DEPARTMENT OF SPACE AND CLIMATE PHYSICS
MULLARD SPACE SCIENCE LABORATORY

An alternative approach to calculating the polarisation of synchrotron radiation

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Radiation

acceleration

- Relativistic Larmor formula: $$P = \frac{2q^2\gamma^6}{3c}\left[(\dot{\boldsymbol{\beta}})^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2\right]$$
- Relativistic aberration: $$\sin\theta = \frac{\sin\theta'}{\gamma(1+\beta\cos\theta')}$$

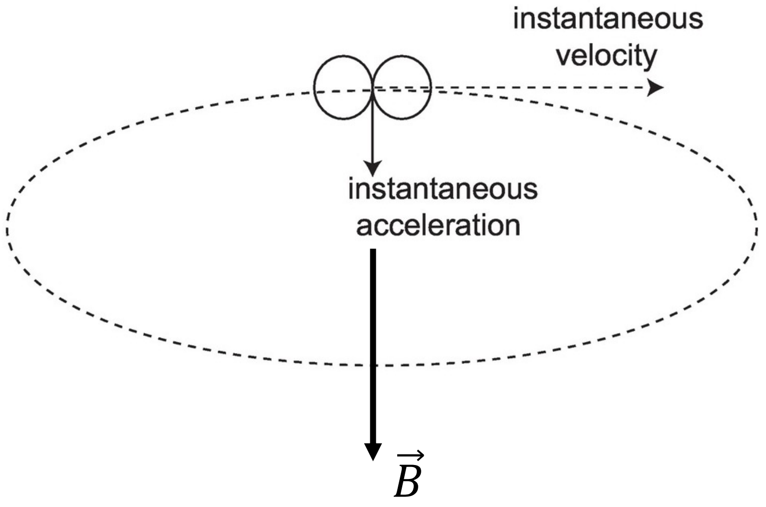
Credit: ERA, NRAO

non-relativistic radiation pattern ($\gamma \approx 1$)

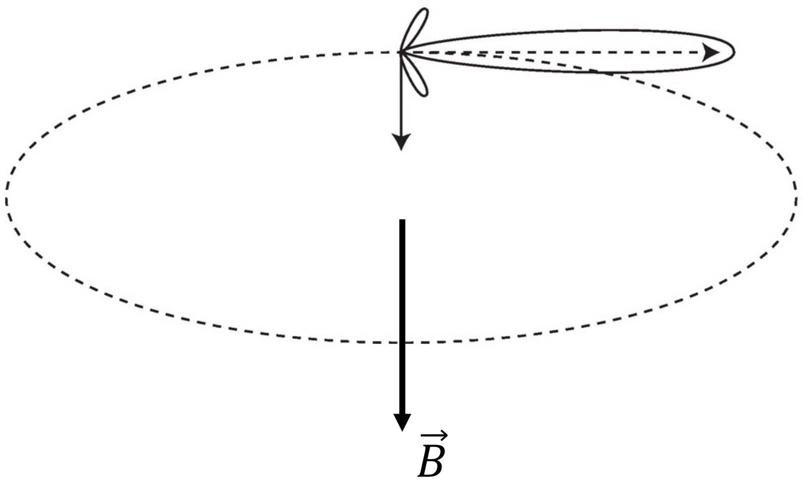
with relativistic aberration ($\gamma = 5$)

Acceleration by a magnetic field

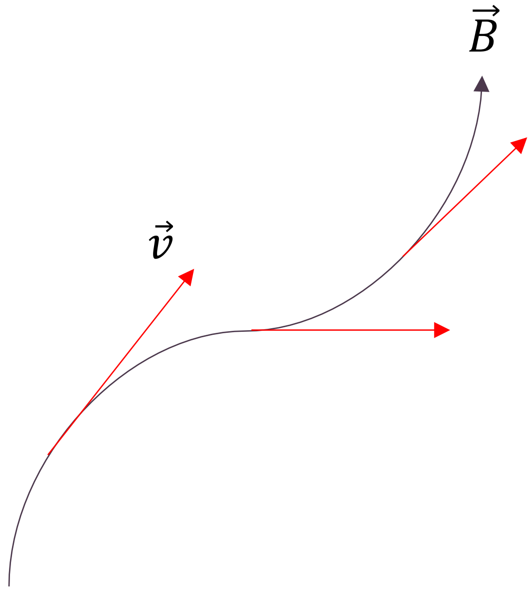
Credit: D. Phil Woodruff



Cyclotron radiation
(non-relativistic particles)



