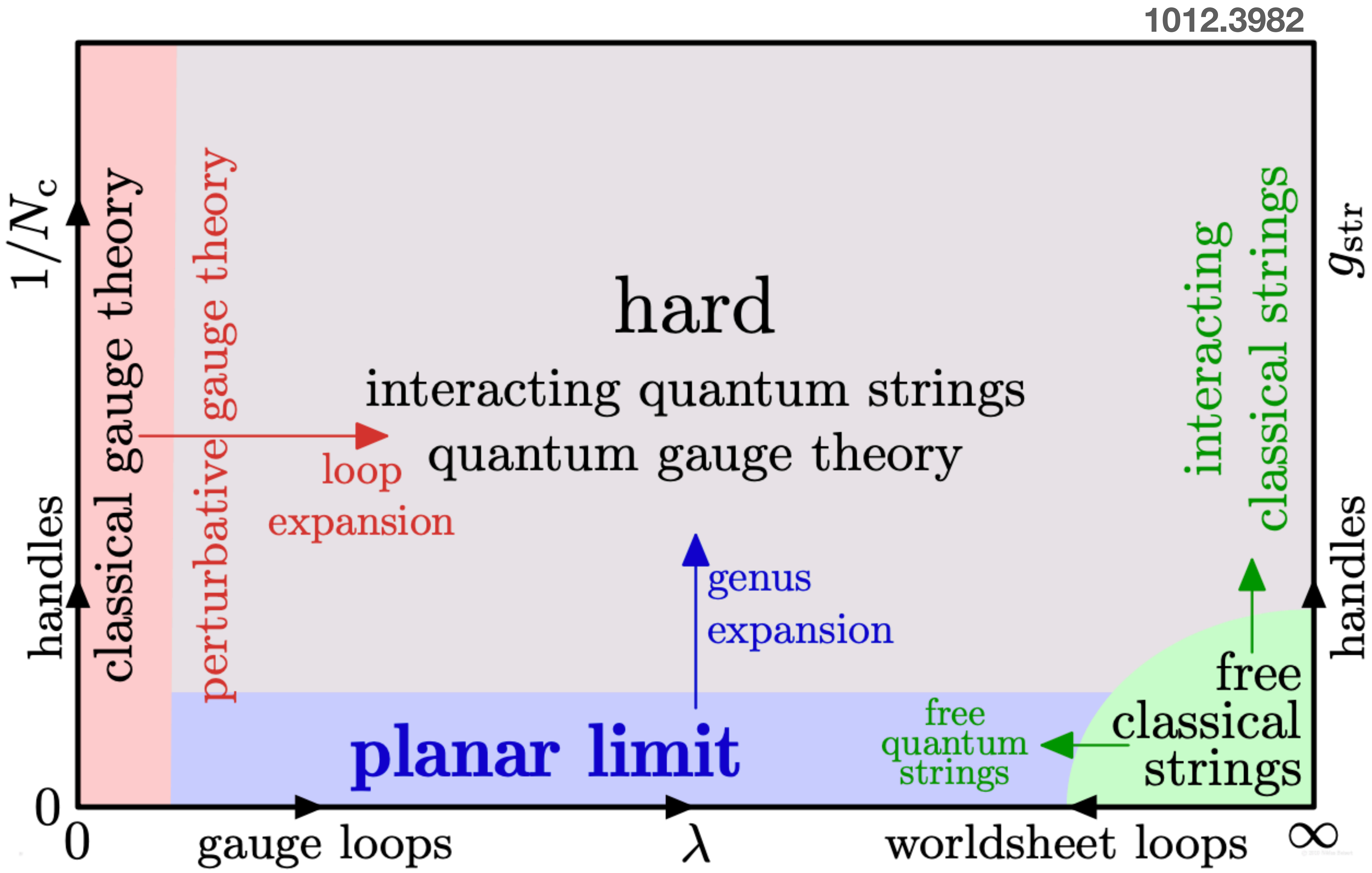
N = 4 SYM parameter space



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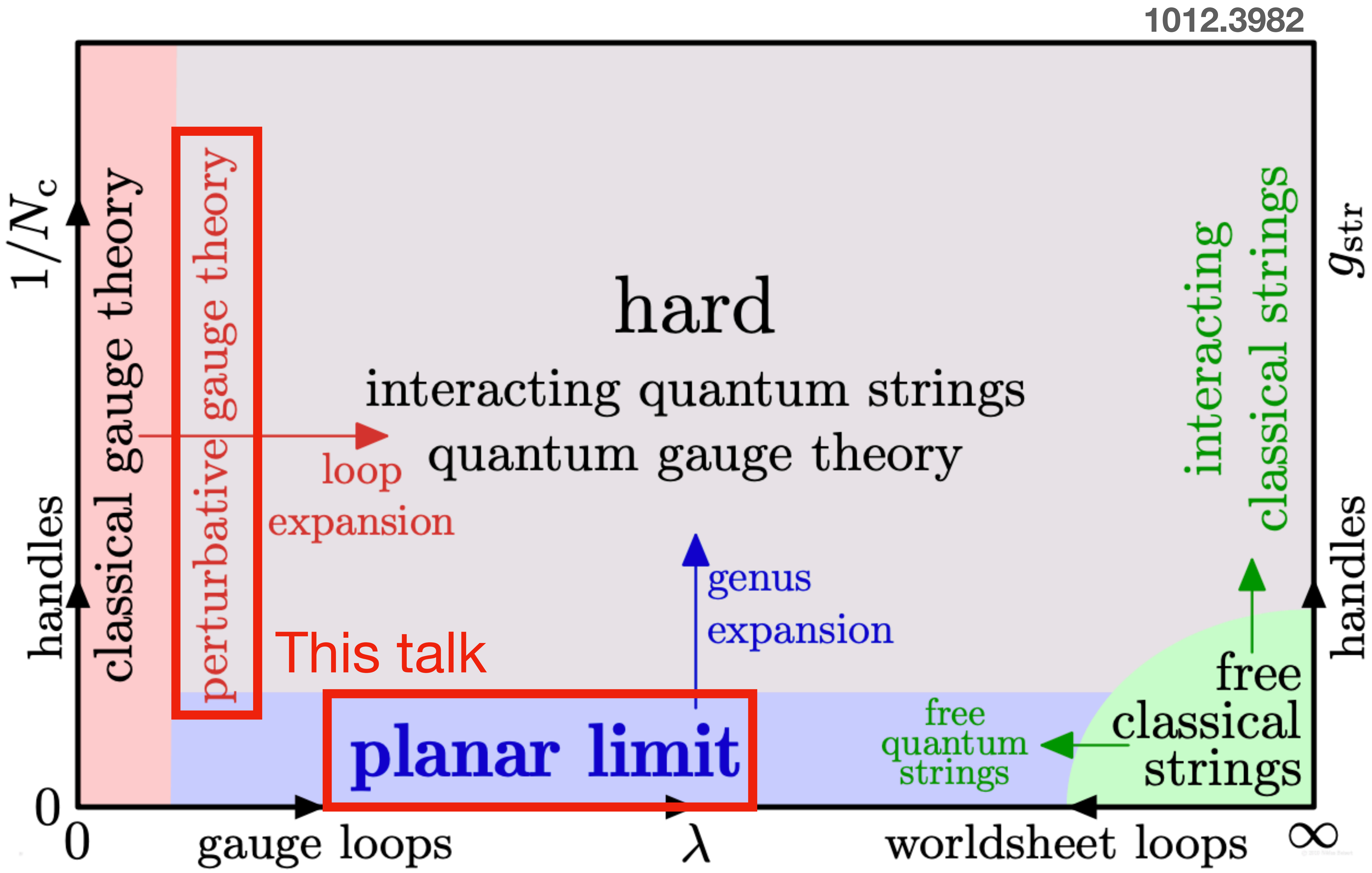

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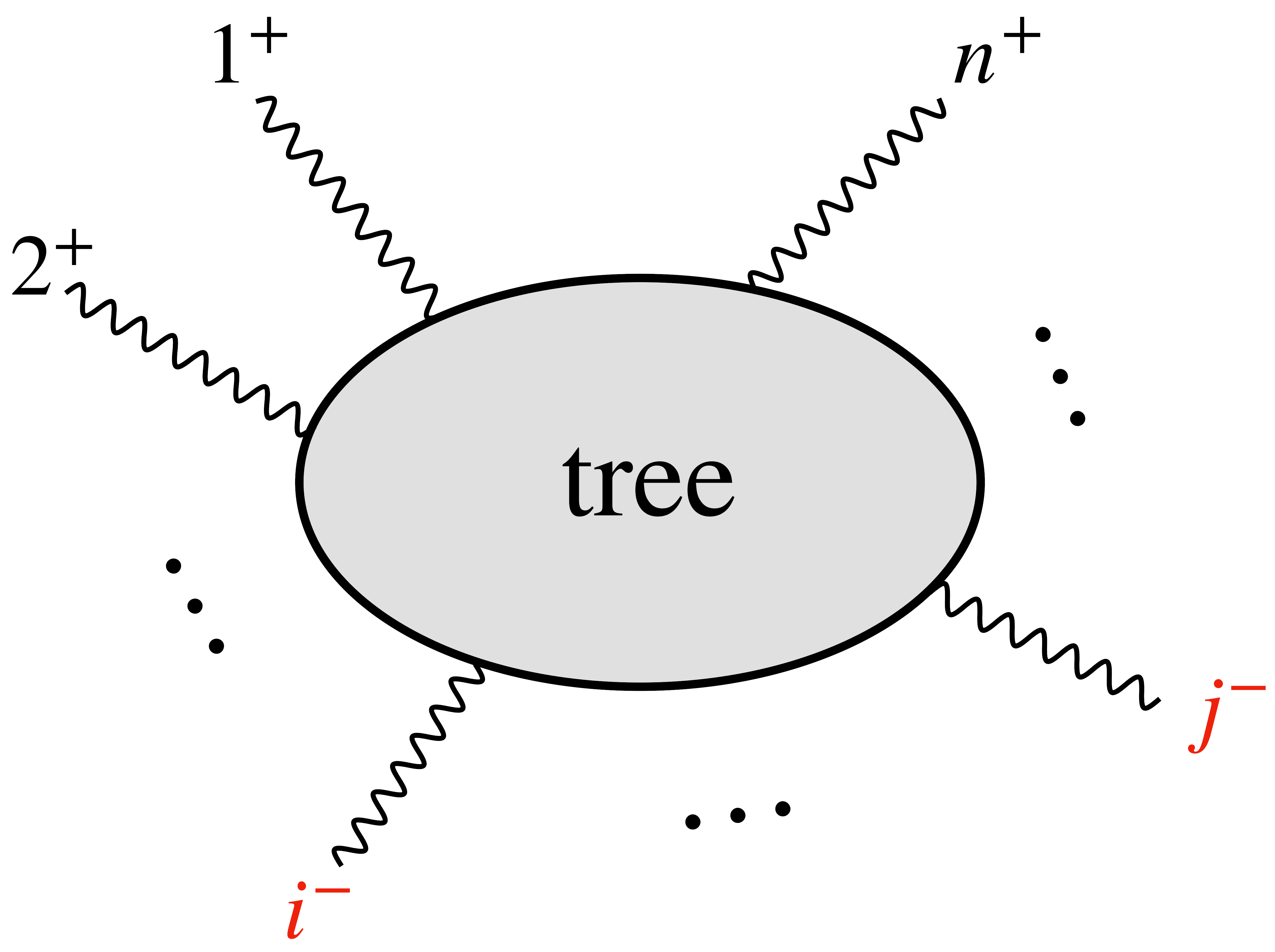
Simplest class of amplitudes: Maximal-Helicity-Violating (MHV)

0-loop



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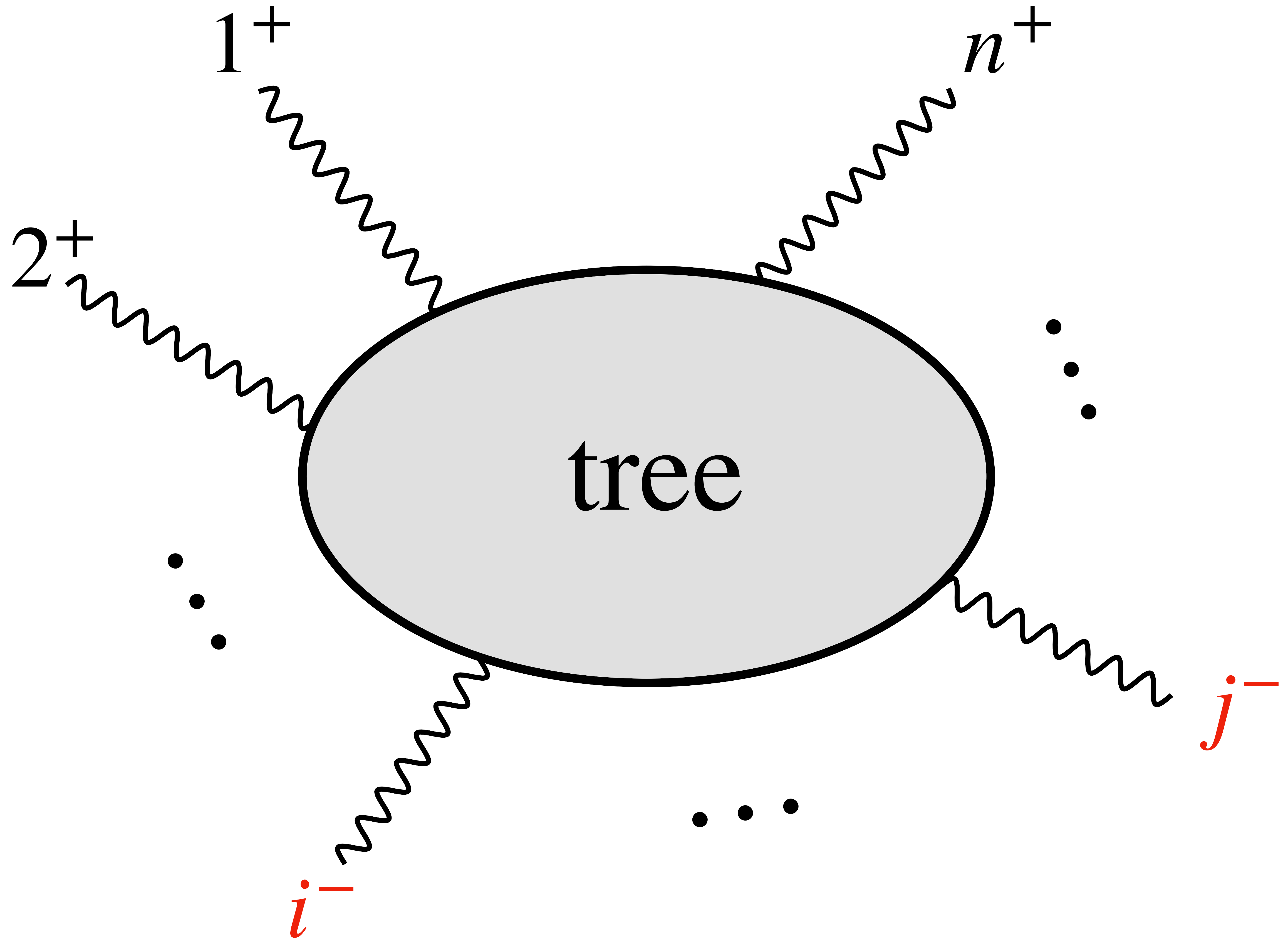

[Parke-Taylor]

$$= \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \mathrm{Tr}\left(T^{a_1} T^{a_2} \ldots T^{a_n}\right)$$

$$+ \ldots$$

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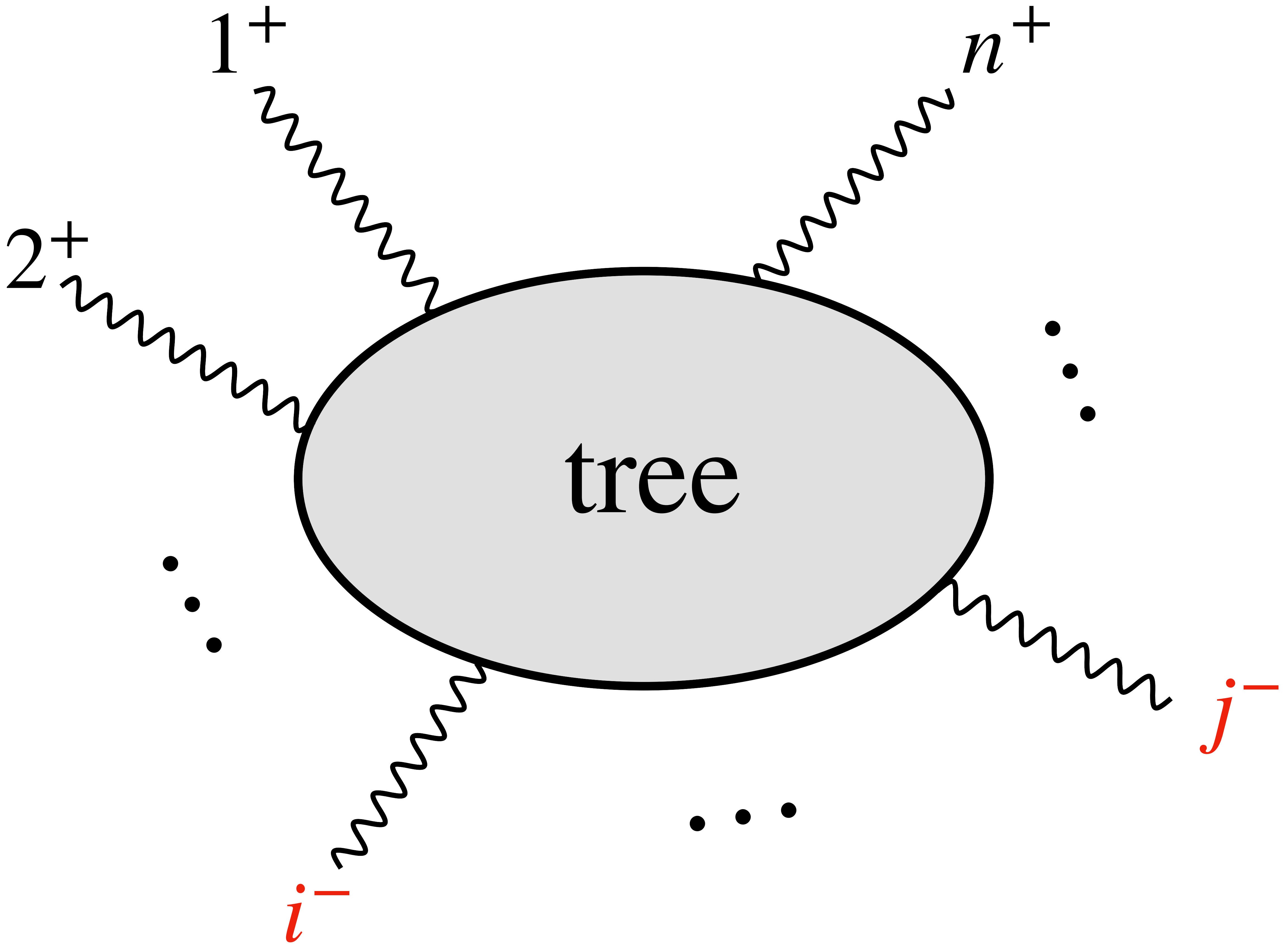

