Perturbative quasinormal mode frequencies

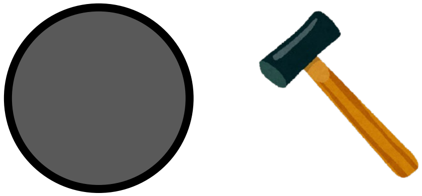
Masashi Kimura
(Daiichi Inst. Tech.)

2023 NCTS Annual Theory Meeting: Particle Physics, String Theory, Cosmology, and Astrophysics
15th Dec 2023@NCTS

Let us consider linear gravitational perturbation around black holes(BHs)
There exist specific modes:
Quasi Normal Modes (QNMs)

Similar to specific sounds for a drum

If we “beat” BHs, and “hear” QNMs, then we can obtain information of BHs



Introduction: master eq

In general, it is difficult to study linear perturbation of BHs $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon h_{\mu\nu}$

Linear gravitational perturbation on a highly symmetric BH usually reduces to a single master equation

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V(x) \right] \tilde{\Phi} = 0$$

$x \to \infty$: spatial infinity

$x \to -\infty$: horizon

Introduction: master eq

e.g.) Schwarzschild case (odd mode):

$$\left[-\frac{\partial^2}{\partial t^2} + \underline{f\frac{\partial}{\partial r}\left(f\frac{\partial}{\partial r}\right)} - V\right]\tilde{\Phi} = 0 \qquad (f = 1 - 2M/r)$$

$$= \frac{\partial^2}{\partial x^2} \qquad \left(f\frac{\partial}{\partial r} = \frac{\partial}{\partial x} \right)$$

$$V = f\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right)$$

$$\left[\begin{array}{l} x = r + 2M\ln\left(\frac{r-2M}{2M}\right) \\ x \to \infty \quad (r \to \infty) \\ x \to -\infty \quad (r = 2M) \end{array}\right]$$

$h_{\mu\nu}$ is written as

$$h_0 = f\partial_r(r\tilde{\Phi}) \qquad h_1 = rf^{-1}\partial_t\tilde{\Phi}$$

$$h_{\mu\nu}dx^\mu dx^\nu = 2\sin\theta\partial_\theta Y_{\ell 0}d\phi\left[h_0 dt + h_1 dr\right]$$

Introduction: master eq

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-V(x)\right]\tilde{\Phi}=0$$

$$\tilde{\Phi}(t,x)=e^{-i\omega t}\Phi(x)$$

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi=\omega^2\Phi$$

same form as
1dim (time-independent) Schrödinger eq

Introduction: QNM

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi=\omega^2\Phi$$

Quasinormal mode (QNM) :

solution of the master eq

s.t. purely outgoing at $x\to\infty$ $\quad\Phi\to e^{i\omega x}$

purely ingoing at $x\to-\infty$ $\quad\Phi\to e^{-i\omega x}$

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-V(x)\right]\tilde\Phi=0 \qquad \tilde\Phi(t,x)=e^{-i\omega t}\Phi(x)$$

$$\tilde\Phi \overset{x\to\infty}{\simeq} Ae^{-i\omega(t-x)}+\cancel{Be^{-i\omega(t+x)}}$$

$$\tilde\Phi \overset{x\to-\infty}{\simeq} \cancel{Ce^{-i\omega(t-x)}}+De^{-i\omega(t+x)}$$

Introduction: QNM

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi=\omega^2\Phi$$

Quasinormal mode (QNM) :

solution of the master eq

s.t. purely outgoing at $x\to\infty$ $\Phi\to e^{i\omega x}$

purely ingoing at $x\to-\infty$ $\Phi\to e^{-i\omega x}$

Find a sol with above boundary condition

This is an eigenvalue problem

Similar to quantum mechanics,

only special ω admit this type of solution

Introduction: QNM

e.g.) Schwarzschild case

($\ell = 2$ fundamental mode)

$$\omega_{\mathrm{QNM}} = \frac{0.37365... - i0.0885...}{M}$$

- ω is complex
- damped oscillation ($\tilde{\Phi}(t,x) = e^{-i\omega t}\Phi(x)$)
- Infinite number of QNM ω exist

Introduction: QNM

How to calculate QNM ω?