Synchrotron radiation
(relativistic particles)



Curvature radiation
(strong magnetic field)

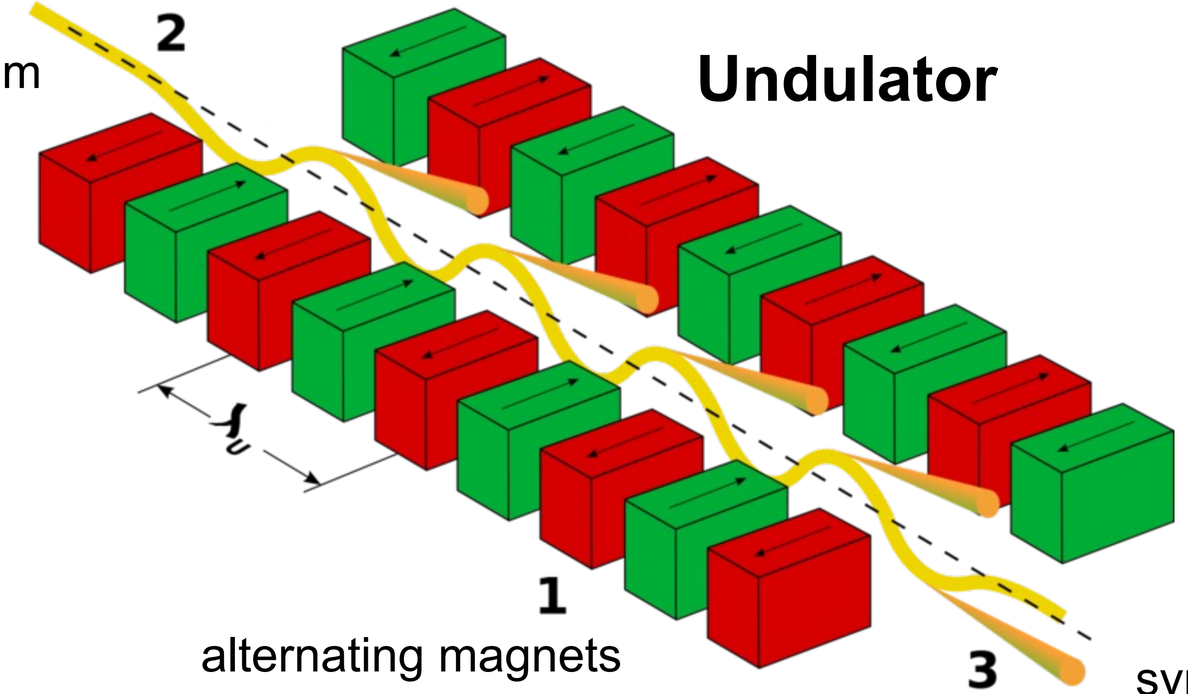
Synchrotron radiation in laboratories

- To produce high-intensity X-rays (~keV; ~10^{-10} m)
- Applications: material science, nuclear physics, etc.

electron beam

Undulator

(Wikimedia: Harp)