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No multi-particle poles

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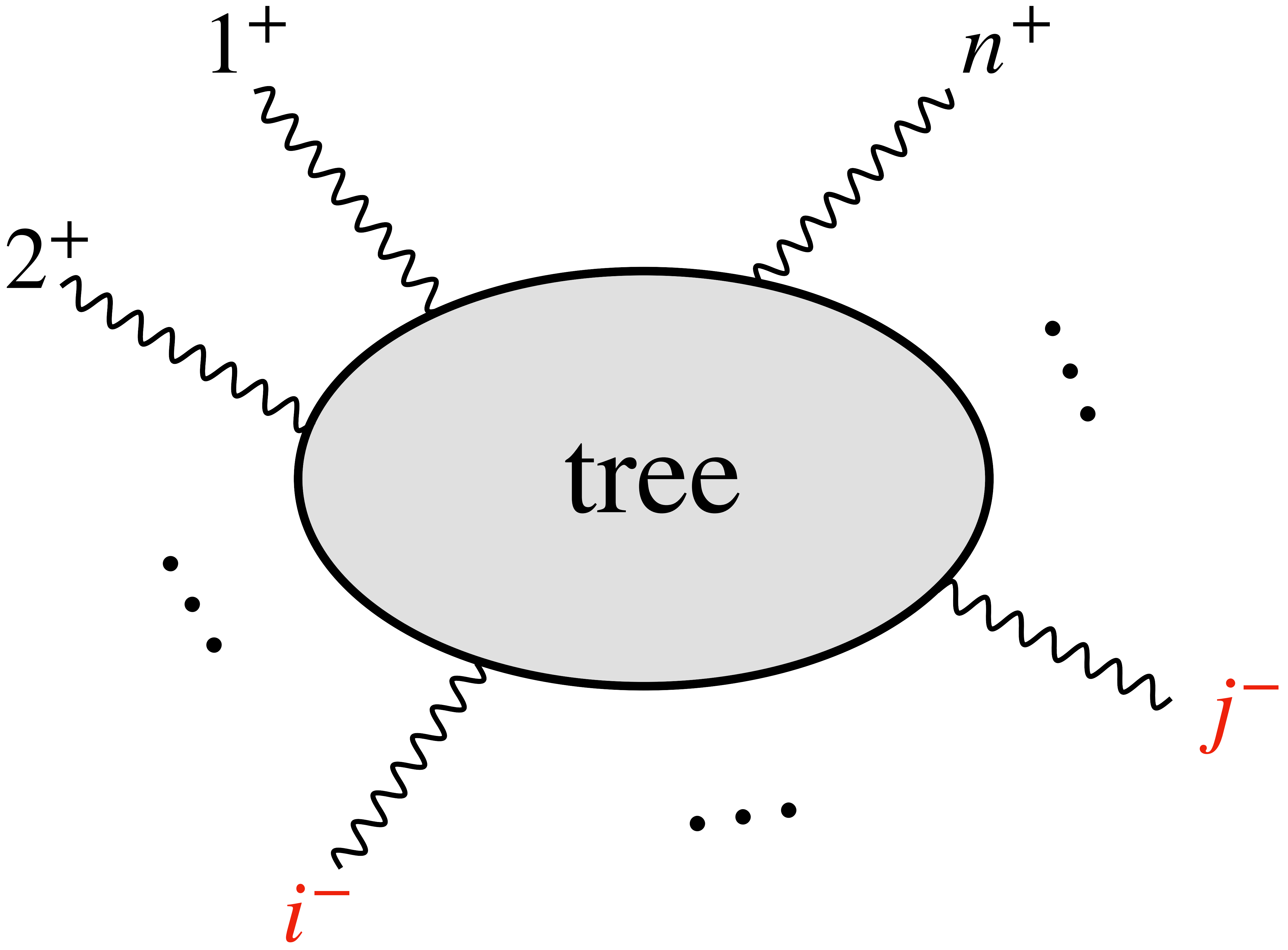

Same as QCD

[Parke-Taylor]

$$= \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\mathrm{Tr}\left(T^{a_1}T^{a_2}\ldots T^{a_n}\right)$$

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Amplitudes are ***not*** conformal invariant - asymptotic states

$\geq$ **1 loops**

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Amplitudes are ***not*** conformal invariant - asymptotic states

IR divergences are simple

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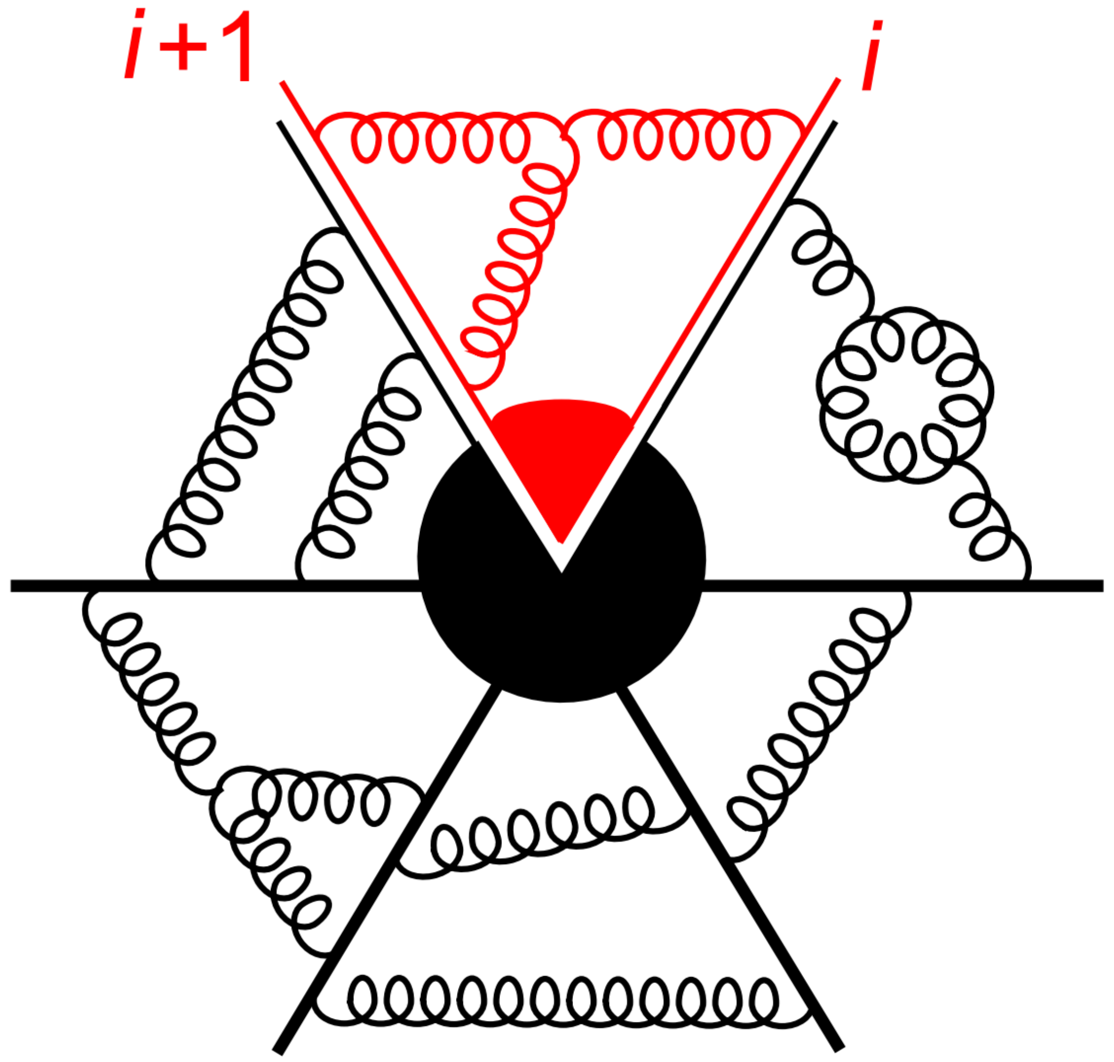

hep-th/0505205

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Amplitudes are ***not*** conformal invariant - asymptotic states

IR divergences are simple

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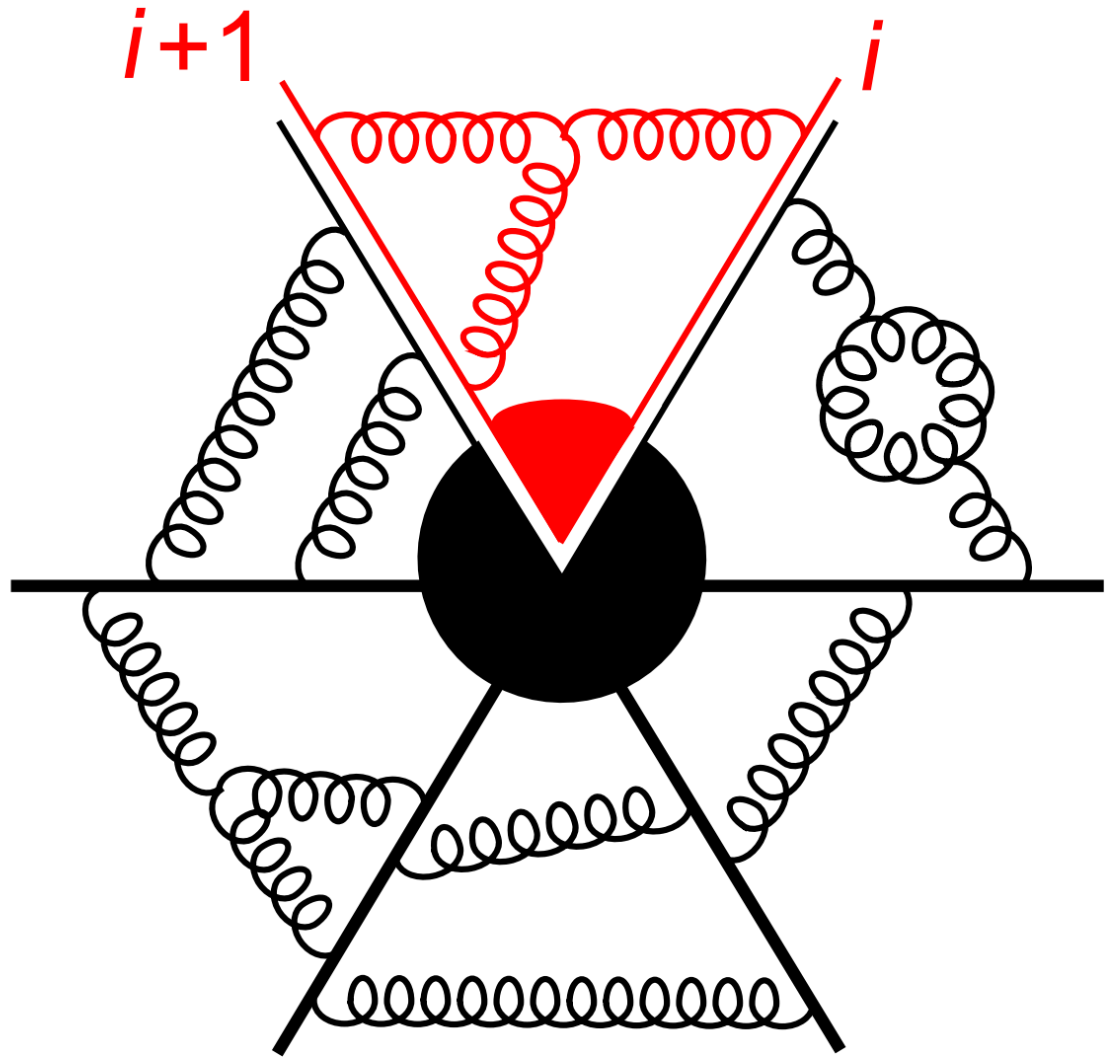
A = [Tree] × [Exponentiated IR] × Exp(**Remainder**)

BDS Ansatz hep-th/0505205

Remainder ***is*** conformal invariant



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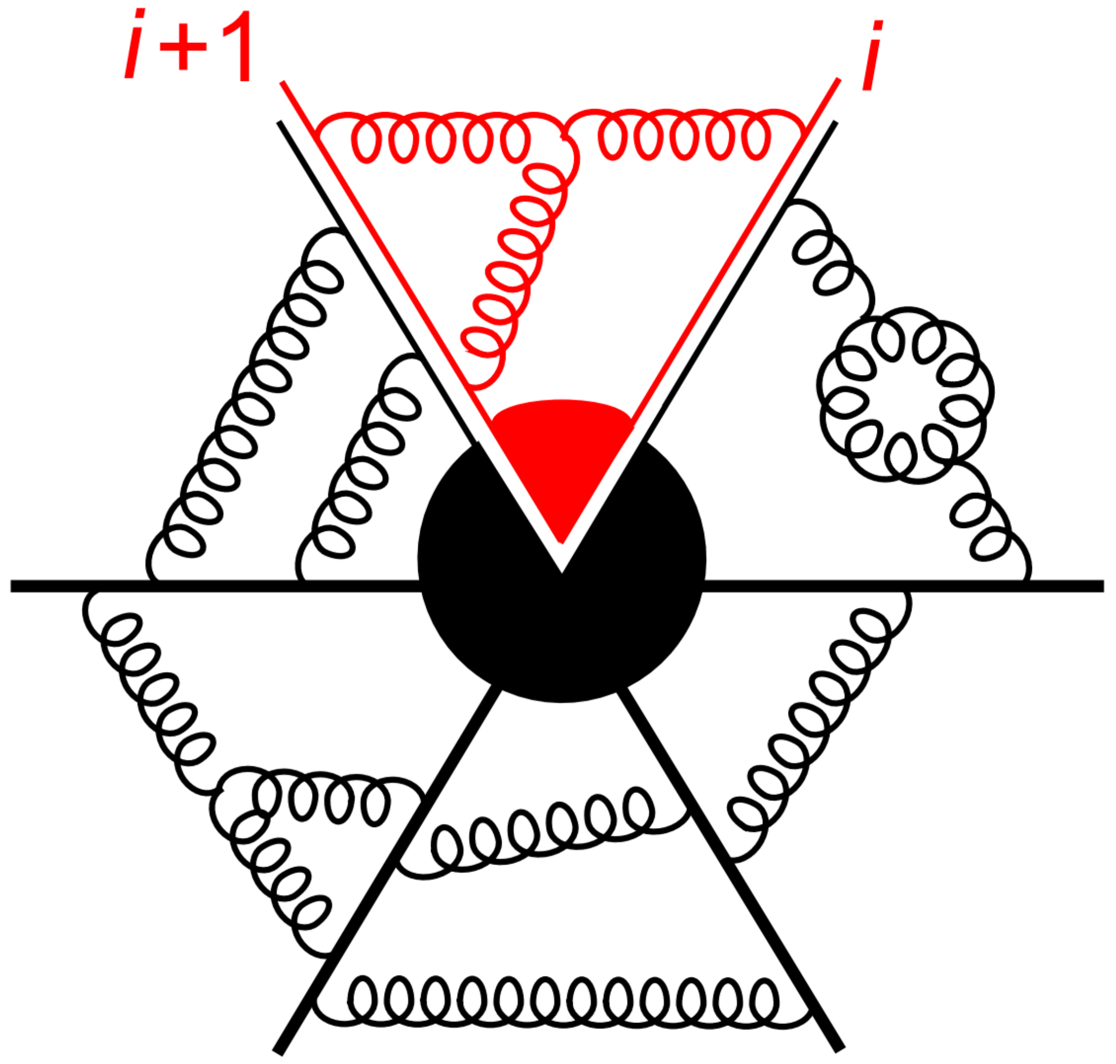

A = [Tree] $\times$ [Exponentiated IR] $\times$ Exp(**Remainder**)

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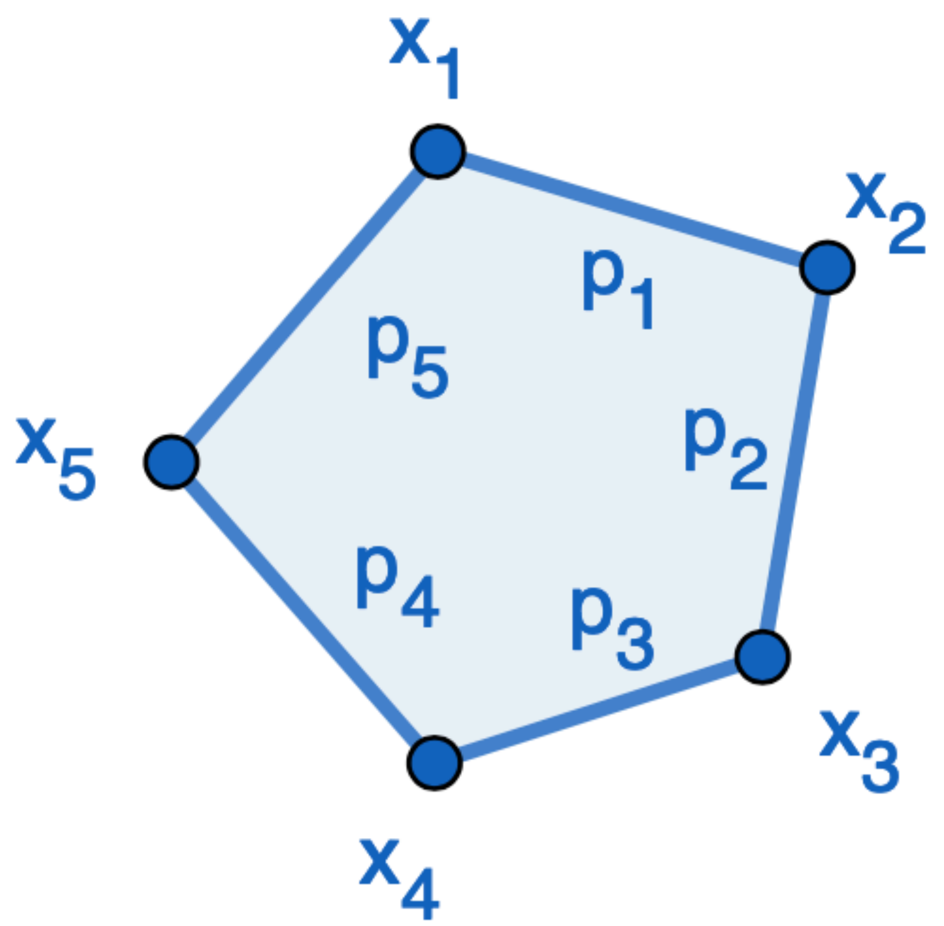
hep-th/0505205

Remainder *is* conformal invariant

[Alday, Maldacena, Drummond, Henn, Korchemsky, Sokatchev, Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich]

Hidden symmetry: ***Dual*** conformal symmetry

(Amplitude = Wilson Loop)



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Remainder depends on conformal cross ratios

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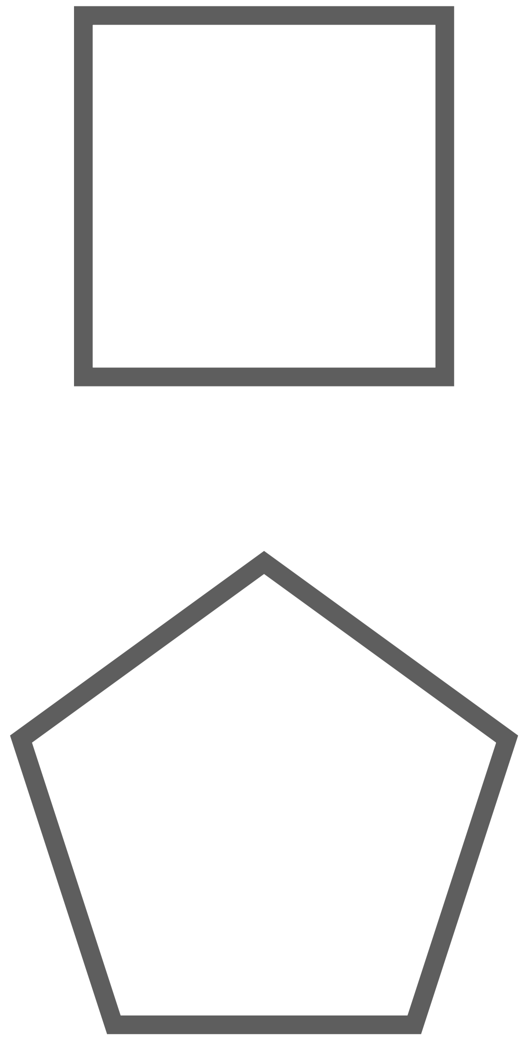
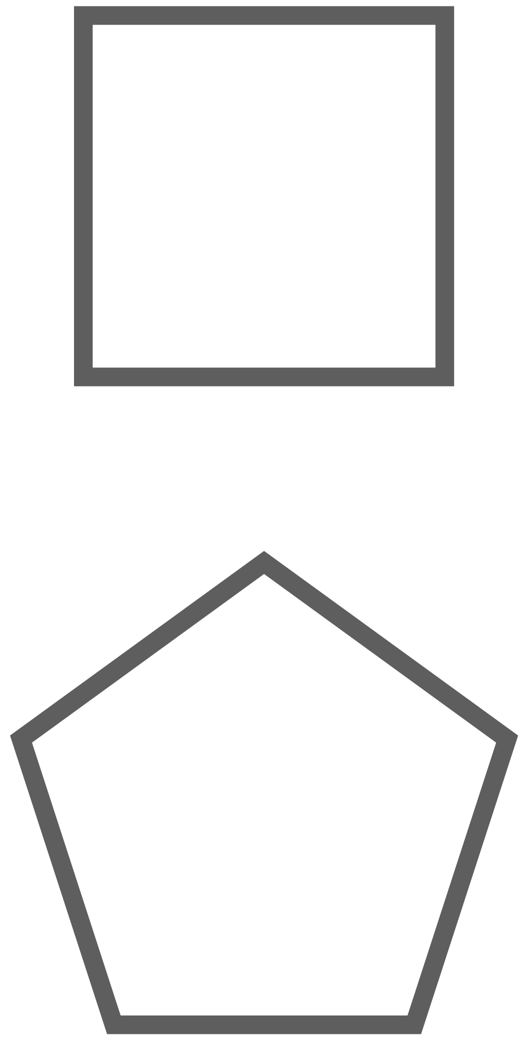


