Approximate recovery in Operator Algebra Quantum Error Correction for local erasure

Parthiv Haldar
National Center for Theoretical Sciences (NCTS)

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Quantum Error Correction: Why is it interesting in Quantum Field Theory?

- Quantum error correction codes are ways to protect quantum information, correlation against noises/error that may result due to various reasons, like interaction with environment.

- QEC is at the center of fault tolerate quantum computation.

- The basic logic of quantum error correction is to encode the quantum state to be protected redundantly in a larger system on which the error is supposed to occur. But the redundant encoding comes with a dual decoding map such that the error can be reversed.

- The encoding and decoding protocols very depend on the error.

- QEC plays a surprising role in quantum gravity, especially in the context of AdS/CFT holographic duality.

- In particular, it was shown that encoding of bulk observables in AdS in terms of dual boundary CFT observables is a QEC! [Almheiri, Dong, Harlow (2014)]

⊙ Emergence of bulk geometry from boundary CFT can also be understood as a kind of renormalization group flow, often called holographic renormalization. In a sense the Alhmeiri-Doug-Harlow result establishes QEC nature for holographic RG.

$$\boxed{\text{Does RG in generic QFT has QEC properties?}}$$

Does RG in generic QFT has QEC properties?

- There are evidences! But for that we need to look at real space RG.
- Multi Entanglement Renormalization Ansatz (MERA) of lattice models is an approximate QEC against local erasure of lattice.

[Kim and Kastoryano (2017)]

⊙ Kadanoff spin blocking of 2D Ising Model (classical) is actually a classical error correcting code against local excitations. [Furuya, Lashkari, Moussa (2021)]

Q. What about real space RG in QFT? Hints in cMERA, but not satisfactory!! [Furuya, Lashkari, Moussa (2021)]

Q. Can we develop/show these connections in general QFT?

- Lack of general understanding of real space RG in generic QFT

- In QFT there is no local Hilbert space, only local algebra of observables. For generic QFT, this is a type III von Neumann algebra, most notorious type of vN algebra!

How to do QEC with arbitrary vN algebra?

Outline

- Quantum Error Correction
  - Schrödinger Picture, Knill-Laflamme Cond$^{\underline{n}}$
  - Heisenberg Picture (Operator Älgebra QEC)
- Approximate quantum error correction (AQEC)
- Quantum Erasure and local erasure
- Approximate operator algebra QEC
- Afterthoughts

Quantum Error Correction

Schrödinger Picture QEC

$\mathcal{K}_L$: Logical Hilbert space,

$\mathcal{H}_S$: Physical Hilbert space (state space)

$\mathcal{H}_C$: Code subspace of $\mathcal{H}_S$; $\mathcal{H}_C \subseteq \mathcal{H}_S$

W : Encoding map, isometry

$W : \mathcal{K}_L \hookrightarrow \mathcal{H}_S$ [isometric embedding]

$W(\mathcal{K}_L) \cong \mathcal{H}_C$

$W^\dagger$: Decoding map

$WW^\dagger = \Pi_C$: proj. to $\mathcal{H}_C$

$\mathcal{E} \rightarrow$ Error/Noise

$\mathcal{R} \rightarrow$ Recovery

$$
\begin{tikzcd}
 & \mathcal{K}_L \arrow[dr, "W"] & \\
\mathcal{H}_s \arrow[ur, "W^\dagger"] & \mathcal{H}_s \arrow[l, "\mathcal{R}"'] & \mathcal{H}_s \arrow[l, "\mathcal{E}"']
\end{tikzcd}
$$

Knill-Laflamme condition [Exact QEC]

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho \qquad \forall \rho = \Pi_c \rho \Pi_c$$

$$\rho \in \mathcal{B}(\mathcal{H}_s)$$

Heisenberg picture Error Correction

- In Heisenberg picture, the states do not evolve. The operators/observables evolve. We have now algebra of operators which can be corrupted by error and need to be protected.

- $\mathcal{B}$: Logical algebra $\mathcal{A}$: state algebra

  i : encoding superoperator α : decodingg superoperator

$$\begin{tikzcd} & \mathcal{B} \arrow[dl, "i"'] & \\ \mathcal{A} \arrow[r, "\mathcal{R}"] & \cdot \arrow[r, "\Phi"] & \mathcal{A} \arrow[ul, "\alpha"'] \end{tikzcd}$$

$\boxed{\text{Kraus Representation}}$

$$i(b) = W b W^{\dagger} \qquad \alpha(a) = W^{\dagger} a W$$

$$\mathcal{R}(a) = \sum_r R_r^{\dagger} a R_r \qquad \Phi(a) = \sum_r V_r^{\dagger} a V_r$$

$\boxed{\text{Exact QEC Criterion}}$

$$\alpha \circ \Phi \circ \mathcal{R} \circ i\,(b) = b, \quad \forall\, b \in \mathcal{B}$$

[Bény, Kempf and Kribbs, 2007]

- However this Heisenberg picture QEC, also called Operator Algebra QEC, is for operators on finite dimensional Hilbert space as well as countably infinite dim. In the latter case, we consider bounded operator.