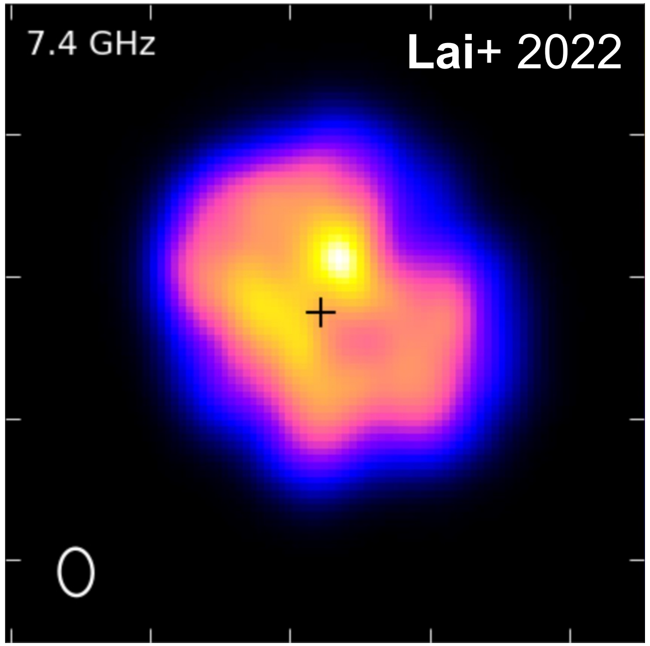


alternating magnets

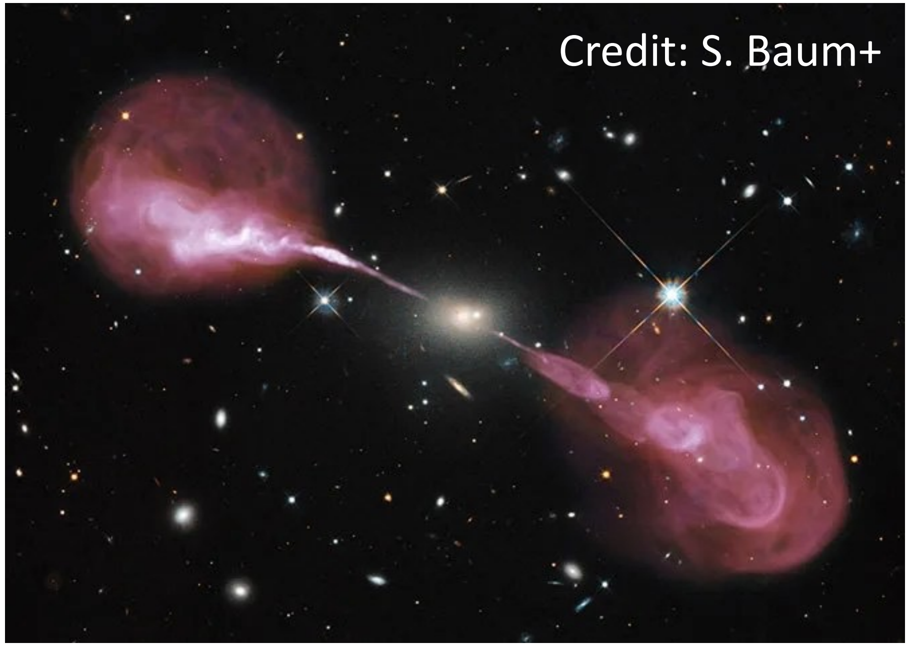
synchrotron X-ray beam

Synchrotron radiation in astrophysics

- Magnetic fields are everywhere
- Relativistic electrons are also everywhere

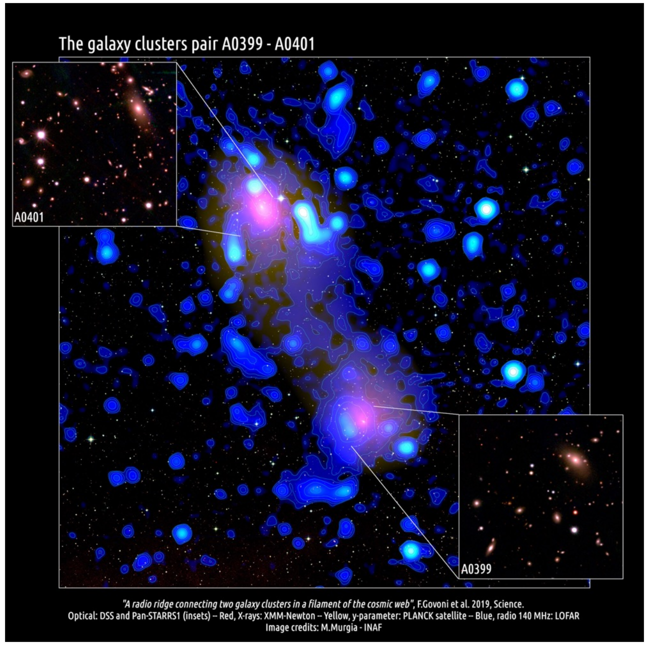


pulsar wind nebulae



radio galaxies

Credit: F. Govoni, M. Murgia, INAF

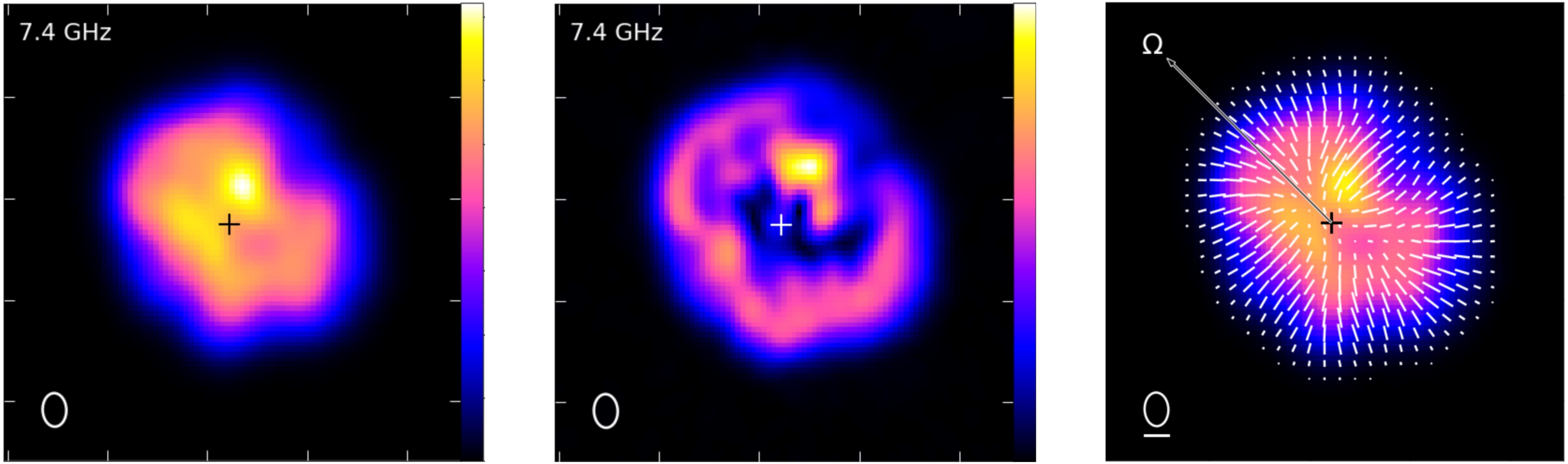


galaxy clusters

Synchrotron polarisation in astrophysics

pulsar wind nebula G21.5−0.9

Lai+ 2022



total intensity

polarised intensity

total intensity and
the projected magnetic field

Equations for synchrotron radiation

electron energy distribution

Total emissivity: $$\varepsilon_I(\nu,\theta) = \frac{\sqrt{3}e^2\nu_{\rm G}\sin\theta}{2c}\int_1^\infty {\rm d}\gamma\; \underline{n_e(\gamma)}\, F\left(\frac{\nu}{\nu_c}\right) \quad [{\rm erg\,s^{-1}\,cm^{-3}\,Hz^{-1}\,sr^{-1}}]$$