No kinematic DOF

Remainder depends on conformal cross ratios



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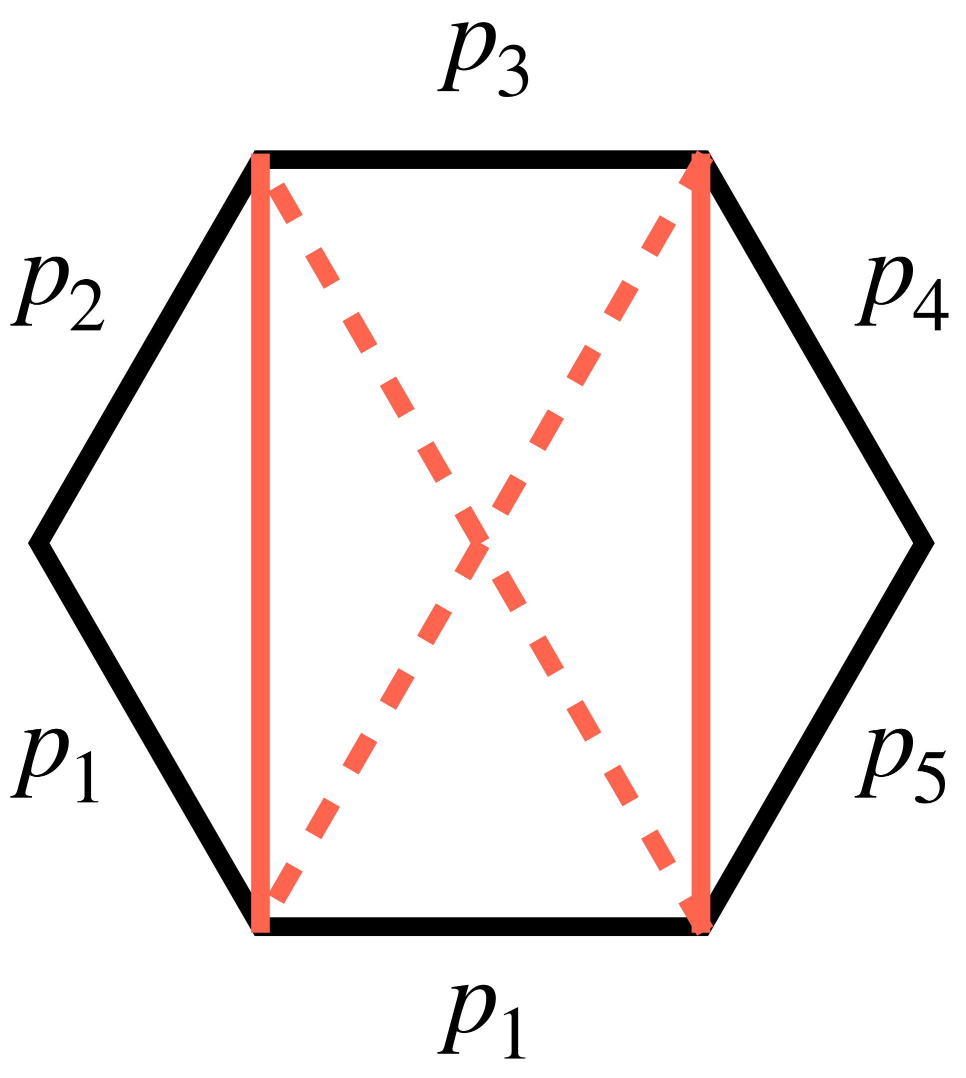

e.g. $u_1 = \frac{s_{12}s_{45}}{s_{123}s_{456}}$

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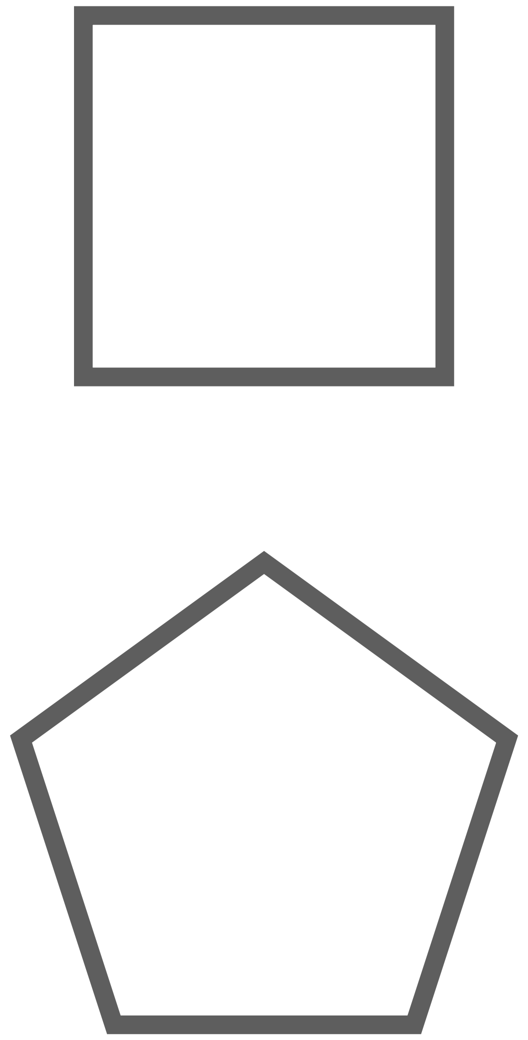

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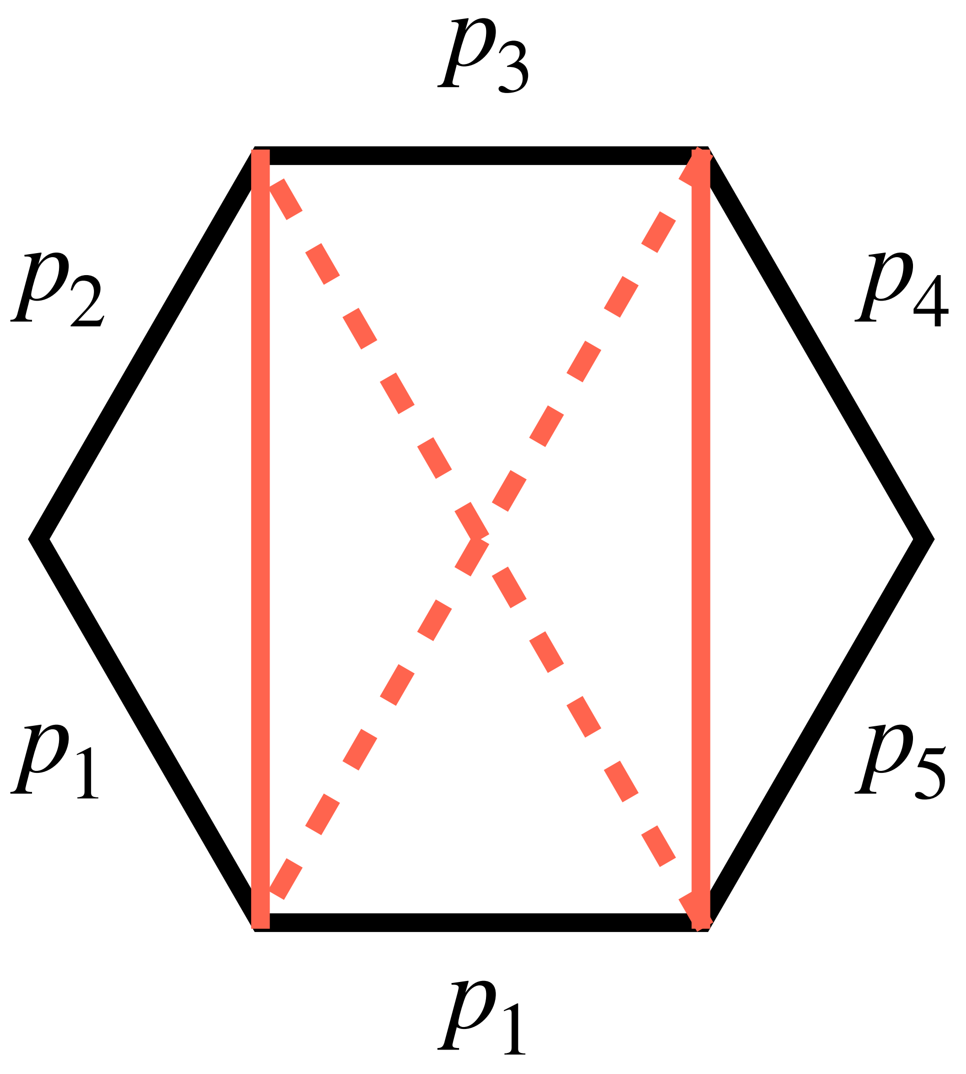

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3 cross ratios
3 DOF

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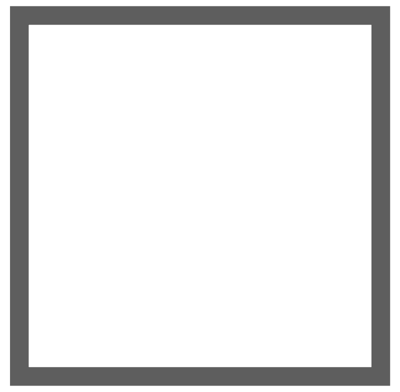

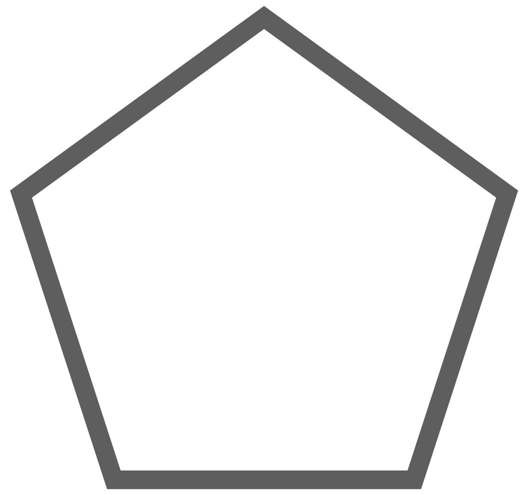

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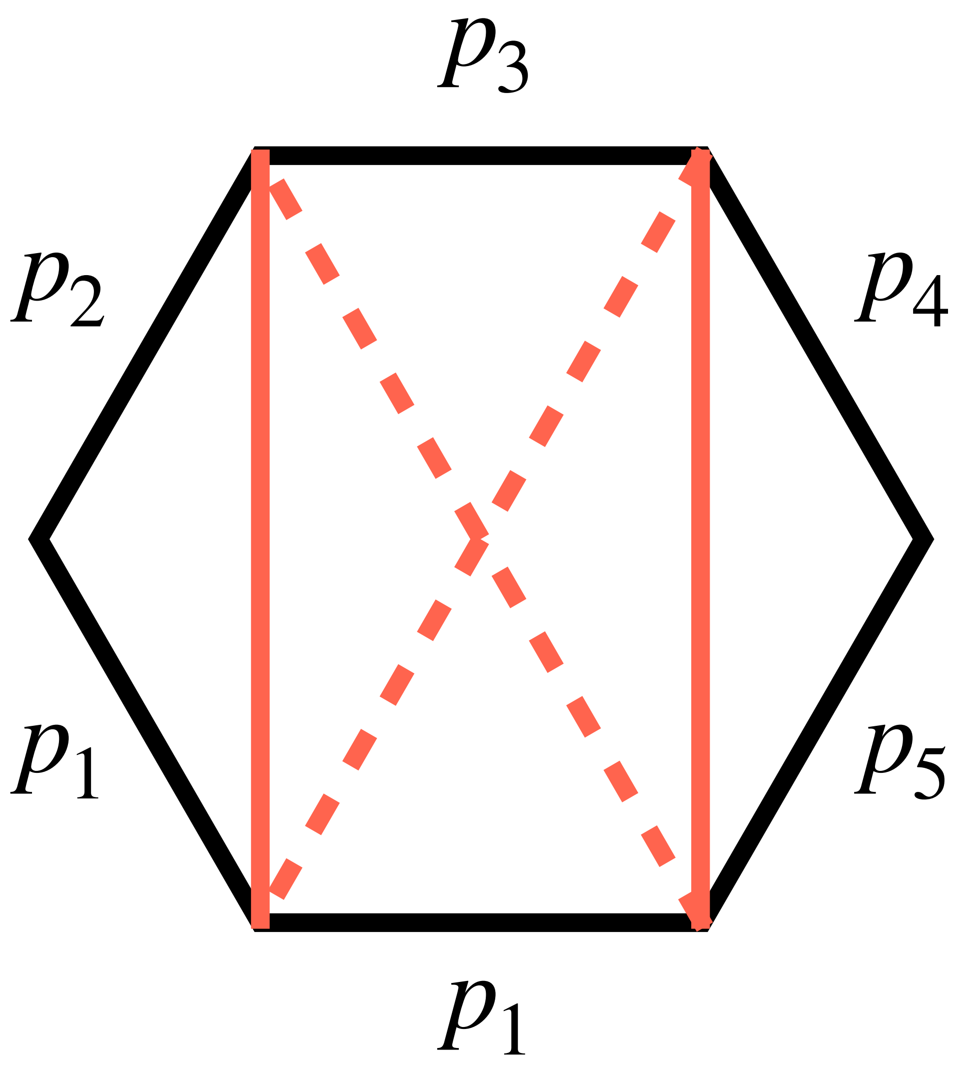

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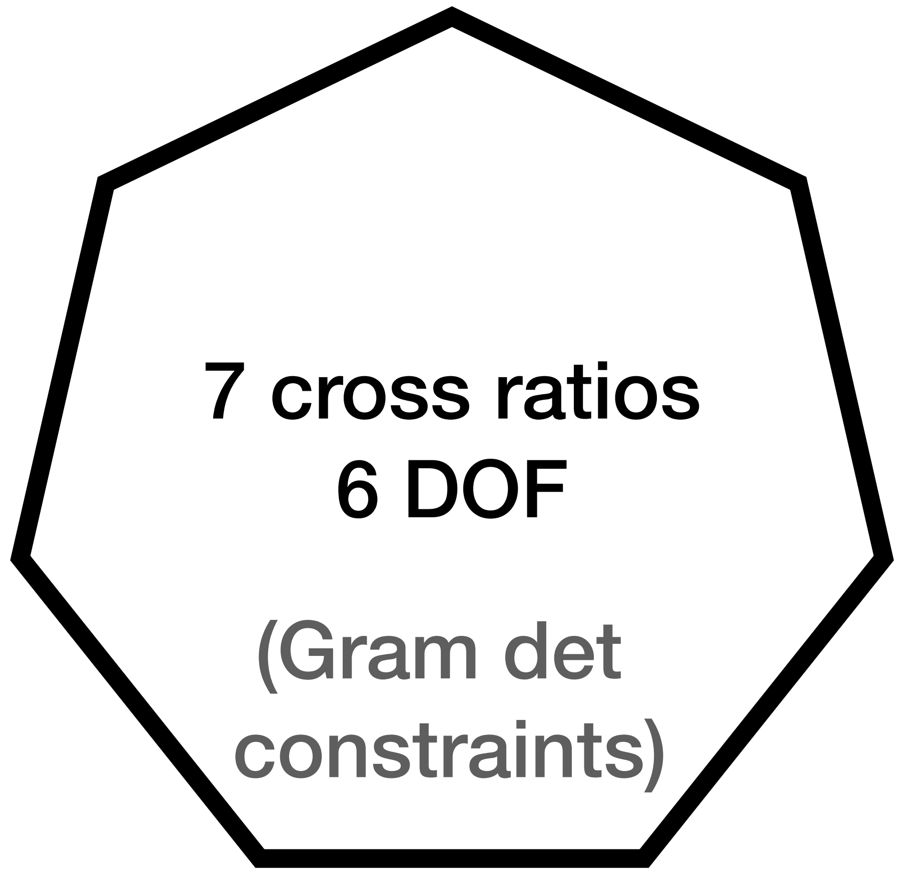

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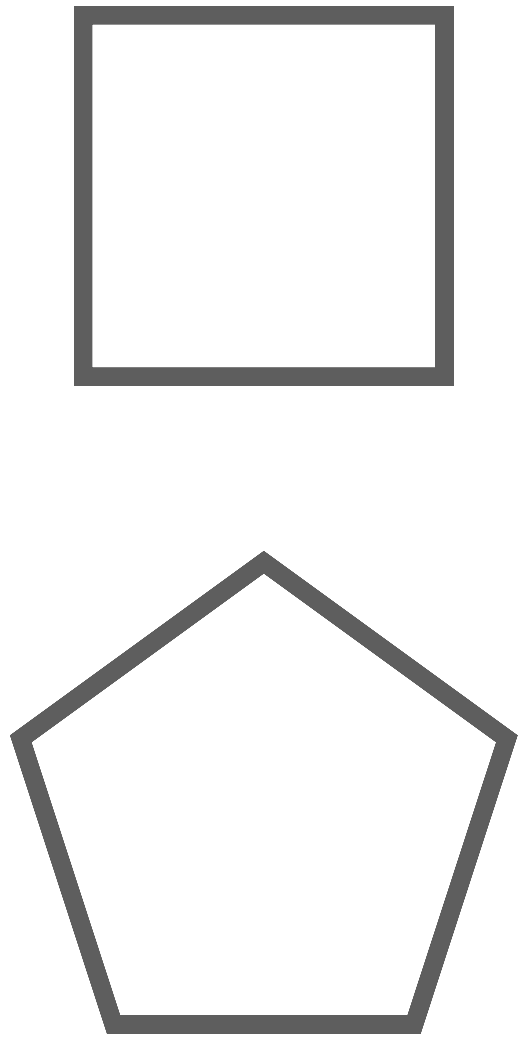