- In these cases the operator algebra is what is known as type I von-Neumann algebra.

- The important feature of these algebra is that trace as well as the factorization of Hilbert space, on which the algebra acts irreducibly, is also well defined.

- QFT : Trace or Factorization not well defined !

  Density matrix (local) does not exist !!

Approximate Quantum Error Correction

- Exact QEC codes can be resource costly.
- Approximate error correction is sufficient for many practical purposes.
- Let us give a general statement in terms of Kraus operators of the error map.

$$\mathcal{E}(\cdot) = \sum_k M_k (\cdot) M_k^\dagger$$

$M_k \rightarrow$ Kraus operator

$$\boxed{\Pi_c M_k^\dagger M_\ell \Pi_c = \lambda_{k\ell} \Pi_c + \delta_{k\ell}}$$

, $\lambda_{k\ell} \in \mathbb{C}$

$\delta_{k\ell} \rightarrow$ small

Quantum Erasure

- We will be interested in a particular type of quantum error called erasure.
- Erasure corresponds to a situation where one knows the position of the error.
- The simplest example is a quantum secret sharing code involving 3 qutrits.

  [Cleve, Gottesmann, Lo (1999)]

- Suppose Alice wants to send a quantum secret in the form of a qutrit state to Bob. But they know that some agent can intercept and make exactly one measurement, erasure of one qutrit by tracing out. How does she protect the secret against this?

- Encode in a 3-qutrit space

$$|0\rangle \mapsto |\tilde{0}\rangle = \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right)$$

$$|1\rangle \mapsto |\tilde{1}\rangle = \frac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right)$$

$$|2\rangle \mapsto |\tilde{2}\rangle = \frac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right)$$

- Now it is easy to see that these states are maximally entangled if one traces out one qutrit.

$$\mathrm{Tr}_1 |\tilde{\psi}\rangle\langle\tilde{\psi}| \propto \mathbb{I}$$

Tr_1 : Tracing out one qutrit.

Thus if one does only one measurement of qutrit then one gains no information about $|\psi\rangle$. The state is protected against the erasure of one qutrit.

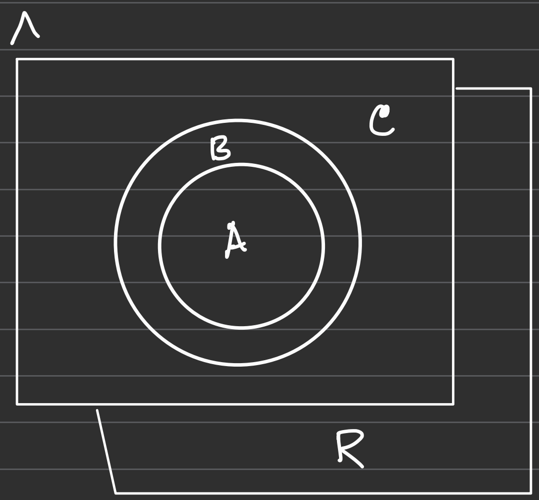
⊙ However the state can be reconstructed by a 2 qutrit operation. For example
- There exists a 2-qutrit unitary operation U_{12} acting on first and second qutrit s.t.

$$U_{12} \otimes I_3 |\tilde{\Psi}\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{3}} \left[|00\rangle + |11\rangle + |22\rangle \right]$$

Here $U_{12} \otimes I_3 \longrightarrow$ decoding map

Local Erasure and Local Recoverability

(*)



$\Lambda \rightarrow$ lattice

A, B, C disjoint systems

$R \rightarrow$ purifying system for Λ

The region $A \subset \Lambda = ABC$, with $AB = A^{+\ell}$ is the region that includes all qubits within a dist. ℓ of A, and C is the compliment of of AB in Λ. The region is

[Flammia et. al. 2016]

(δ, ℓ) correctable if there exist a Recovery Channel $\mathcal{R}_B^{AB}$

$$B_1\left(\mathcal{R}_B^{AB}(\rho^{BCR}), \rho^{ABCR}\right) \leq \delta$$

$B_1 \rightarrow$ Bures distance

$$B_1(\rho,\sigma)^2 = 2\left(1 - \sqrt{F(\rho,\sigma)}\right)$$

$$F(\rho,\sigma) = \left[\mathrm{tr}\left(\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)\right]^2$$

Information disturbance tradeoff

$$\delta_e(A) := \min_{\omega_A} \sup_{\rho^{ABCR}} \mathcal{B}\left(\omega^A \otimes \rho^{CR}, \rho^{ACR}\right)$$

$$\inf_{\mathcal{R}_B^{AB}} \sup_{\rho^{ABCR}} \mathcal{B}\left(R_B^{AB}(\rho^{BCR}) \otimes \rho^{ABCR}\right) = \delta_e(A)$$

⊙ These results are for finite dim. Hilbert spaces, or equivalently for type I vN algebra. We attempt a generalization of this to arbitrary vN algebra.