We need to solve the eigen value problem for the master eq

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi=\omega^2\Phi$$

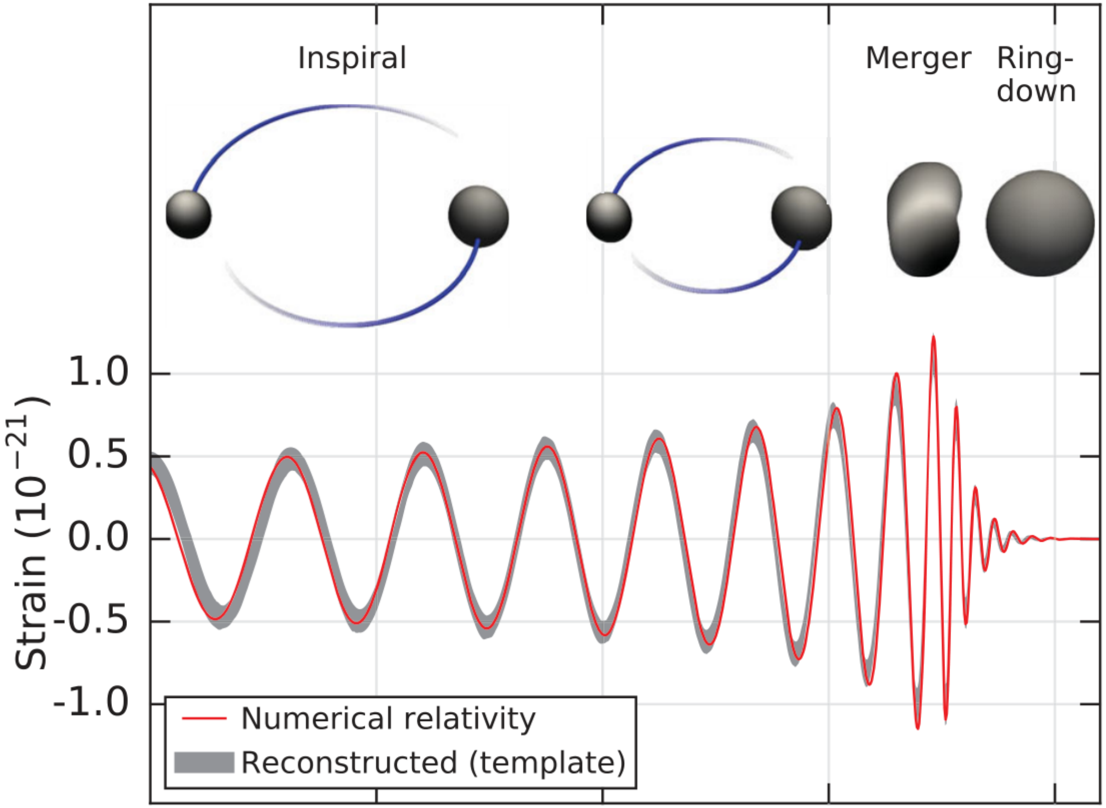
We can calculate QNM ω numerically

- Leaver's method
- Direct integration
- WKB approximation

etc…

Introduction: QNM

GWs emitted from the last stage of binary BH coalescence is well approximated by QNMs



[PRL **116**, 061102 (2016)]

Introduction: motivation

In GR, QNM ω is determined only from the mass and spin of the final state BH

checking ↑ = test for GR

We want to test GR by (future) observation

We also want to put a constraint on gravity theories

Contents

QNM for non-GR

We want to discuss QNMs for non-GR gravity theories

usual approach:

fix a specific gravity theory

→ find a black hole solution

→ derive master eq and find QNMs

We want to discuss QNMs for many gravity theories systematically

QNM for non-GR

We assume that the theory is almost GR, i.e., (vacuum) GR + small correction

We focus on spherically sym (static) case

We expect that the master equation is RW or Zerilli eq + small correction

$$\left[-\frac{d^2}{dx^2}+V_{\mathrm{GR}}+\delta V\right]\Phi=\omega^2\Phi$$

$$\omega_{\mathrm{QNM}}=\omega^{(\mathrm{GR})}_{\mathrm{QNM}}+\delta\omega$$

Parametrized potential

Schwarzschild case + correction term

$$f\frac{d}{dr}\left(f\frac{d\Phi}{dr}\right)+[\omega^2-V]\Phi=0 \qquad f=1-\frac{r_H}{r}$$

$$V=V_\pm+\delta V_\pm$$

$$\left(\begin{array}{c} V_-=f\left(\frac{\ell(\ell+1)}{r^2}-\frac{3r_H}{r^3}\right) \quad V_+=f\frac{9\lambda r_H^2 r+3\lambda^2 r_H r^2+\lambda^2(\lambda+2)r^3+9r_H^3}{r^3(\lambda r+3r_H)^2} \\ \lambda=\ell^2+\ell-2 \end{array}\right)$$

$$\delta V_\pm=\frac{f}{r_H^2}\sum_{j=0}^{\infty}\alpha_j^\pm\left(\frac{r_H}{r}\right)^j$$: our assumption