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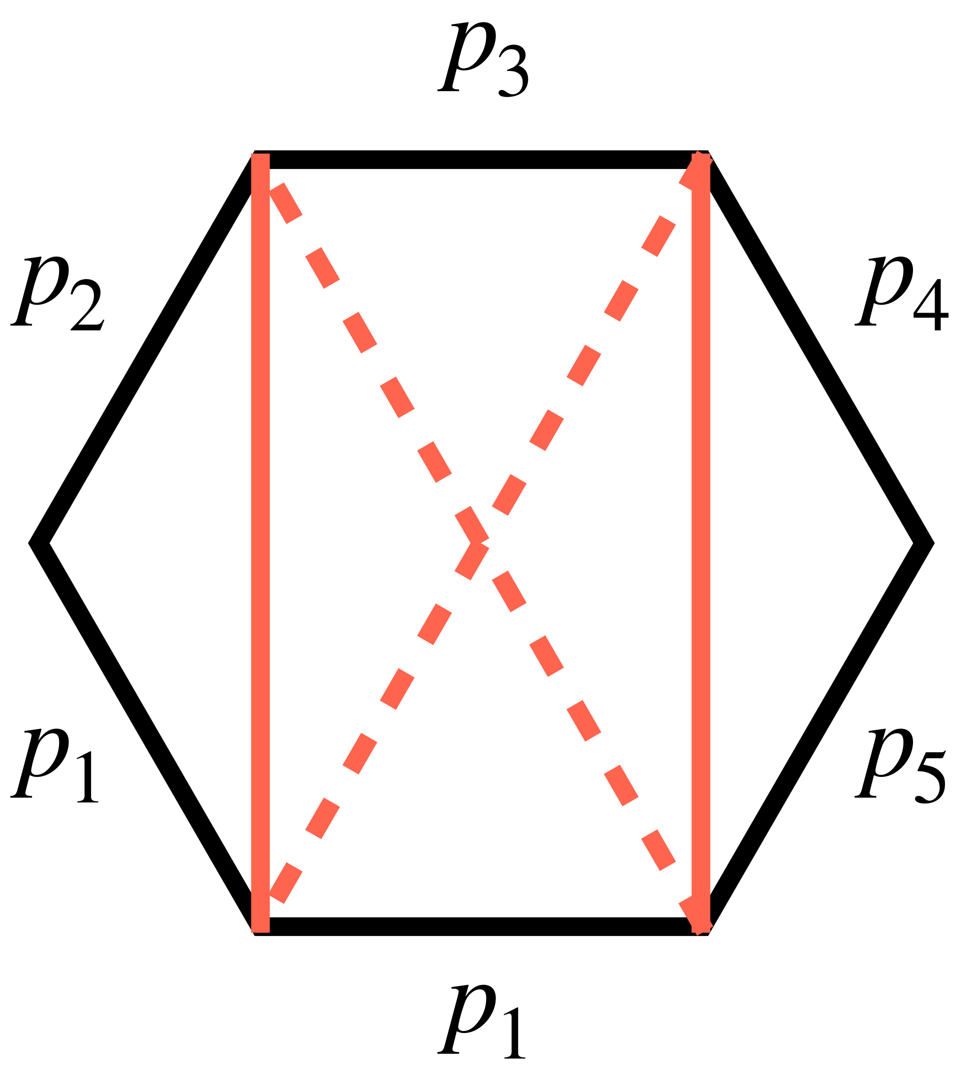

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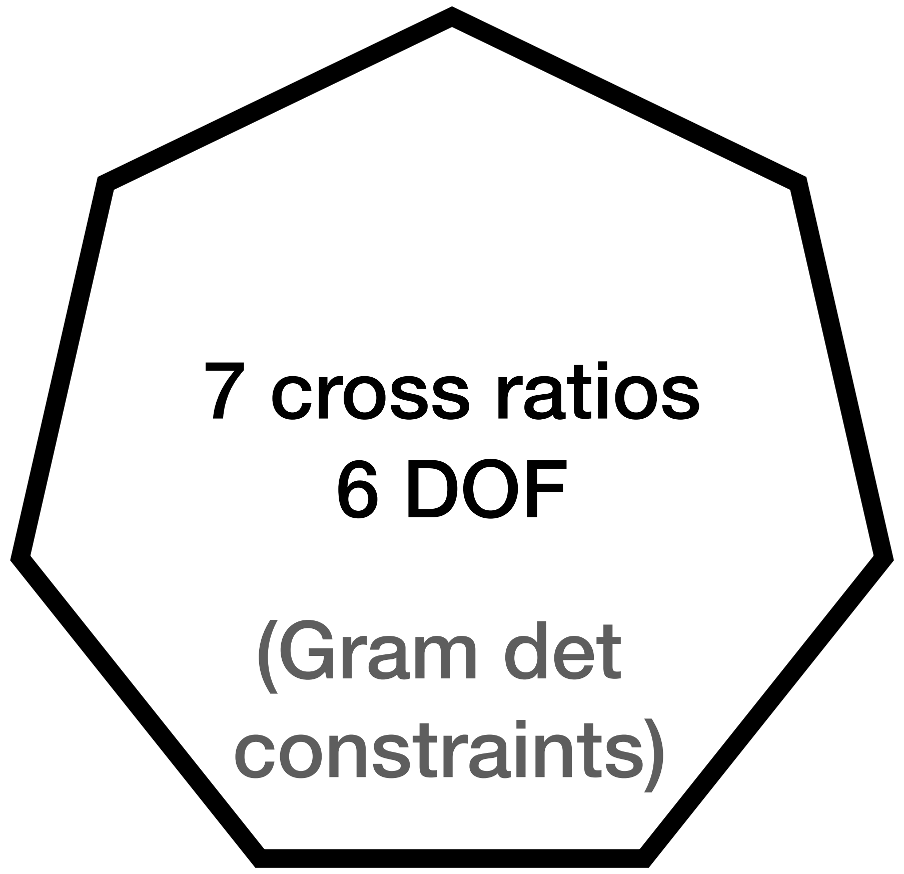

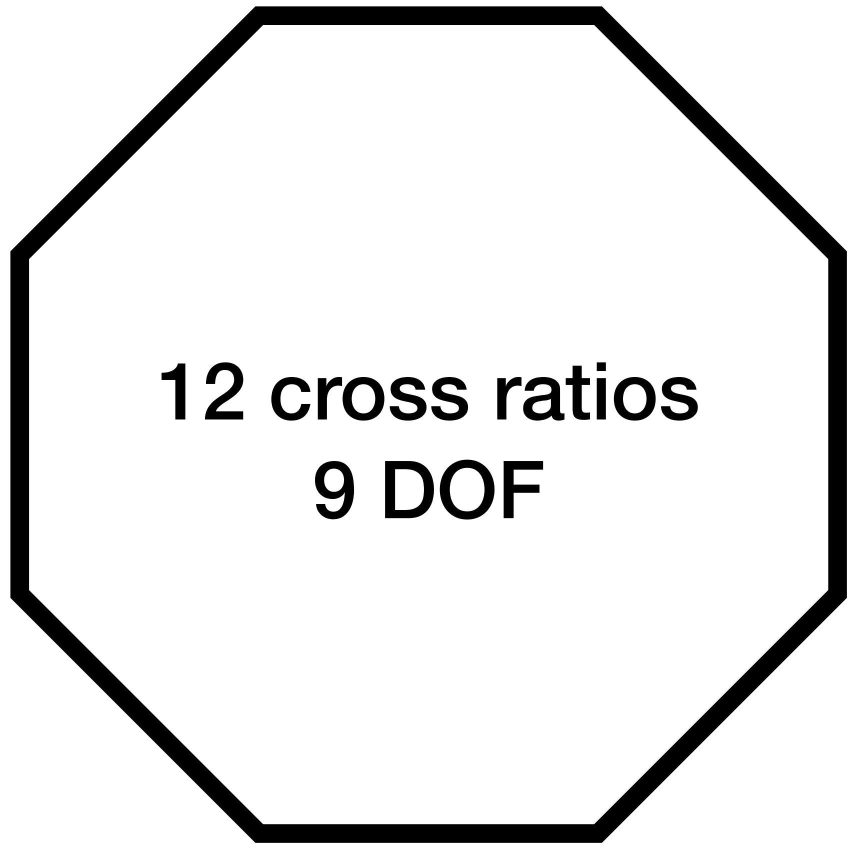

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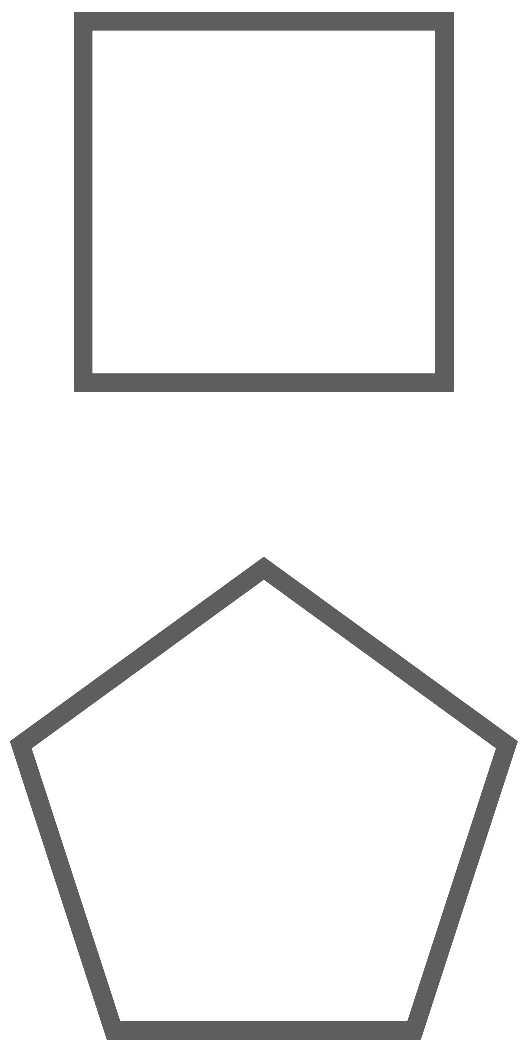

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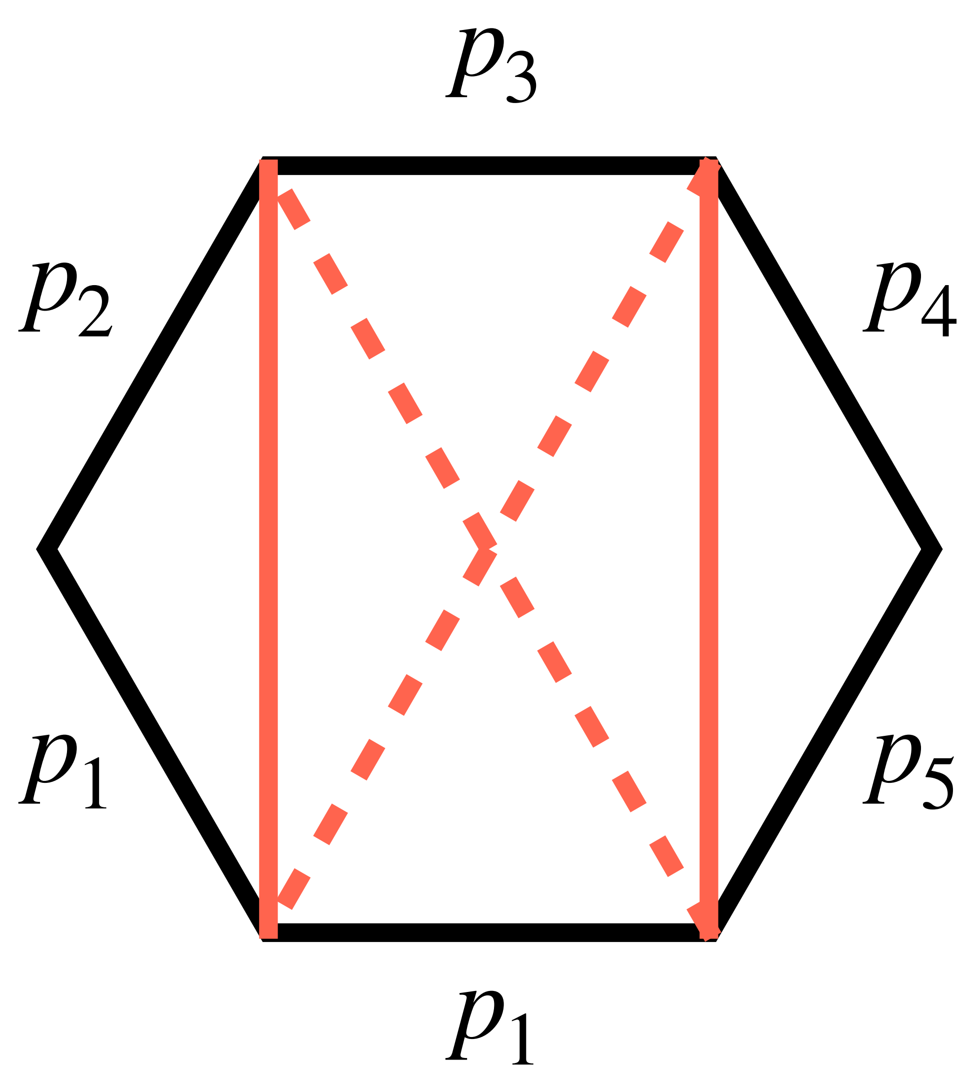

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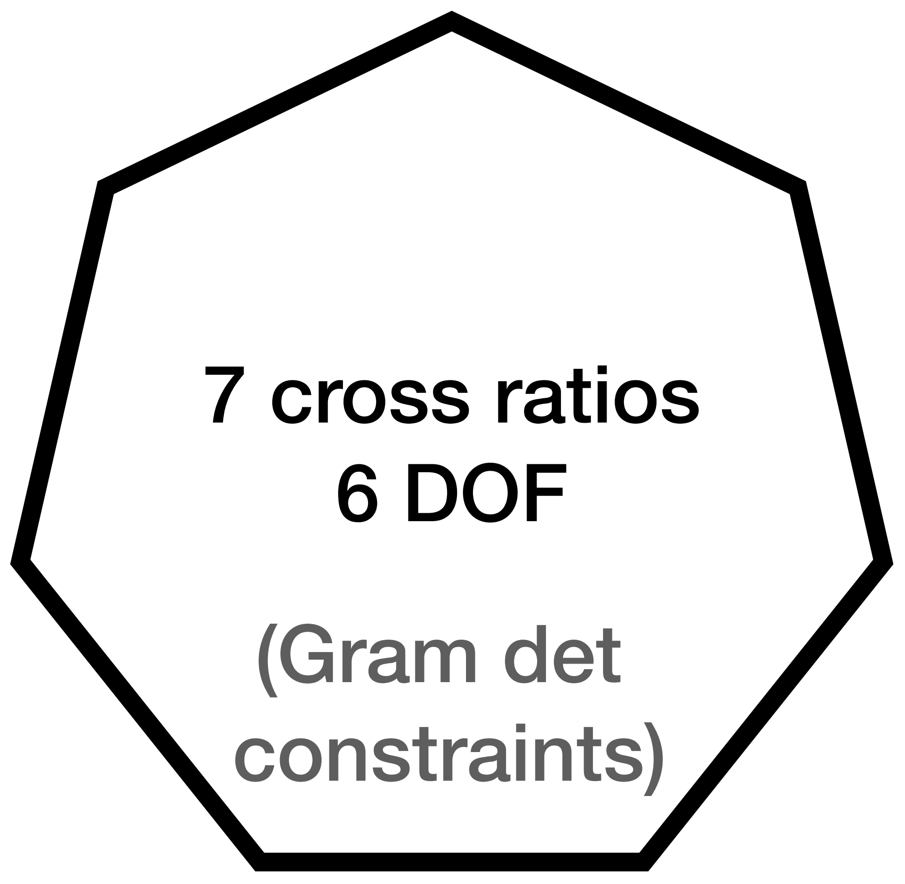

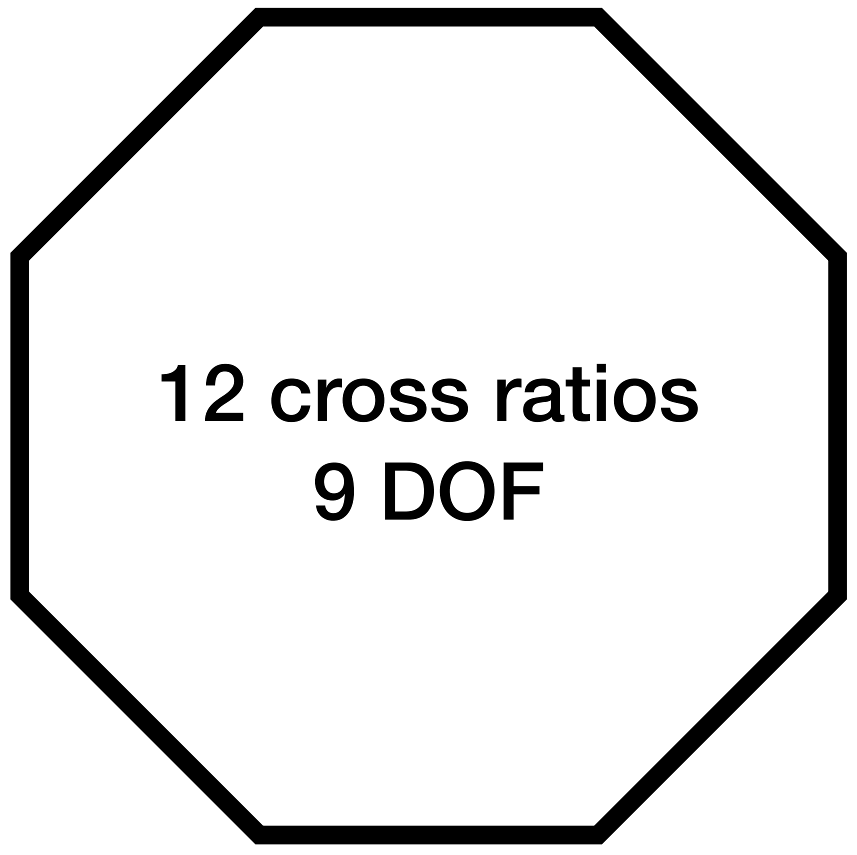

Dimension of dual conformal group = 15 =
10 (Poincare) + 1 (dilation) + 4 (special conformal)