Polarised emissivity: $$\varepsilon_Q(\nu,\theta) = \frac{\sqrt{3}e^2\nu_{\rm G}\sin\theta}{2c}\int_1^\infty {\rm d}\gamma\; \underline{n_e(\gamma)}\, G\left(\frac{\nu}{\nu_c}\right) \quad [{\rm erg\,s^{-1}\,cm^{-3}\,Hz^{-1}\,sr^{-1}}]$$

Polarisation degree: $$PD = \frac{\text{polarised emission}}{\text{total emission}} = \frac{\varepsilon_Q}{\varepsilon_I} = \frac{\int_1^\infty {\rm d}\gamma\; n(\gamma)\, G(\frac{\nu}{\nu_c})}{\int_1^\infty {\rm d}\gamma\; n(\gamma)\, F(\frac{\nu}{\nu_c})}$$

A new approach: $$PD \approx \frac{\alpha_\nu + 1}{\alpha_\nu + 5/3},$$

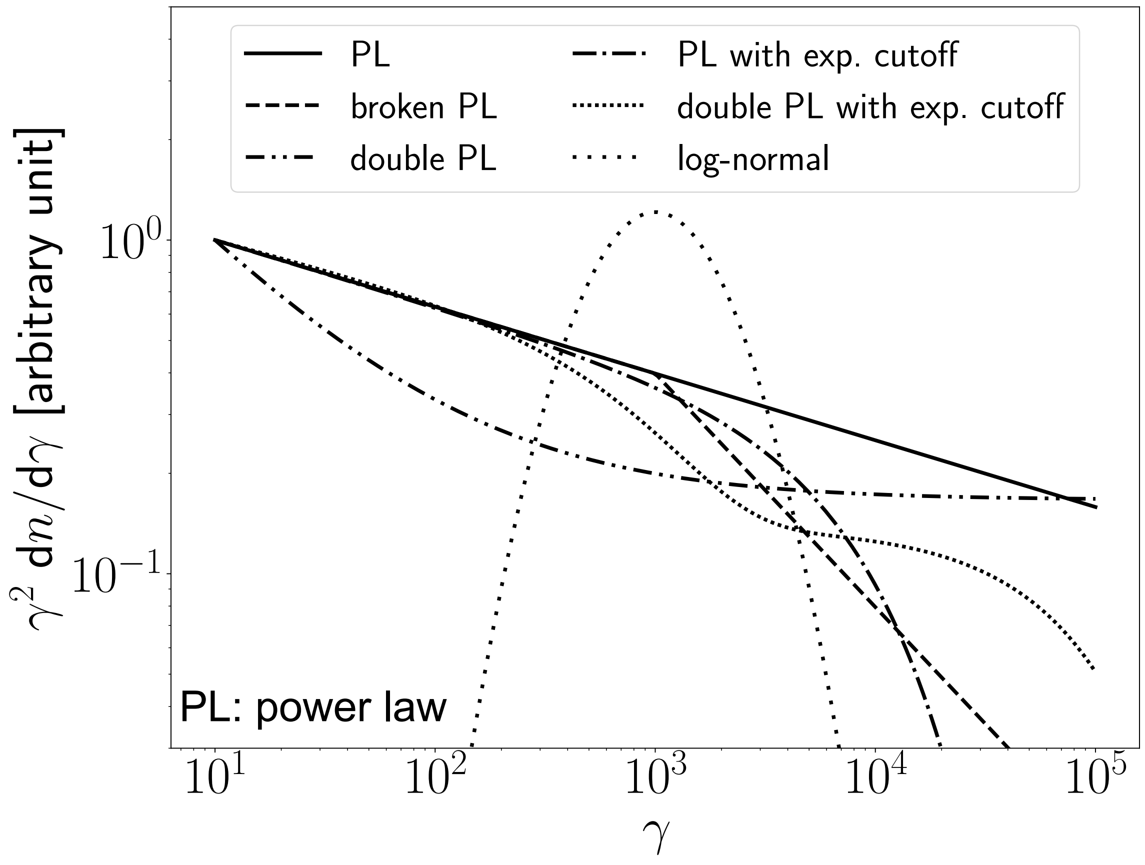
$$\alpha_\nu = -\frac{{\rm d}\log\varepsilon_I}{{\rm d}\log\nu}$$ is the spectral index