$\alpha_j^\pm$: small parameters

Parameterized QNM formalism

[V. Cardoso, A. Maselli, MK, E. Berti, C. F. B. Macedo, R. McManus, 2019]

Expanding ω as a series of $\alpha_j^{\pm}$

$$\omega_{\mathrm{QNM}}^{\pm} = \omega_{\mathrm{GR}}^{\pm} + \sum_{j=0}^{\infty} \alpha_j^{\pm} \underline{e_j^{\pm}} + \sum_{j,k=0}^{\infty} \alpha_j^{\pm} \alpha_k^{\pm} \underline{e_{j,k}^{\pm}} + \mathcal{O}(\alpha^3)$$

$e_j^{\pm}$, $e_{j,k}^{\pm}$: model independent coefficients
($\alpha_j^{\pm}$ independent)

We can calculate $e_j^{\pm}$, $e_{j,k}^{\pm}$ numerically

Result: e_j^-

j	$r_H\, e_j^-$
0	0.24725+0.092643i
1	0.15985+0.018208i
2	0.096632-0.0024155i
3	0.058491-0.0037179i
4	0.036679-0.00043870i
10	0.0036853+0.0065244i

for $\ell = 2$ fundamental mode

Application 1 (non-GR)

EFT extension of GR (kind of modified gravity)

(V. Cardoso, M. Kimura, A. Maselli, and L. Senatore, 2018)

$$\frac{d^2\Psi_-}{dr_*^2} + \left[\omega^2 - f(V_- + \delta V_-)\right]\Psi_- = 0$$

$$\delta V_- = \frac{\overbrace{\boxed{18(\ell+2)(\ell+1)(\ell-1)\epsilon_2}}^{\alpha_{10}^-} r_H^8}{r^{10}}$$

$$\omega_{\mathrm{QNM}} = \omega_0 + \alpha_{10}^- e_{10}^- \qquad \left(\begin{array}{l}\text{for } \ell = 2 \\ e_{10}^- = 0.0036853 + i0.0065244\end{array}\right)$$

$$= \omega_0 + \frac{(0.796025 + 1.40927i)\epsilon_2}{r_H} \quad (\ell = 2)$$

Application 2: GW QNM for slow rot Kerr

[MK and Y.Hatsuda 2020]

Wave eq for Kerr case can be written in the Schrödinger form for small spin $\boldsymbol{a}$

$$
\begin{aligned}
M\omega^{\mathrm{QNM}}_{[\mathrm{2nd}]} = & \,(0.3736716844180418 - 0.0889623156889357i) \\
& + (0.0628830795083 + 0.0009979348536i)\frac{ma}{M} \\
& + \Big((0.03591312868219 + 0.00638178925048i) \\
& + (0.00895679029 - 0.00029122129i)m^2\Big)\left(\frac{a}{M}\right)^2 + \mathcal{O}(a^3),
\end{aligned}
$$

This agrees very well with numerical results for small spin parameter $\boldsymbol{a}$

QNM for non-GR

Our formalism is a tool for calculating QNMs if a specific model is fixed

We also want to put constraint on $\alpha_j^{\pm}$ from observational data

$$\omega_{\mathrm{QNM}}^{\pm} = \omega_{\mathrm{GR}}^{\pm} + \sum_{j=0}^{\infty} \alpha_j^{\pm} e_j^{\pm} + \sum_{j,k=0}^{\infty} \alpha_j^{\pm} \alpha_k^{\pm} e_{j,k}^{\pm} + \mathcal{O}(\alpha^3)$$

[cf Völkel, Franchini, Barausse, Berti arXiv:2209.10564]

Constraints on master equations

Recursion relations for $e_j^{\pm}$

I found recursion relations
among $e_j^{\pm}$ with different $j\ (\geq 0)$

[MK, PRD **101**, 064031 (2020)]

For e_j^- :

$$-4(j-1)r_H^2(\omega_0^-)^2 e_j^- - j(-4\ell(\ell+1)+(j-1)(j+1))e_{j+2}^-$$

$$+(2j+1)(-6-2\ell(\ell+1)+(j-1)(j+2))e_{j+3}^- - (j-3)(j+1)(j+5)e_{j+4}^- = 0$$

(Derivation will be shown later)