DOF = 3n - 15

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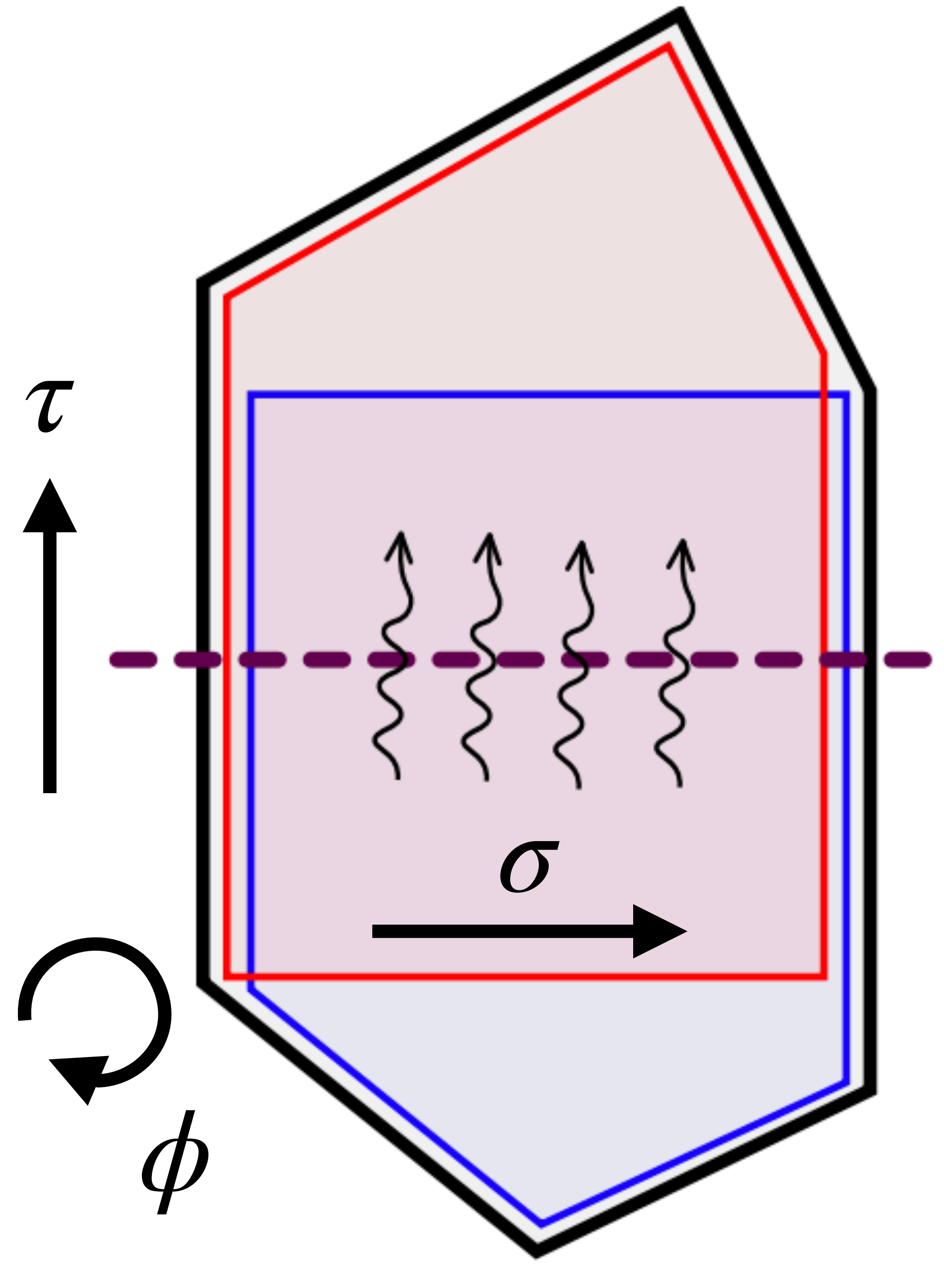
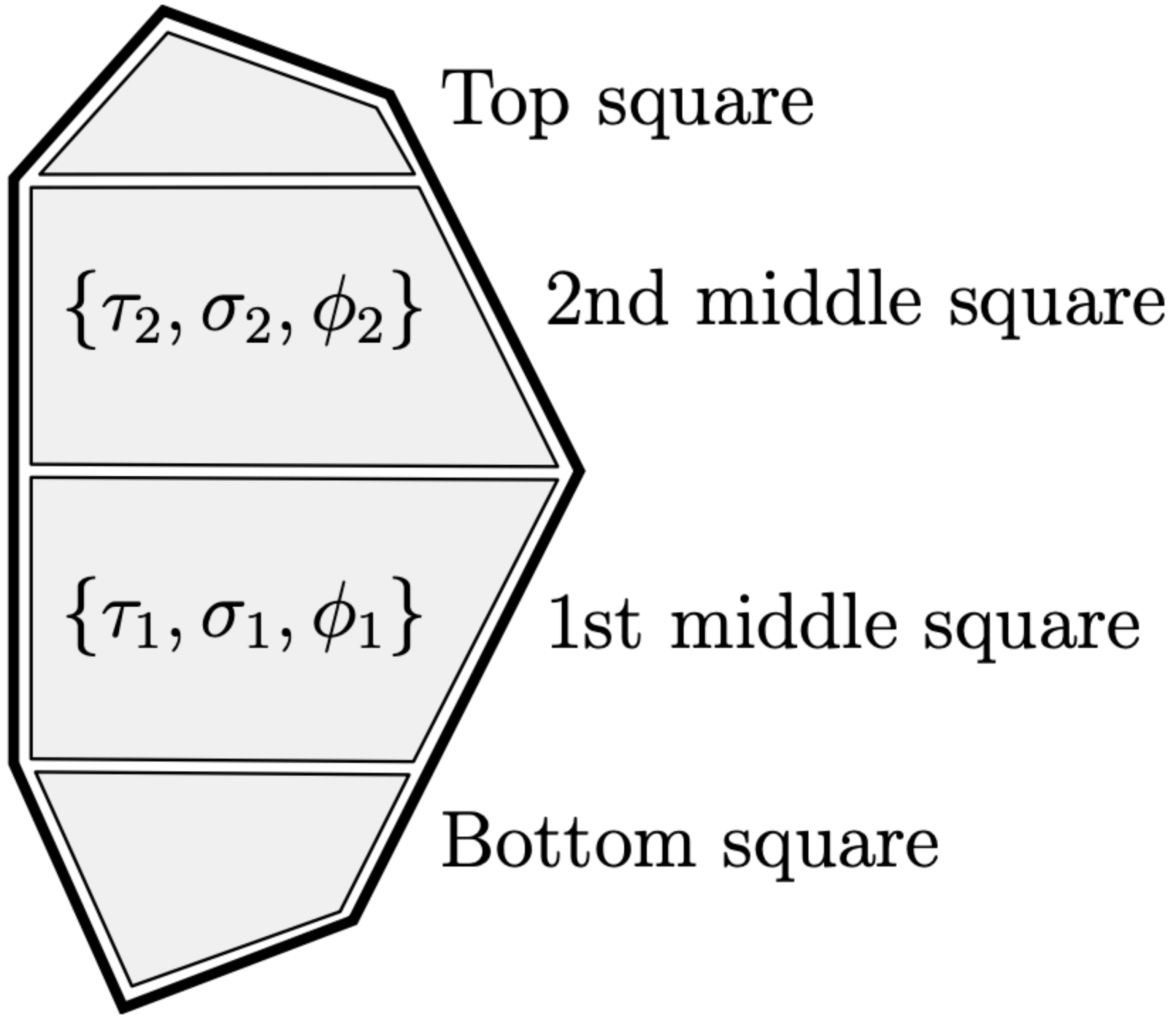
Wilson Loop "OPE" picture

[Alday, Gaiotto, Maldacena, Sever, Vieira, Basso]





1303.1396, 1306.2058

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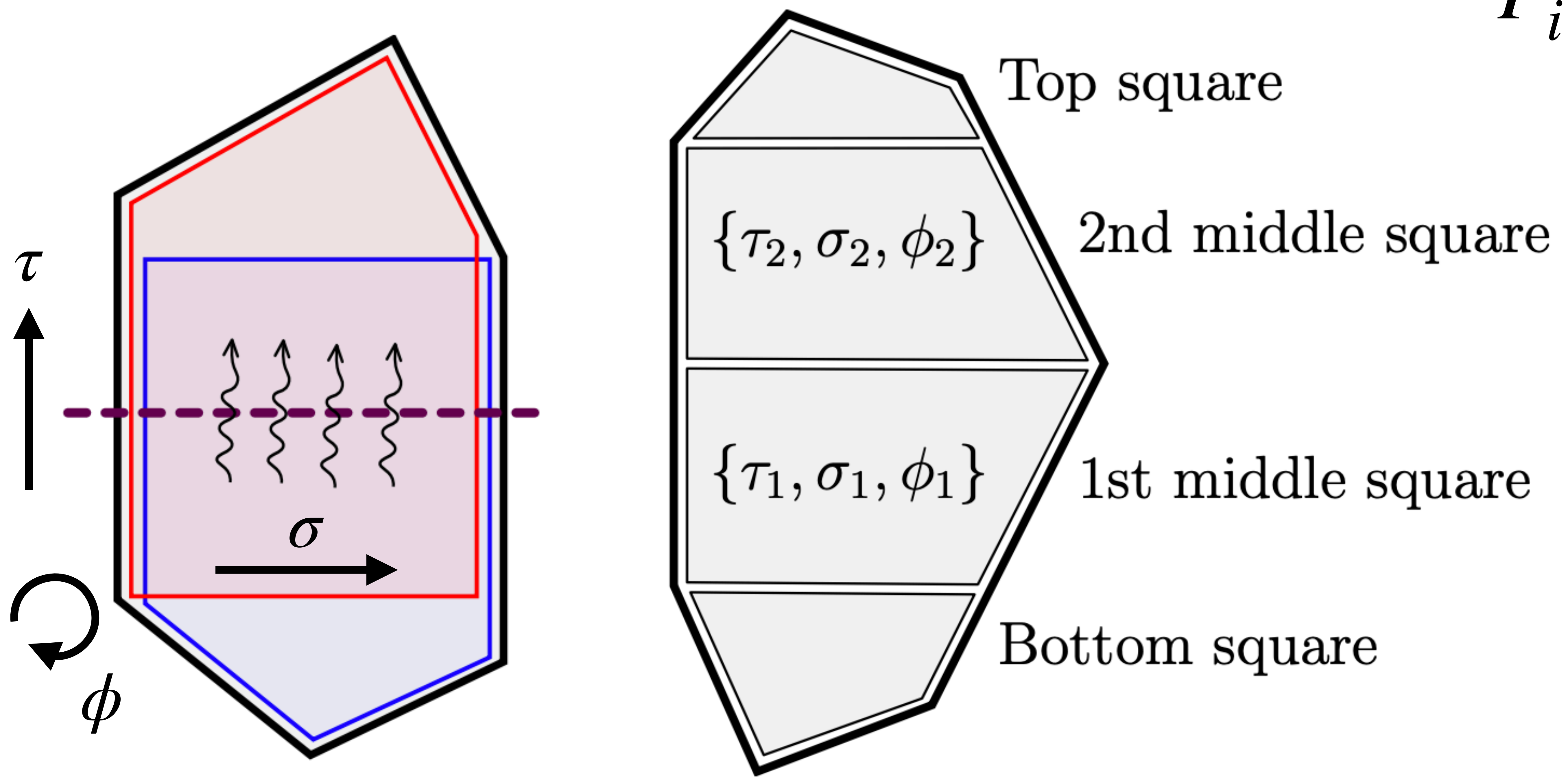
Wilson Loop "OPE" picture

[Alday, Gaiotto, Maldacena, Sever, Vieira, Basso]

$$T_i := e^{-\tau_i}$$

$$S_i := e^{\sigma_i} \qquad i = 1, \ldots, n-5$$

$$F_i := e^{i\phi} \qquad \text{DOF} = 3(n-5)$$



1303.1396, 1306.2058

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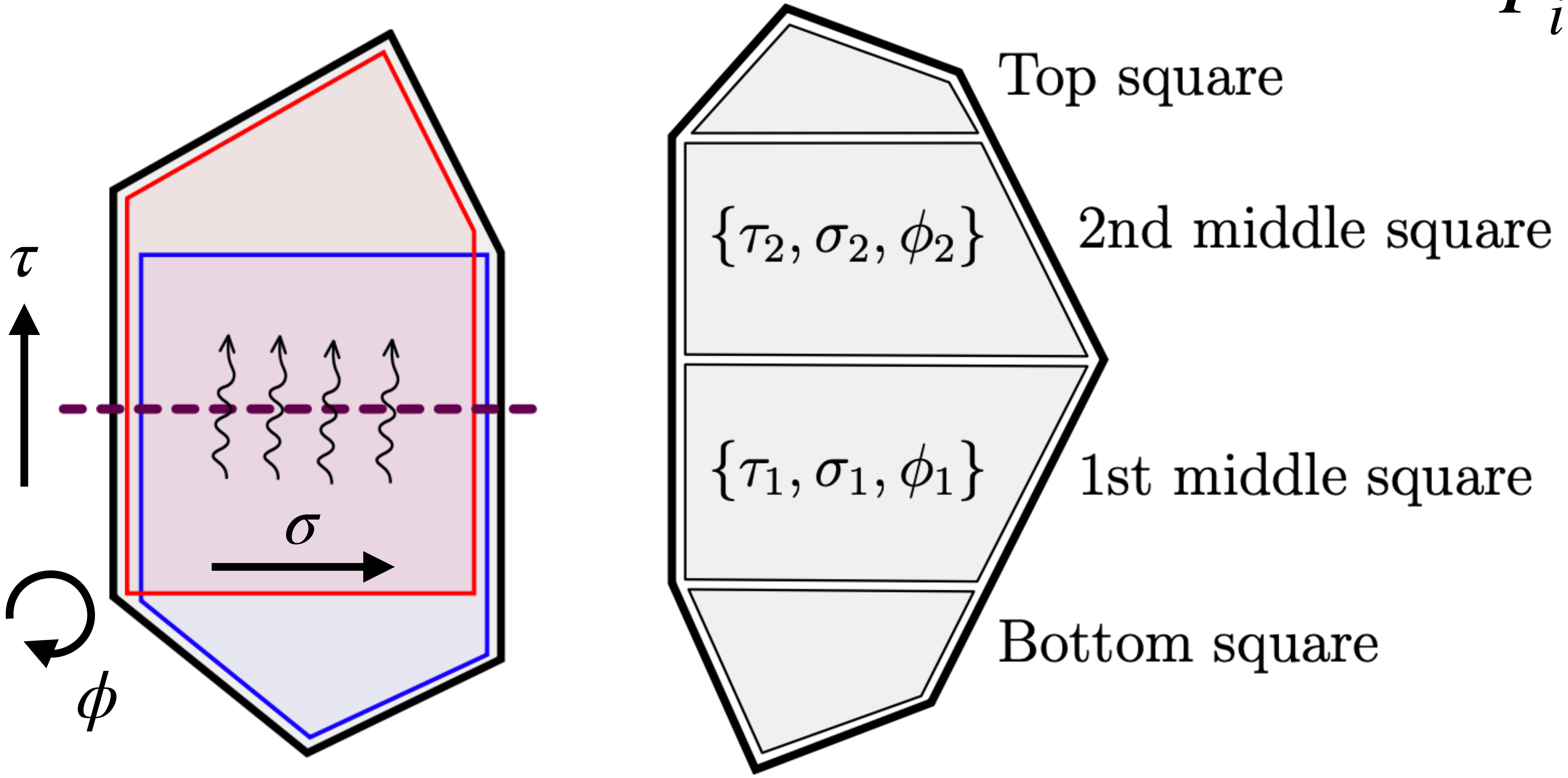

1303.1396, 1306.2058

(2-loop)
Symbol letters are ***rational functions*** of T_i, S_i, F_i

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- Type of function:

L-loop: ***Generalized (Multiple) Polylogarithm*** of weight 2L

[Goncharov, Spradlin, Vergu, Volovich, Golden]
[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]

- Example:

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

$$= -\int_{0<t_2<t_1<x} d\ln(t_1)\ d\ln(1-t_2)$$

weight 2

Iterated $d\log$ integration

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- *Generalized (Multiple) Polylogarithm* F defined by:

$$dF = \sum_{\phi} F^{\phi} \, d \ln \phi$$

- F has weight n if F^{ϕ} have weight $n-1$

- The *symbol* of F is defined by:

$$\mathcal{S}(F) = \sum_{\phi} \mathcal{S}(F^{\phi}) \otimes \phi$$

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e.g. $\mathrm{Li}_n(x) = -\int_{0<t_1<t_2<\ldots<t_n<x} d\ln(1-t_1)\ d\ln(t_2)\ \cdots\ d\ln(t_n)$

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e.g. $$\mathrm{Li}_n(x) = -\int_{0<t_1<t_2<\ldots<t_n<x} d\ln(1-t_1)\ d\ln(t_2)\ \cdots\ d\ln(t_n)$$

$$\mathcal{S}(\mathrm{Li}_n(x)) = -\,(1-x)\otimes x\otimes\cdots\otimes x$$

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First entry:
Discontinuity
(Branch Cuts)

Last entry:
Derivatives

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e.g. $$\mathcal{S}(\ln x \ln y) = x\otimes y + y\otimes x$$

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Properties of the symbol:

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Properties of the symbol:

- Symbols obey identities:

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$$(\ldots \otimes x^n \otimes \ldots) = n \cdot (\ldots \otimes x \otimes \ldots)$$

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$$(\ldots \otimes x^n \otimes \ldots) = n \cdot (\ldots \otimes x \otimes \ldots)$$

- By definition, $\mathcal{S}(\text{const.}) = 0$

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e.g. $$\mathcal{S}(R_n^{(L)}) = \sum_{i_1,\ldots,i_{2L}} c^{i_1,i_2,\ldots,i_{2L}}\, \phi_{i_1} \otimes \phi_{i_2} \otimes \ldots \otimes \phi_{i_{2L}}$$

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symbol letter

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symbol alphabet = $\{\phi_i\}$

symbol letter

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symbol alphabet = $\{\phi_i\}$

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- Generalized polylogarithms have a *Hopf algebra* structure

- ***Antipode*** map S :

$$\phi_{i_1} \otimes \phi_{i_2} \otimes \cdots \otimes \phi_{i_{N-1}} \otimes \phi_{i_N} \mapsto (-1)^N \phi_{i_N} \otimes \phi_{i_{N-1}} \otimes \cdots \otimes \phi_{i_2} \otimes \phi_1$$

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[Goncharov, Spradlin, Vergu, Volovich, Golden]

- For 6-point amplitude, there are **9** symbol letters:

$\{u_1, u_2, u_3\} \cup \{1-u_1, 1-u_2, 1-u_3\} \cup$ {3 parity odd}

- For 7-point amplitude, there are **42** symbol letters:

$\{u_1 \ldots\} \cup \{1-u_1 \ldots\} \cup \{1-u_1u_4 \ldots\} \cup$

$\{1-u_1u_4-u_3u_6 \ldots\} \cup$ {2 x 7 parity odd}

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6-point example

$$T := e^{-\tau}$$

$$S := e^{\sigma}$$

$$F := e^{i\phi}$$

$$u_1 = \frac{F}{F + FS^2 + ST + F^2ST + FT^2},$$

$$u_2 = \frac{FS^2}{\left(1 + T^2\right)\left(F + FS^2 + ST + F^2ST + FT^2\right)},$$

$$u_3 = \frac{T^2}{1 + T^2}, \quad 1 - u_1, \quad 1 - u_2, \quad 1 - u_3, \quad \{\text{parity odd} \times 3\}$$

Parity Flip:

$$F \to \frac{1}{F}$$

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Planar N = 4 SYM, MHV amplitudes

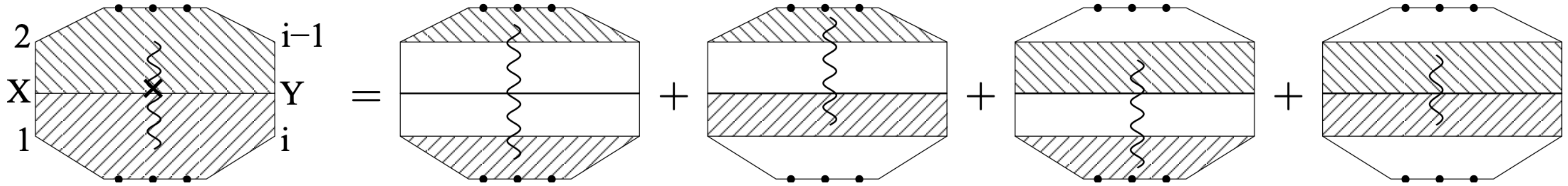
A = [Tree] × [Exponentiated **1-loop**] × Exp(**Remainder**)

Starts at 2 loops

$$\mathcal{S}(R_n^{(2)}) = \sum_{i,j,k,l} r^{ijkl} \phi_i \otimes \phi_j \otimes \phi_k \otimes \phi_l$$

uniformly weight 4

Known for all n [Caron-Huot 1105.5606]



Andy Liu
Stanford University

Antipodal **Symmetry** [AL 2207.11815]

- First, take the parity even part of $R_n^{(2)}$ (i.e. setting $F_i = 1$)
- **Conjecture:** there exists matrix A_n^{ij} such that

$$\mathcal{S}(R_{n,e}^{(2)}) \sim S\left(\mathcal{S}(R_{n,e}^{(2)}) \Big|_{\ln \phi_i \mapsto A_n^{ij} \ln \phi_j} \right)$$

Antipode map $S : \phi_i \otimes \phi_j \otimes \phi_k \otimes \phi_l \mapsto \phi_l \otimes \phi_k \otimes \phi_j \otimes \phi_i$

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$\mathcal{S}(R_{n,e}^{(2)})$ **symmetric** under *letter map* A_n^{ij} plus *antipode map* S

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ϕ_i → **Discontinuity**

ϕ_l → **Derivatives**

Andy Liu
Stanford University

6-point example

$\ln \phi_i \mapsto A_6^{ij} \ln \phi_j :$

$$u_1 \mapsto \frac{u_2 u_3}{(1-u_2)(1-u_3)}, \quad 1-u_1 \mapsto \frac{u_1}{(1-u_2)(1-u_3)}, \quad \ldots$$

Unique if require dihedral symmetry

A_n^{ij} explicitly found for $n = 6,7,8$

Andy Liu
Stanford University

6-point example

$$u_1 = \frac{1}{1+S^2+2ST+T^2},$$

$$u_2 = \frac{S^2}{\left(1+T^2\right)\left(1+S^2+2ST+T^2\right)},$$

$$u_3 = \frac{T^2}{1+T^2}, \quad 1-u_1,\ 1-u_2,\ 1-u_3,$$ {parity odd = 1}

Parity even:

$$F = 1$$

Equivalent alphabet: $\{S,\ T,\ S+T,\ 1+T^2,\ 1+ST+T^2,\ 1+S^2+2ST+T^2\}$

Andy Liu
Stanford University

6-point example

$\ln \phi_i \mapsto A_6^{ij} \ln \phi_j$: (equivalently)

$$S \mapsto \frac{1}{ST}, \quad T \mapsto \frac{T}{S},$$

$$S + T \mapsto \frac{1+T^2}{ST}, \quad 1 + T^2 \mapsto \frac{(S+T)^2}{S^2},$$

$$1 + ST + T^2 \mapsto \frac{1 + S^2 + 2ST + T^2}{S^2}, \quad 1 + S^2 + 2ST + T^2 \mapsto \frac{\left(1 + ST + T^2\right)^2}{S^2T^2}$$