modified Bessel functions of the second kind

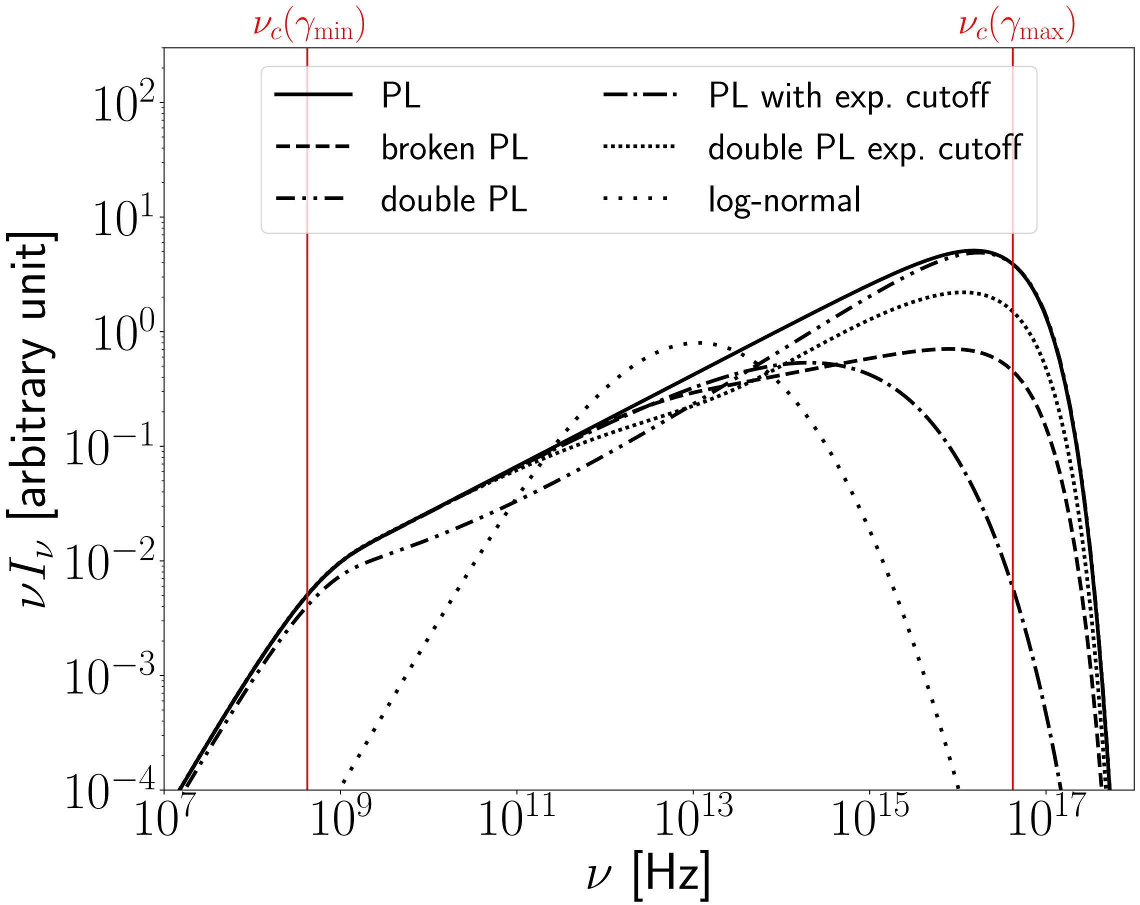
$$F(x) = x\int_x^\infty {\rm d}\eta\; K_{5/3}(\eta)$$