- Barring technical details, I will pin-point the general argument.

In this picture we will now attach a vN algebra to each of the regions. We are still keeping the background space to be lattice.

- The main problem for vN algebra beyond type I (the matrix algebra) is that they don't have irreducible representation on any Hilbert space.

- That means that given an algebra, constructing Hilbert space gives us many options.

- One such construction is GNS construction. In this given an algebra we construct a Hilbert space on which the algebra is realized as an algebra of bounded operators.

GNS Construction

i) Algebra : $\mathcal{A}$

ii) State : linear functional on algebra

$$\rho : \mathcal{A} \to \mathbb{C} \quad , \quad \rho(\mathbb{1}) = 1$$

iii) Cyclic separating vector : For a representation π of vN algebra on a Hilbert space H, an element $\xi \in H$ is cyclic if $\{\pi(x)\xi : x \in \mathcal{A}\}$ is dense in H w.r.t the Hilbert space norm. It is separating if

$$\pi(a)\xi = 0 \Rightarrow a = 0 .$$

So the basic strategy is to declare some state to cyclic, separating for the algebra.

Then

$$a \mapsto \langle \pi(a)\xi, \xi \rangle \equiv \rho \qquad \text{(state)}$$

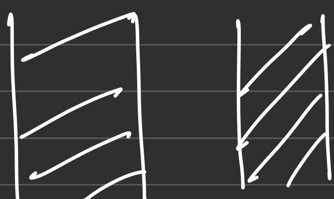
(*) These states on the algebra generalizes density matrix from usual finite dim. quantum information.

(*) And it turns out that the unique cyclic-separating vector generalizes purification of density matrix.

(*) In fact this cyclic and separating vector can be understood analogous to thermofield double state.

Recall thermofield double state looks like

$$|\psi_{TFD}\rangle = \frac{1}{\sqrt{Z}} \sum_{i=1}^{N} e^{-\beta E_i/2} \, |\bar{i}\rangle_l \otimes |i\rangle_r$$



l r

$$\rho_r^\beta = \mathrm{Tr}_{H_l} |\psi_{TFD}\rangle\langle\psi_{TFD}|$$

$$= \frac{1}{Z} e^{-\beta H_r}$$

$|\psi_{TFD}\rangle$ is a purification of ρ_r^β.

(*) Also notice that one can write $|\psi_{TFD}\rangle$ as a $N \times N$ matrix which can be identified with $(\rho_r^\beta)^{1/2}$.

$$|\psi_{TFD}\rangle \sim (\rho_r^\beta)^{1/2}$$

(*) This generalizes to GNS construction

(*) In particular given an algebra $\mathcal{A}$ we have state ρ_A and $|\rho_A^{1/2}\rangle$ as the cyclic, separating vector.

(*) One can generalize the notion of trace to a "renormalized trace" which gives finite value in type II vN algebra, where usual trace does not make sense. However for type III algebra even this renormalized trace does not give anything finite!

(*) Anyway, we can now generalize the information - disturbance tradeoff to type I and type II vN algebra using the GNS Hilbert space of the algebras.

(*) The analysis however does not work for type III due to the very explicit use of trace function. Needs finer analysis.

Afterthoughts

⟨i⟩ We take small steps towards generalizing QEC theory for arbitrary von Neumann algebra. The ultimate goal being to examine whether we can understand real space RG in QFT as this general QEC.

⟨ii⟩ But this will require some general understanding of real space RG in QFT. But one can nevertheless try testing in models like cMERA.

<iii> Can we understand symmetry invariance in RG via QEC? Perhaps using a generalization of what is called co-variant QEC?

<iv> Apart from these formal aspects, on a more practical point, one can work towards perhaps emulating RG, rather its various qualitative feature, via QEC. In fact this

has been done recently in a problem of quantum phase recognition in 1D SPT phases. [Lake, Balasubramanian, Choi (2022)]

<V> Finally can we make a connection between the QEC picture and the usual momentum space RG, especially Polchinski-Wilson RG equation.

Thank You