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6-point example

$\ln \phi_i \mapsto A_6^{ij} \ln \phi_j$: (equivalently)

A_n^{ij} compatible with *tropical geometry* of symbol letters

$$S \mapsto \frac{1}{ST}, \quad T \mapsto \frac{T}{S},$$

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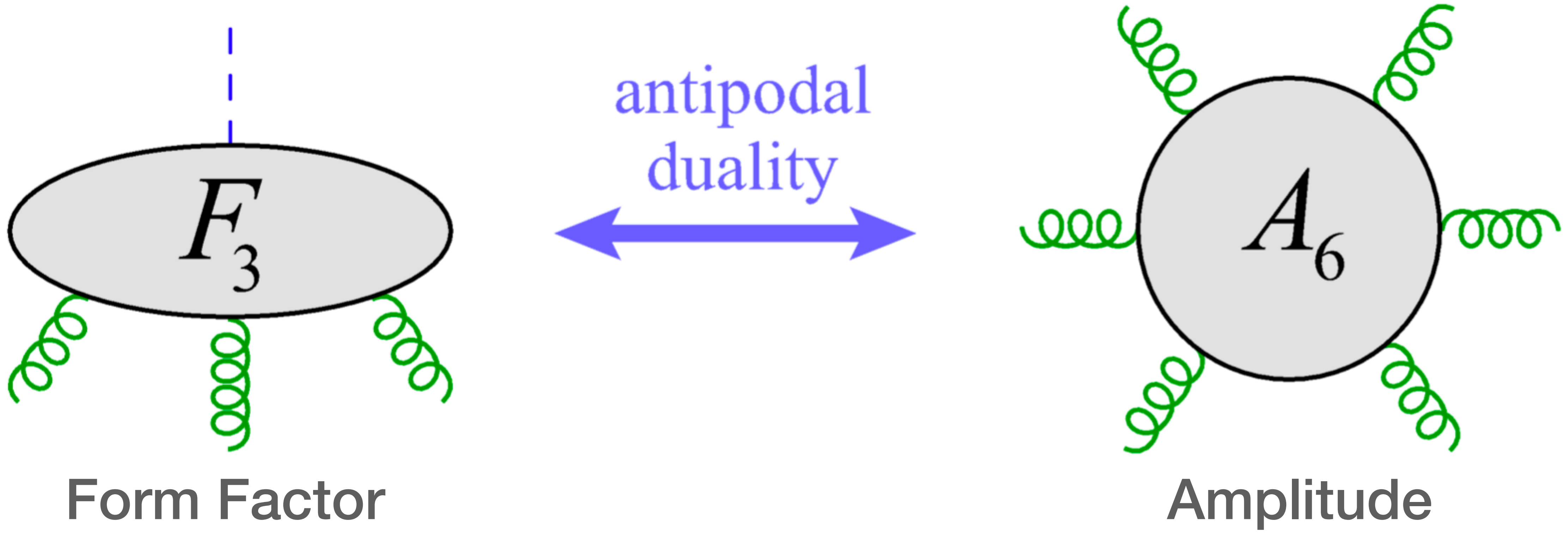
Antipodal <u>Duality</u>

[Dixon, Gürdogan, McLeod, Wilhelm 2112.06243]

$$\left\langle ggg \,\middle|\, \mathrm{Tr}\left(F_{SD}^2\right) \middle|\, 0 \right\rangle =$$



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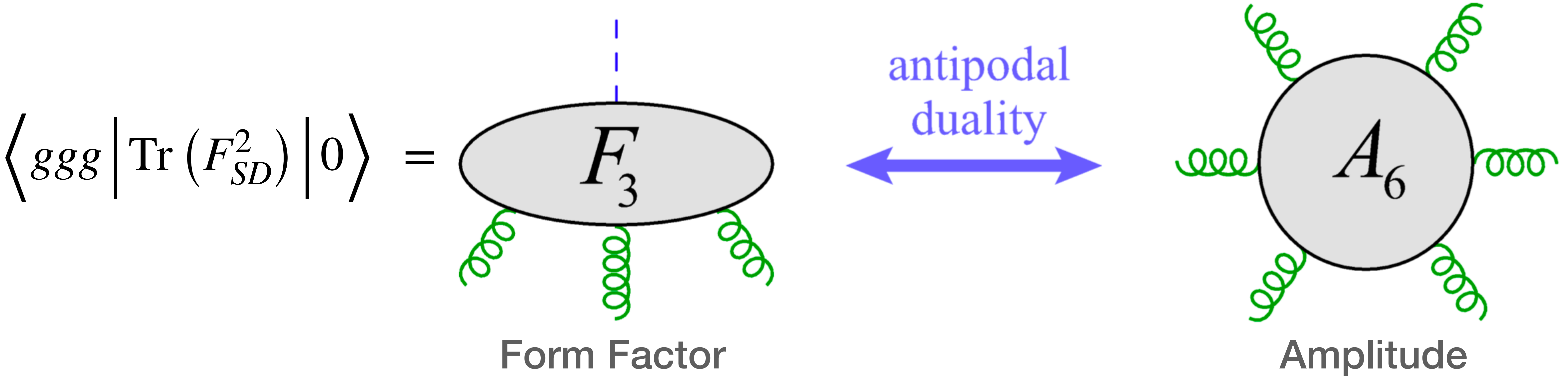
Antipodal Duality

[Dixon, Gürdogan, McLeod, Wilhelm 2112.06243]



- Holds for the parity even part of $R_6^{(2)}$ (i.e. setting $\hat{F} = 1$)

- **Conjecture:** there exists map $\hat{T}(T,S), \hat{S}(T,S)$ such that

$$\mathcal{S}(R_{3,FF}^{(L)})\Big|_{T,S} = S\left(\mathcal{S}(R_{6,e}^{(L)})\Big|_{\hat{T}(T,S),\hat{S}(T,S)} \right)$$

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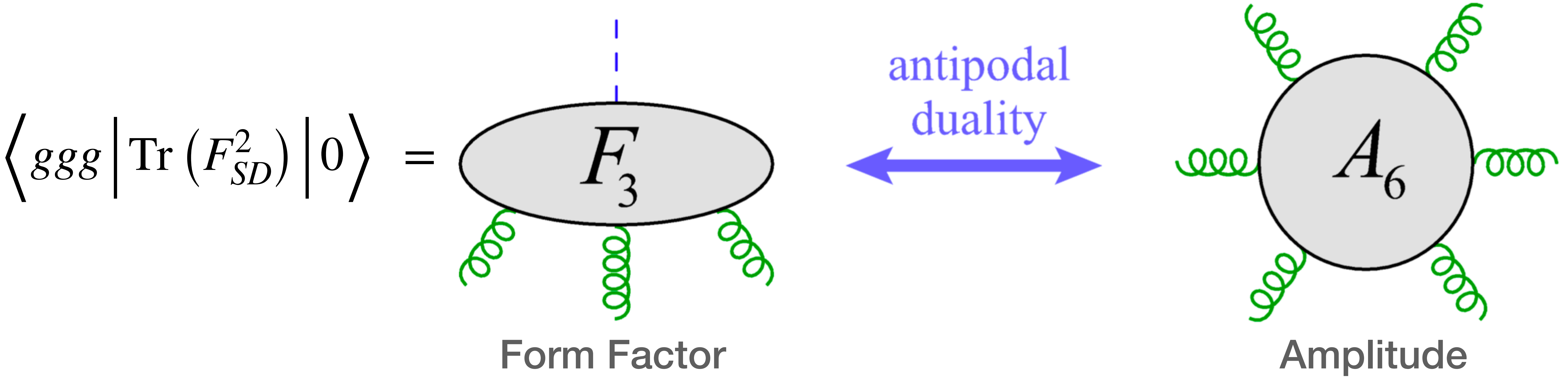
Antipodal <u>Duality</u>

[Dixon, Gürdogan, McLeod, Wilhelm 2112.06243]



$R_{6,e}^{(L)} \leftrightarrow R_{3,FF}^{(L)}$

under *antipode map* S

plus *kinematic map*

$T, S \leftrightarrow \hat{T}, \hat{S}$

- Holds for the parity even part of $R_6^{(2)}$ (i.e. setting $\hat{F} = 1$)

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Stanford University

[AL 2207.11815]

all # of particles n

- Antipodal **Symmetry**: $R_{n,e}^{(2)} \leftrightarrow R_{n,e}^{(2)}$

$$u_1 \mapsto \frac{u_2 u_3}{(1-u_2)(1-u_3)}, \quad 1-u_1 \mapsto \frac{u_1}{(1-u_2)(1-u_3)}, \quad \cdots$$

letter map

all # of loops L

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kinematic map

[Dixon, Gürdogan, McLeod, Wilhelm 2112.06243]

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Stanford University

[AL 2207.11815]

all # of particles n

- Antipodal **Symmetry**: $R^{(2)}_{n,e} \leftrightarrow R^{(2)}_{n,e}$

letter map

$$u_1 \mapsto \frac{u_2 u_3}{(1-u_2)(1-u_3)}, \quad 1-u_1 \mapsto \frac{u_1}{(1-u_2)(1-u_3)}, \quad \ldots$$

$n = 6$

all # of loops L

- Antipodal **Duality**: $R^{(L)}_{6,e} \leftrightarrow R^{(L)}_{3,FF}$

kinematic map

$$\hat{u}_1 \mapsto \frac{u_2 u_3}{(1-u_2)(1-u_3)}, \quad 1-\hat{u}_1 \mapsto \frac{u_1}{(1-u_2)(1-u_3)}, \quad \ldots$$

[Dixon, Gürdogan, McLeod, Wilhelm 2112.06243]

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# legs / # loops	n = 6	n = 7	n = 8	......
L = 2		Antipodal **Symmetry**		
L = 3	Antipodal **Duality**			
L = 4				
......				

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# legs / # loops	n = 6	n = 7	n = 8	......
L = 2	***Relation?***	Antipodal **Symmetry**		
L = 3	Antipodal **Duality**			
L = 4				
......				

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# legs / # loops	n = 6	n = 7	n = 8	......
L = 2	***Relation?***	Antipodal **Symmetry**		
L = 3	Antipodal **Duality**	***???***		
L = 4		***Generalization?***		
......		***???***		

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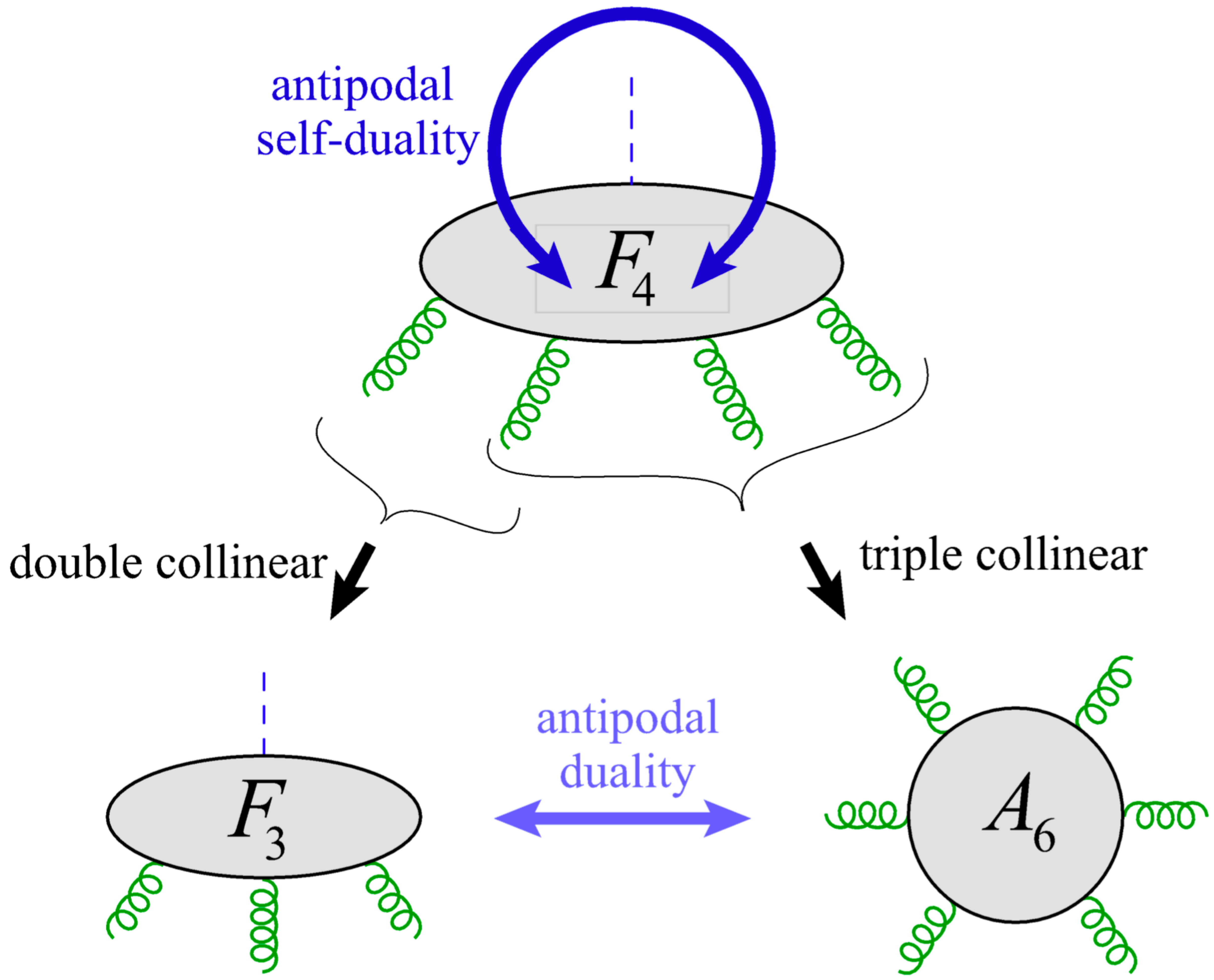
[Dixon, Gürdogan, AL, McLeod, Wilhelm 2212.02410]

Actually, the *duality*

$$R_{6,e}^{(L)} \leftrightarrow R_{3,FF}^{(L)}$$

is implied by a *self-duality*

$$R_{4,FF}^{(L)} \leftrightarrow R_{4,FF}^{(L)}$$



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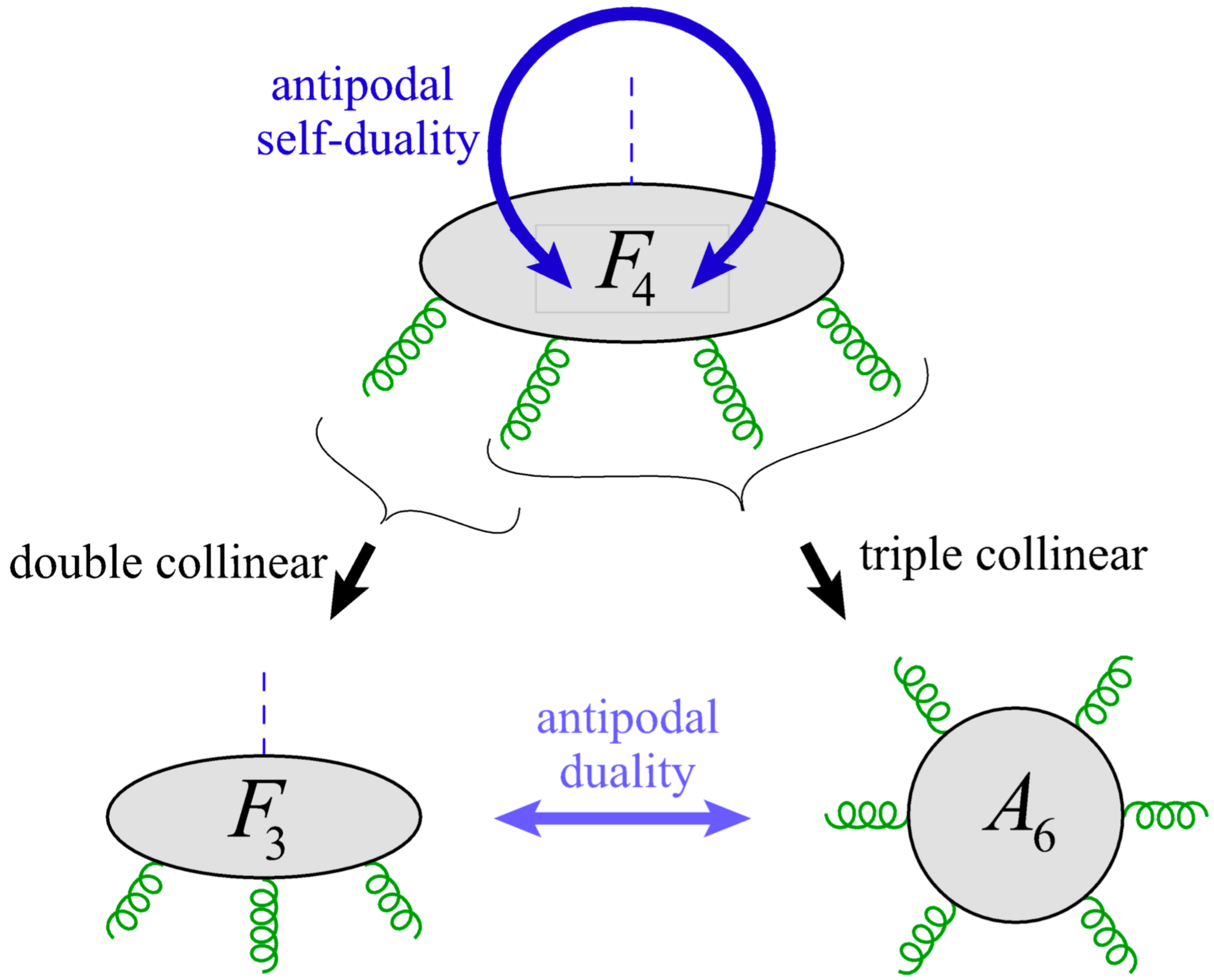
2 loops



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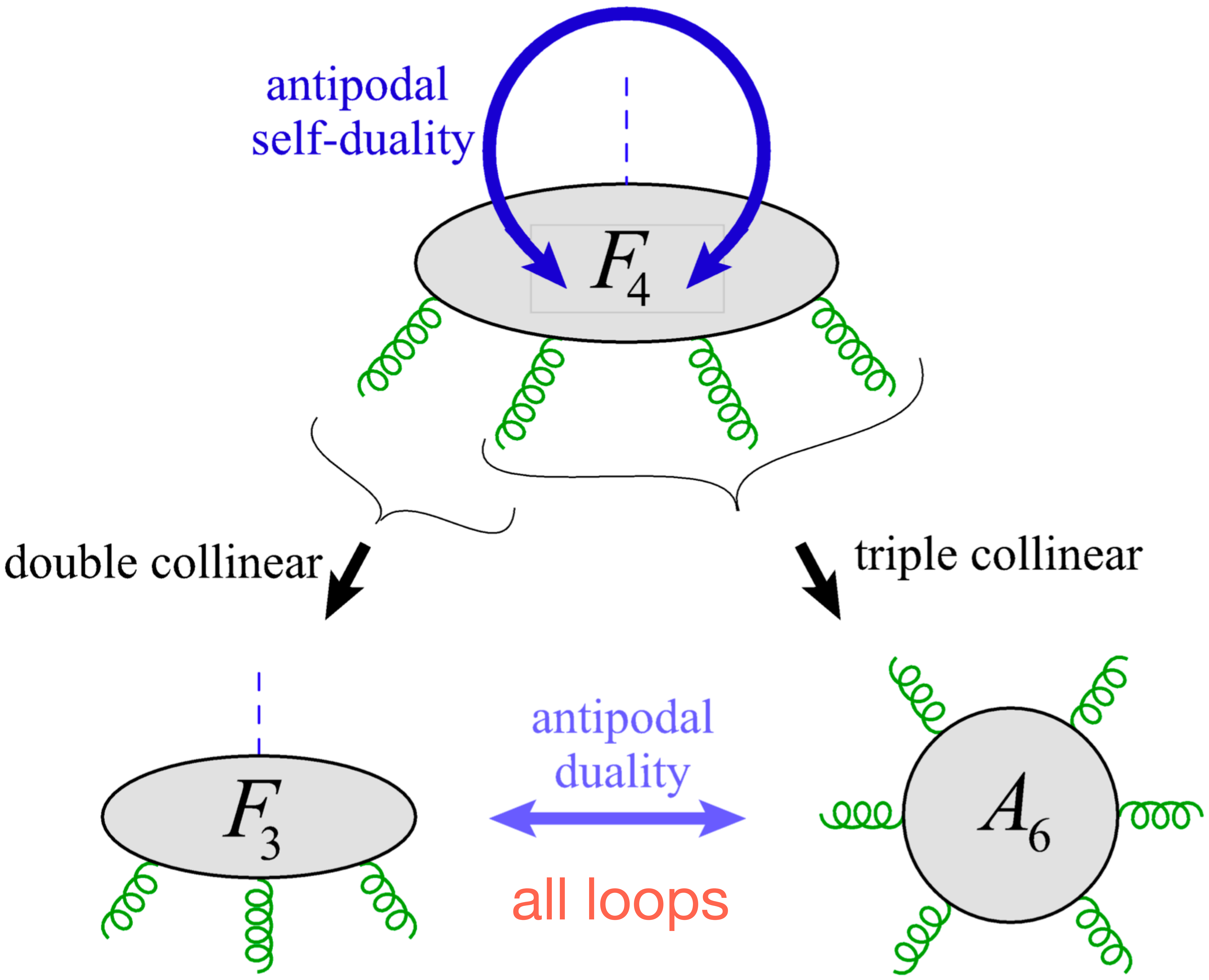