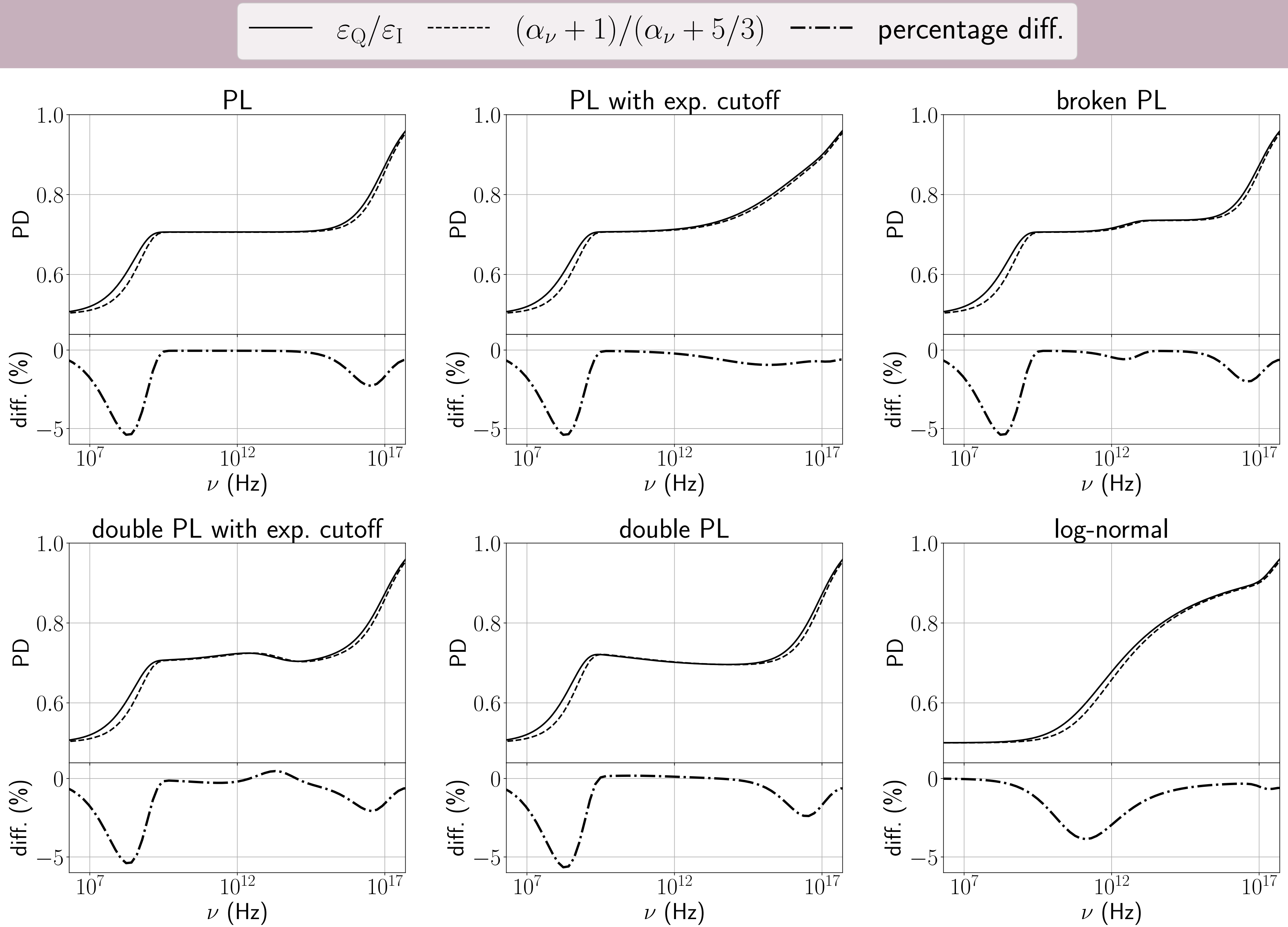
$$G(x) = x\, K_{2/3}(x)$$

electron energy distributions



synchrotron total intensity




$\varepsilon_Q/\varepsilon_I$
$(\alpha_\nu+1)/(\alpha_\nu+5/3)$
percentage diff.
PL
PL with exp. cutoff
broken PL
double PL with exp. cutoff
double PL
log-normal
PD
diff. (%)
ν (Hz)
1.0
0.8
0.6
0
−5
10^7
10^{12}
10^{17}

Lai+ in prep.