We can calculate e_j^- only from $e_0^-, e_1^-, e_2^-, e_7^-$

We can calculate e_j^- only from $e_0^-, e_1^-, e_2^-, e_7^-$

$$e_3^- = \frac{2(\ell-1)\ell(\ell+1)(\ell+2)(\ell^2+\ell-1)r_H^2(\omega_0^-)^2}{(\ell-1)^2\ell^2(\ell+1)^2(\ell+2)^2+36r_H^2(\omega_0^-)^2}e_0^- - \frac{12(\ell^2+\ell-2)r_H^2(\omega_0^-)^2}{(\ell-1)^2\ell^2(\ell+1)^2(\ell+2)^2+36r_H^2(\omega_0^-)^2}e_2^-,$$

$$e_4^- = -\frac{4}{15}r_H^2(\omega_0^-)^2e_0^- + \frac{2}{15}(4+\ell+\ell^2)e_3^-,$$

$$e_5^- = -\frac{1}{15}(3+\ell+\ell^2)r_H^2(\omega_0^-)^2e_0^- + \frac{1}{30}(12+\ell(\ell+1)(2+\ell+\ell^2))e_3^-,$$

$$e_6^- = -\frac{2}{35}(3+\ell+\ell^2)r_H^2(\omega_0^-)^2e_0^- + \frac{\ell(\ell+1)(\ell^2(\ell+1)^2+8)-4(6+5r_H^2(\omega_0^-)^2)}{35(\ell^2+\ell-2)}e_3^-$$

$\vdots$

We can also use these recursion relations to estimate the numerical calculation error

Ambiguity of potential

$$\left[-\frac{d^2}{dx^2}+V_{\mathrm{GR}}\right]\Phi=\omega^2\Phi \qquad dx:=\frac{dr}{f}$$ tortoise coord

GR case, spectra are known

Choosing new master variable Ψ as

$$\Psi=(1+X)\Phi+Y\frac{d\Phi}{dx} \qquad X=\mathcal{O}(\alpha) \quad Y=\mathcal{O}(\alpha)$$

Derivatives of Ψ contains higher derivatives but they can be reduced to lower order

$$\frac{d\Psi}{dx}=\cdots+Y\frac{d^2\Phi}{dx^2}$$
$$=\cdots-Y(\omega^2-V_{\mathrm{GR}})\Phi$$

Ambiguity of potential

$$\left[-\frac{d^2}{dx^2}+V_{\mathrm{GR}}\right]\Phi=\omega^2\Phi \qquad \Psi=(1+X)\Phi+Y\frac{d\Phi}{dx}$$

After some calculations,

$$\frac{d^2\Psi}{dx^2}+\underline{(\cdots)}\frac{d\Psi}{dx}+(\omega^2-V_{\mathrm{GR}}-\delta V)\Psi=0$$

$= \mathbf{0}$ $\rightarrow$ eq is Schrodinger form

(this condition is a differential eq for X, Y)

Spectra are same as GR case,
but correction terms exist in V

Ambiguity of potential

Spectra are GR, but correction terms exist

$$\frac{d^2\Psi}{dx^2} + (\omega^2 - V_{\mathrm{GR}} - \delta V)\Psi = 0$$

$$\delta V = \frac{f}{r_H}\sum_j \alpha_j \left(\frac{r_H}{r}\right)^j$$

read off α_j

$$\omega_{\mathrm{QNM}} = \omega_{\mathrm{QNM}}^{(\mathrm{GR})} \boxed{+ \sum_j \alpha_j e_j}$$

$$= 0$$

because spectra are same as GR

Recursion relation for e_j

Change of master variable for GR case leads to correction terms in V

$$\omega_{\mathrm{QNM}} = \omega_{\mathrm{QNM}}^{(\mathrm{GR})} \underbrace{\boxed{+\sum_j \alpha_j e_j}}_{=\mathbf{0}}$$

We can check the accuracy of e_j^-

Our current numerical results for e_j^- satisfy this $\mathcal{O}(10^{-20})$

Multiple degrees of freedom

In modified gravity theories, sometimes, multiple physical degrees of freedoms are coupled

In many cases, we need to consider master eq with Schrodinger type

$$\left[-\frac{d^2}{dx^2}+V\right]\Phi=\omega^2\Phi$$

where the potential is $n \times n$ matrix, Φ has n components

Our formalism can be extended to coupled system [PRD **100**, 044061 (2019)]

Summary and discussion

- We propose a quick method to estimate QNM ω for a class of parameterized effective potential
- We want to put a constraint on $\alpha_j^{\pm}$ from future QNM observation
- extension to coupled system
- QNM is an entrance of gravitational wave research
- knowledge of quantum mechanics is useful for QNM study