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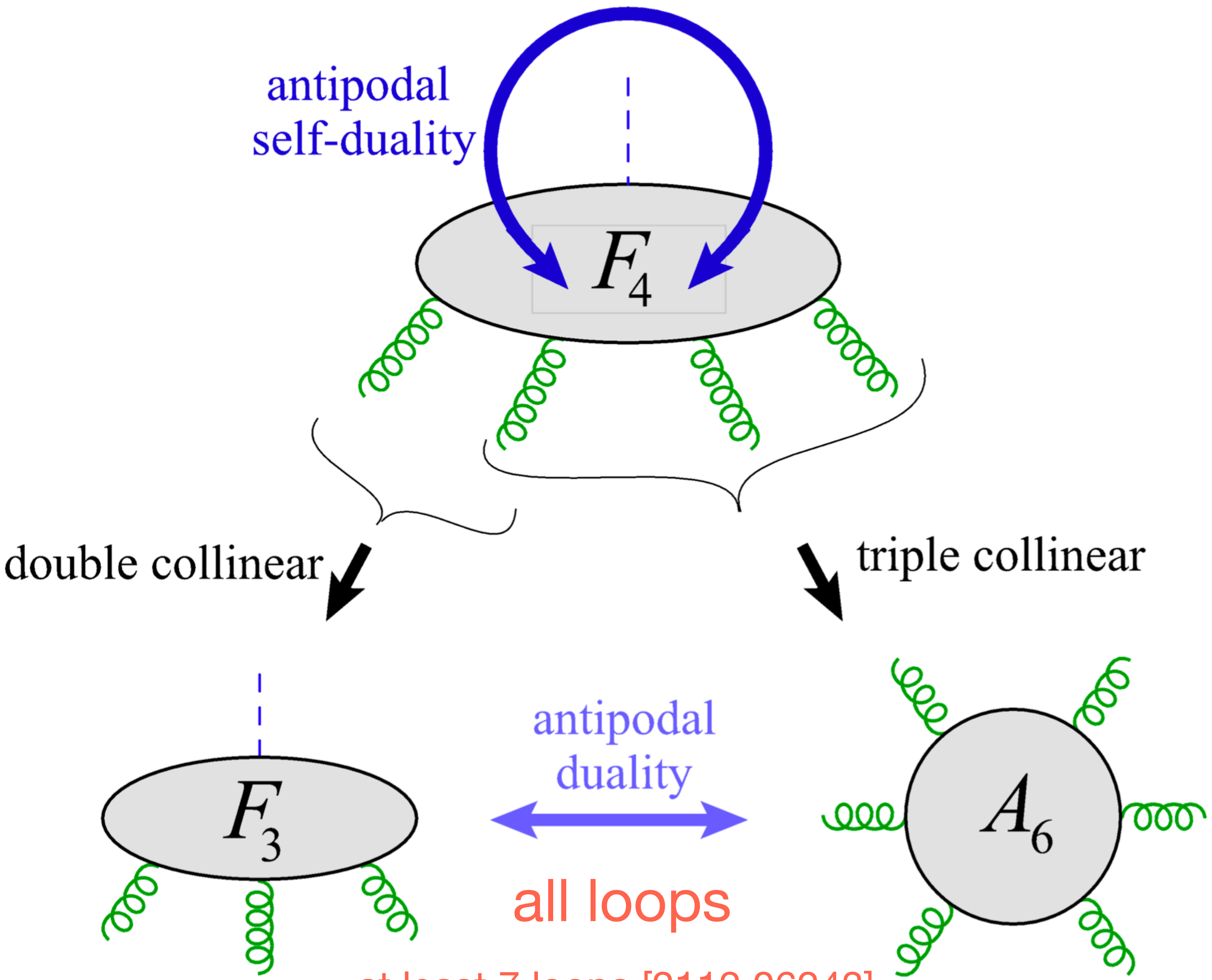

at least 7 loops [2112.06243]
now also 8 loops [Dixon, AL, to appear]

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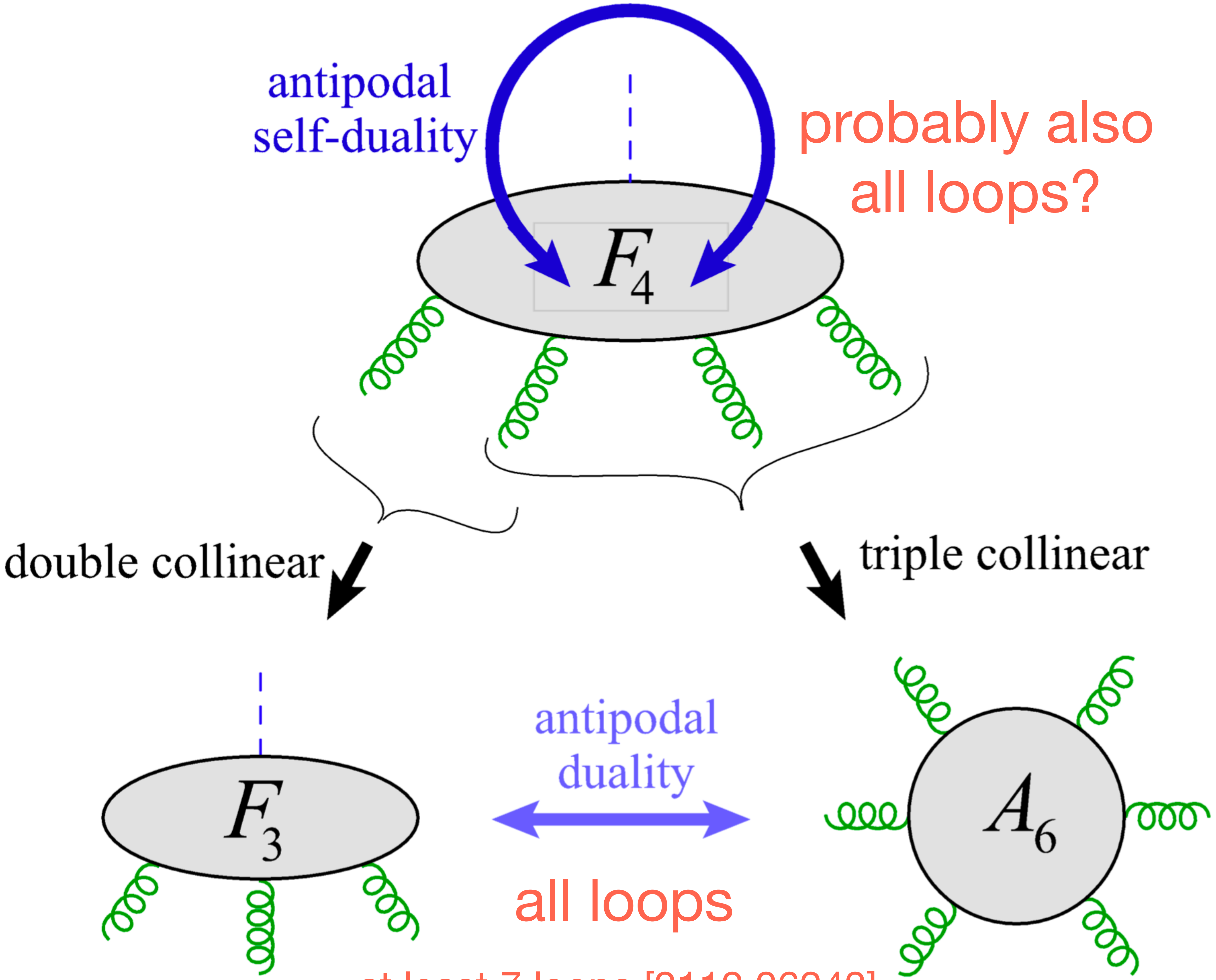
[Dixon, Gürdogan, AL, McLeod, Wilhelm 2212.02410]



at least 7 loops [2112.06243]
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Conclusions

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Conclusions

New symmetry

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Conclusions

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New duality $\Longleftarrow$ self-duality

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Physical origin?

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Antipodal symmetry beyond 2 loops?

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Antipodal symmetry beyond 2 loops?

Antipodal duality beyond 6-pt $\leftrightarrow$ 3-pt?

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Planar N = 4 SYM continues to surprise!

Backup slides

6-point example (Antipodal Symmetry)

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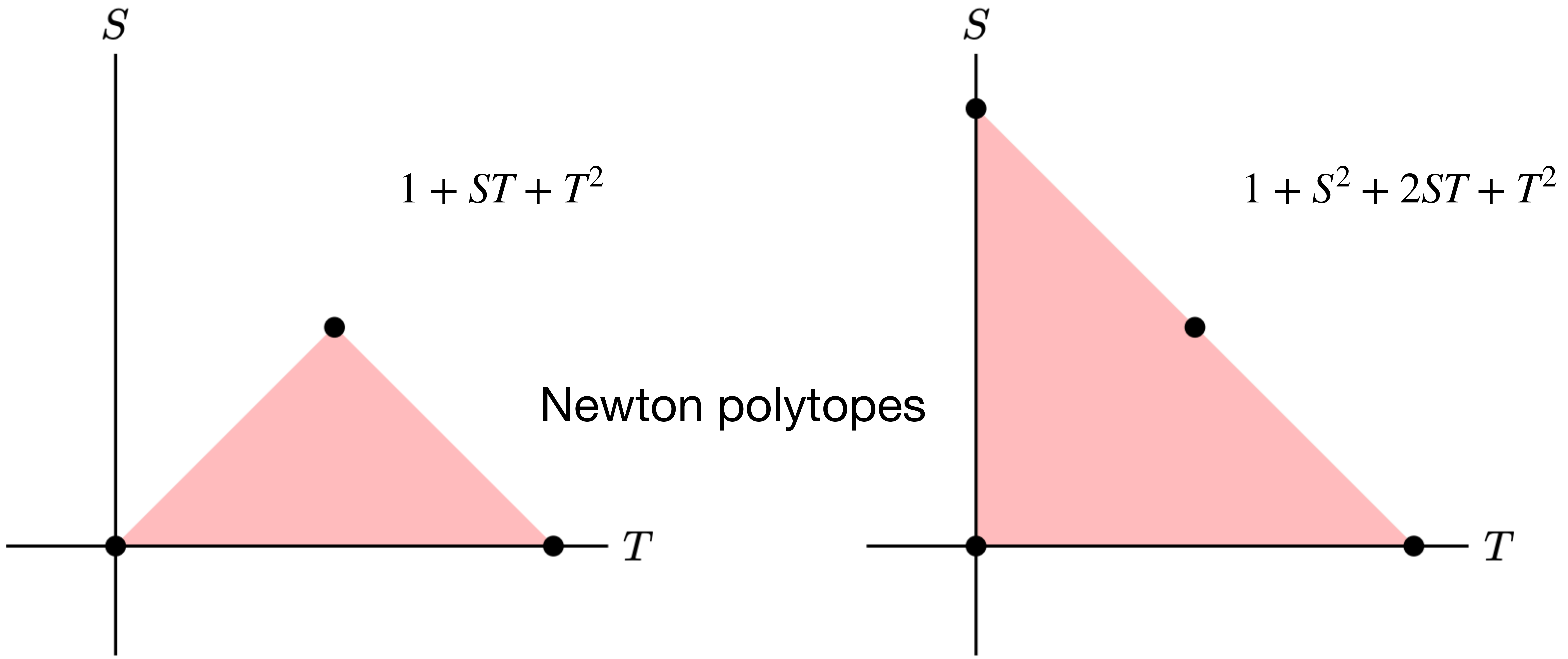

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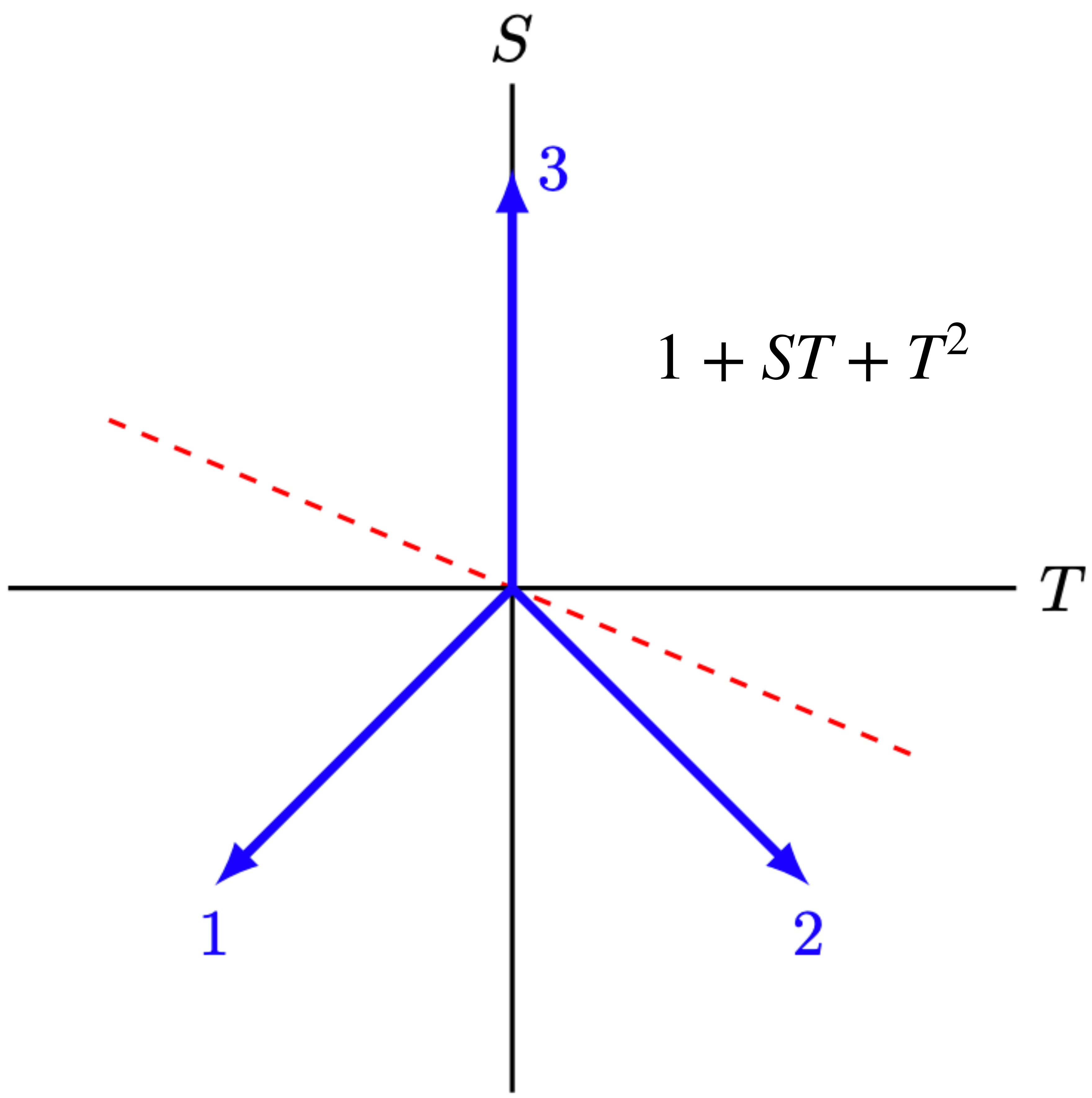
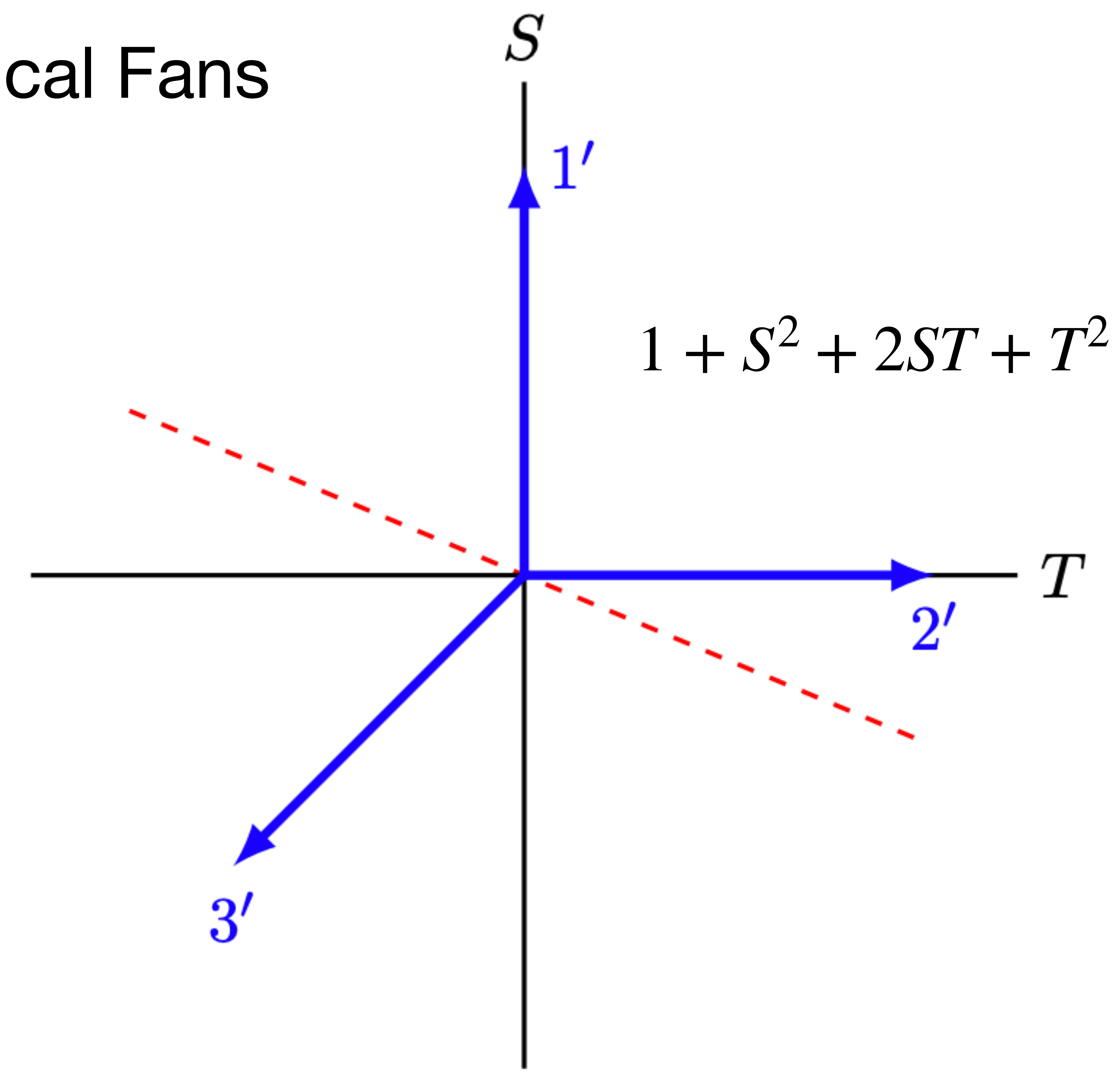
Tropical Fans





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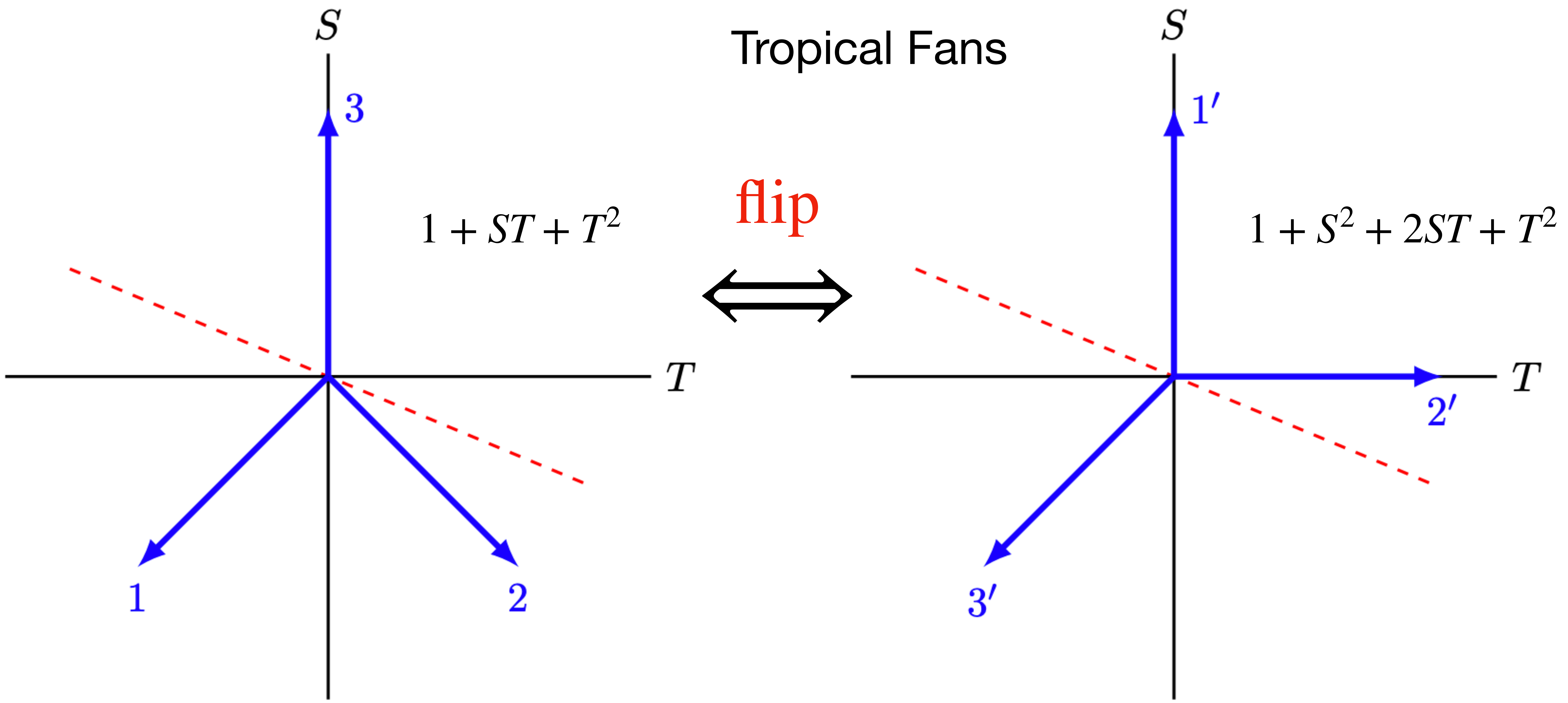

A_n^{ij} generates ***isomorphisms*** on ***tropical fans*** of letters

A_6^{ij} : flip

A_7^{ij} : $\mathrm{flip}_1 \otimes \mathrm{flip}_2$

A_8^{ij} : $\mathrm{flip}_1 \otimes \mathrm{flip}_2 \otimes \mathrm{flip}_3$

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Tropical structure *of amplitudes?*

1912.08222 Arkani-Hamed, Lam, Spradlin
1912.08217 Drummond, Foster, Gürdogan, Kalousios
1912.08254 Henke, Papathanasiou

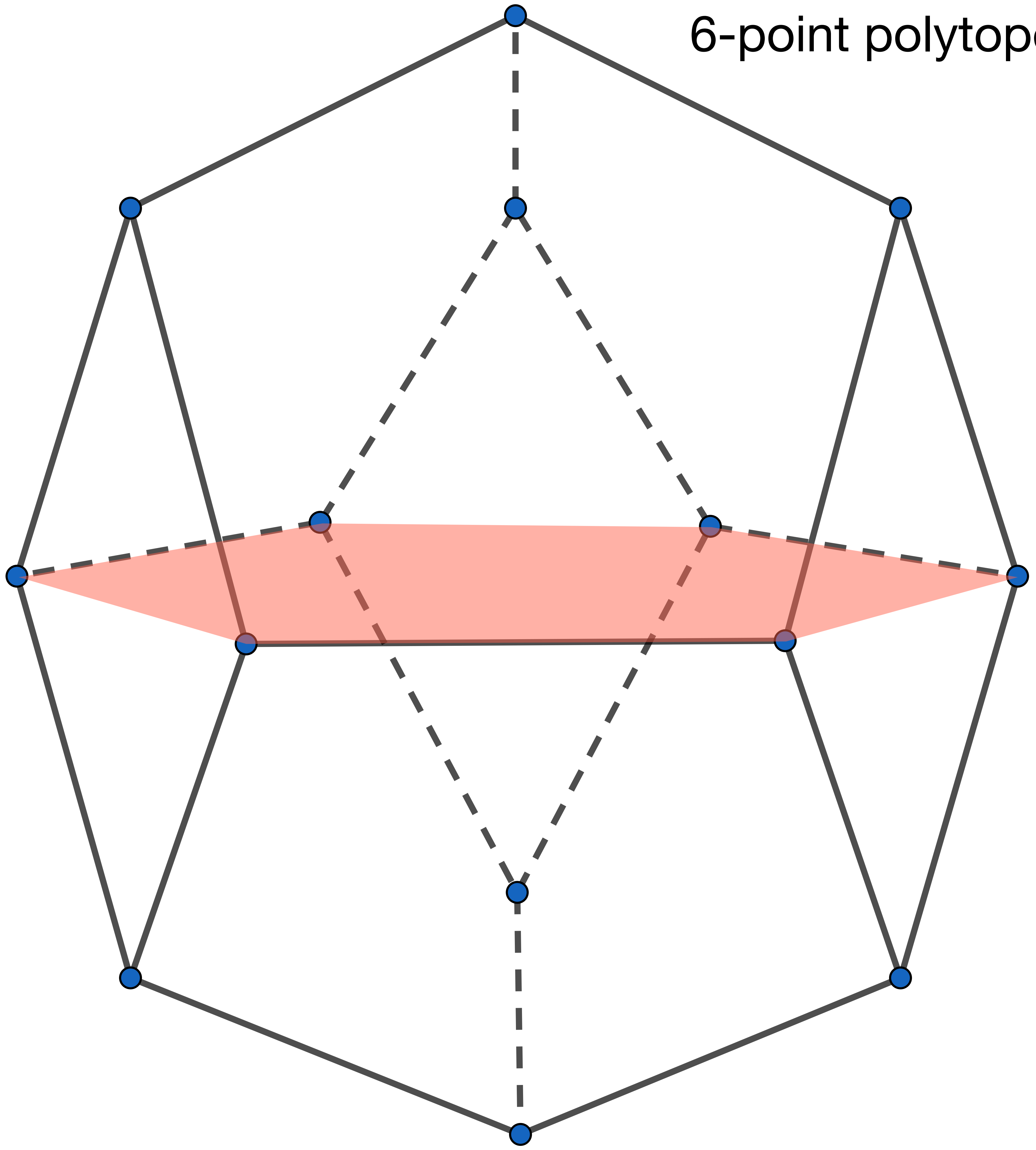
***<u>Tropical structure</u>* of amplitudes?**

Amplitude ⟶ *Polytope*

Encodes information:

e.g. facet ⇔ symbol letter

e.g. adjacency



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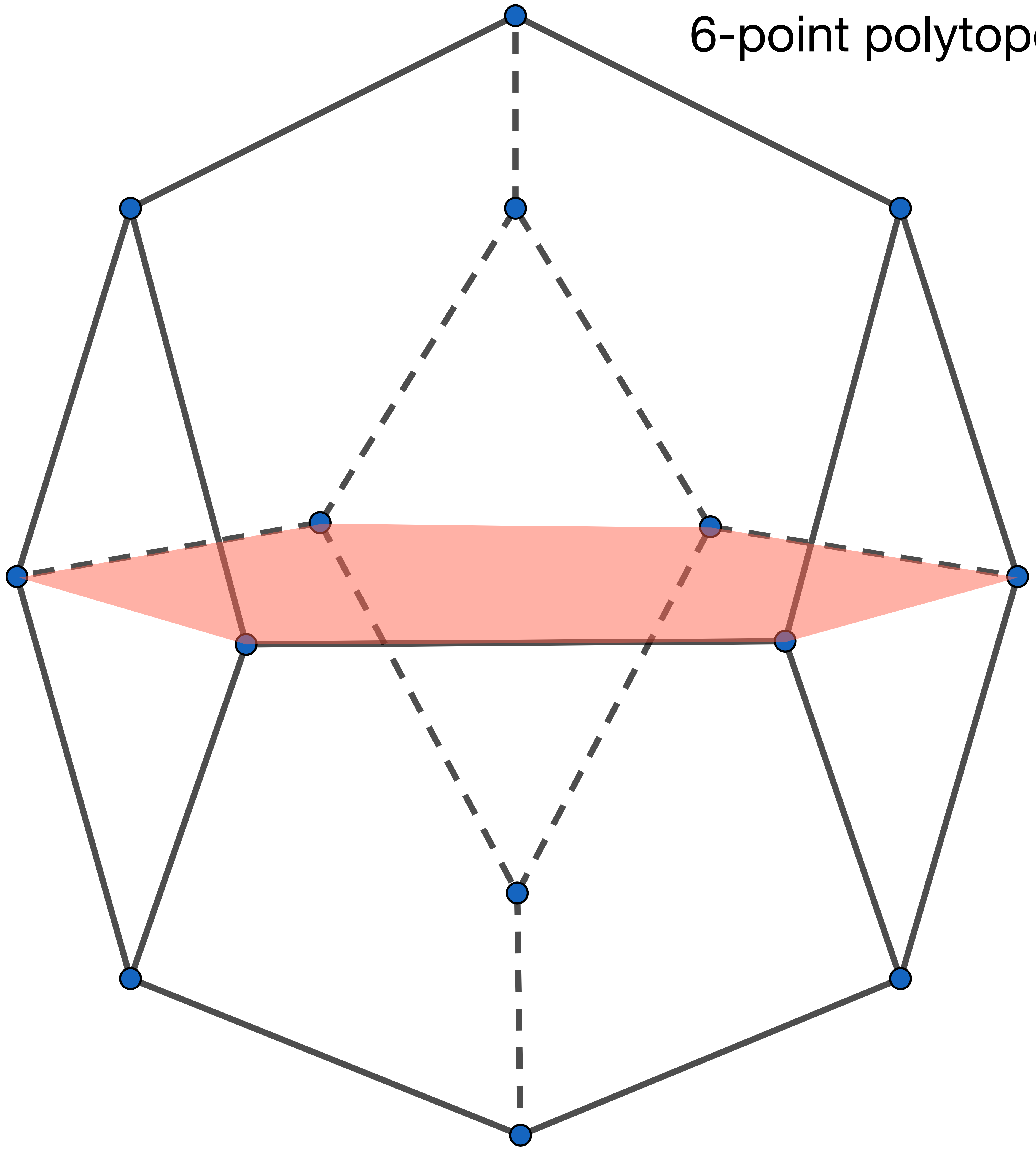
But: very coarse-grained

*(**sum** of geometries of individual letters)*



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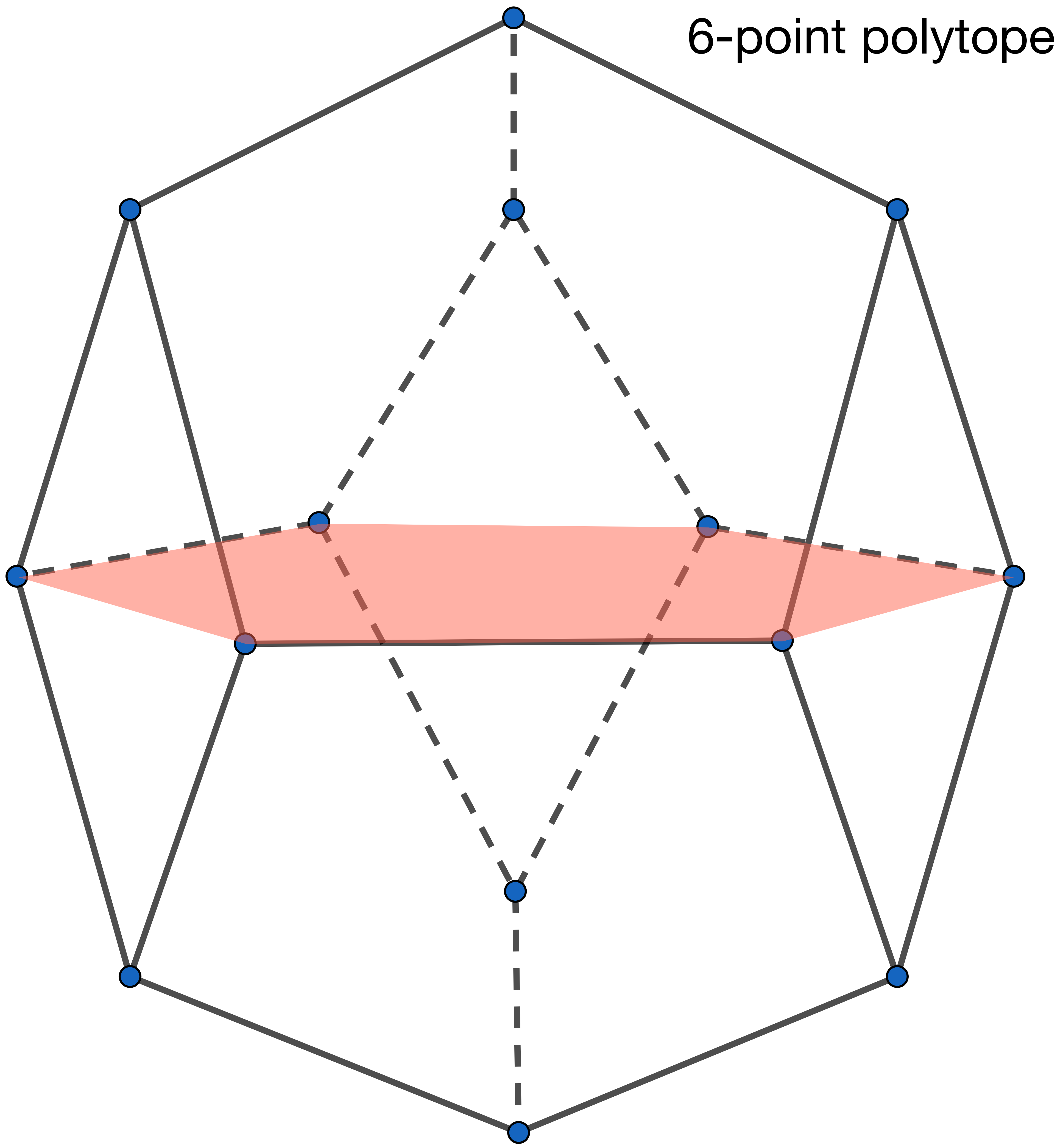
Think: Manifold ⟶ Topological Invariant



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Tropical structure *of amplitudes?*

Amplitude $\longrightarrow$ *Polytope*

Encodes information:

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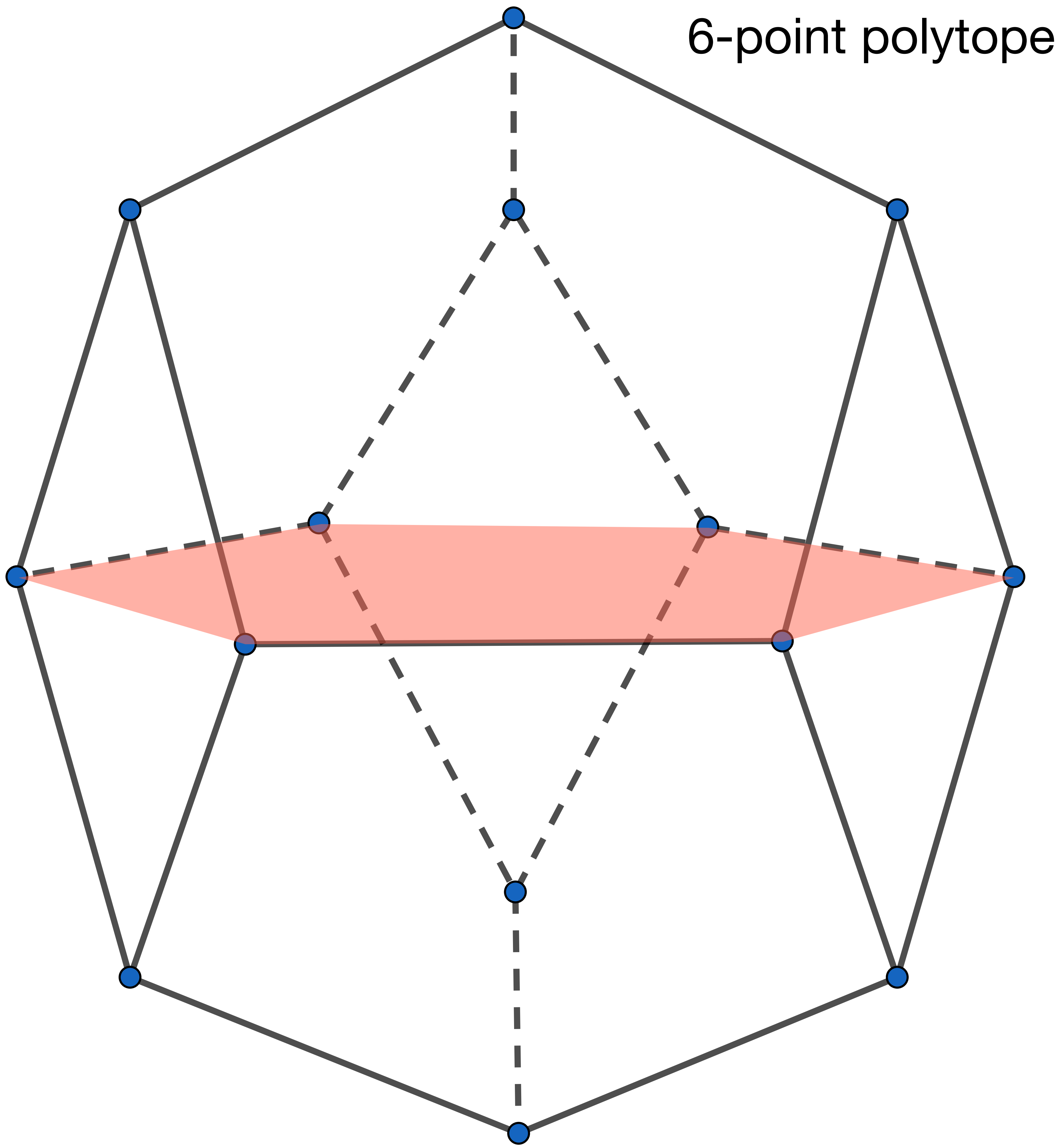
Think: Manifold $\longrightarrow$ *Topological Invariant*



Geometry of **individual** letter has meaning!

(preserved by antipodal letter map) $\Longrightarrow$ Suggests a more ***refined*** tropical description?