Where does this formula come from?

Consider a power-law distribution $n(\gamma) \propto \gamma^{-p}$

In reality $n(\gamma)$ cannot extend to infinity

$$\varepsilon_I \propto \int_1^\infty \mathrm{d}\gamma \; n_e(\gamma) F\left(\frac{\nu}{\nu_c}\right) \approx \int_0^\infty \mathrm{d}\gamma \; n_e(\gamma) F\left(\frac{\nu}{\nu_c}\right) \propto \nu^{-(p-1)/2} = \nu^{-\alpha} \quad \Rightarrow \quad p = 2\alpha + 1$$

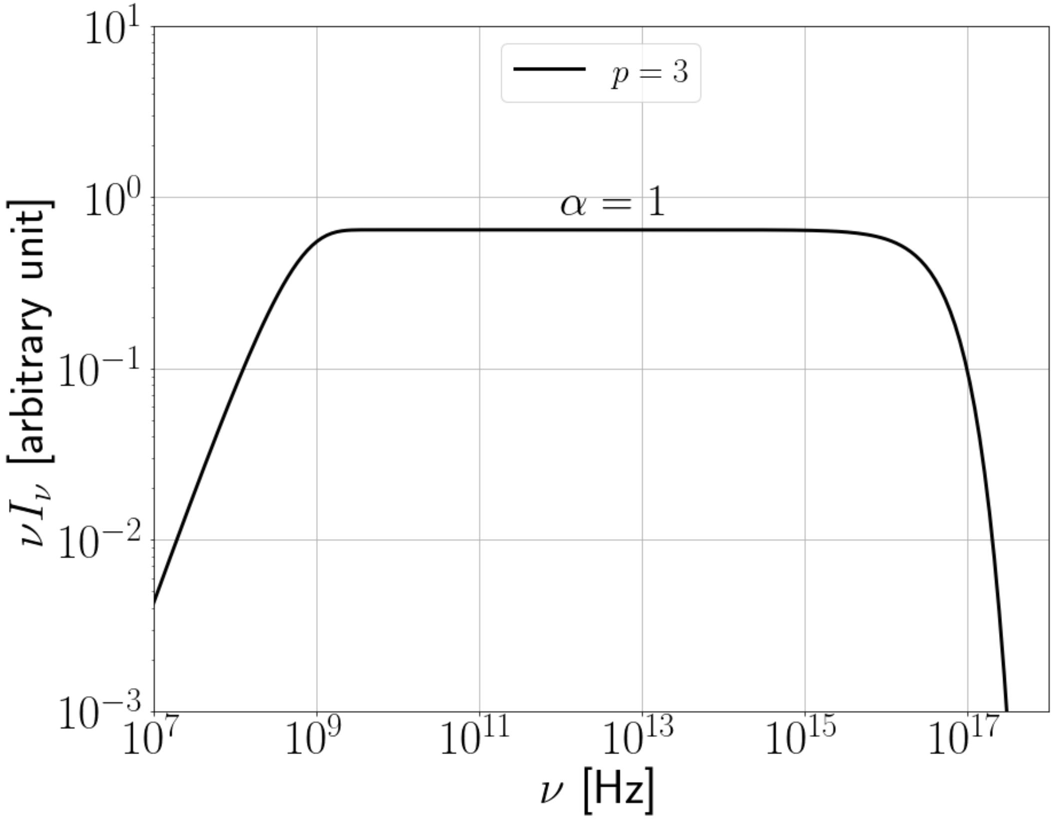
Lorentz factor $\gamma \geq 1$

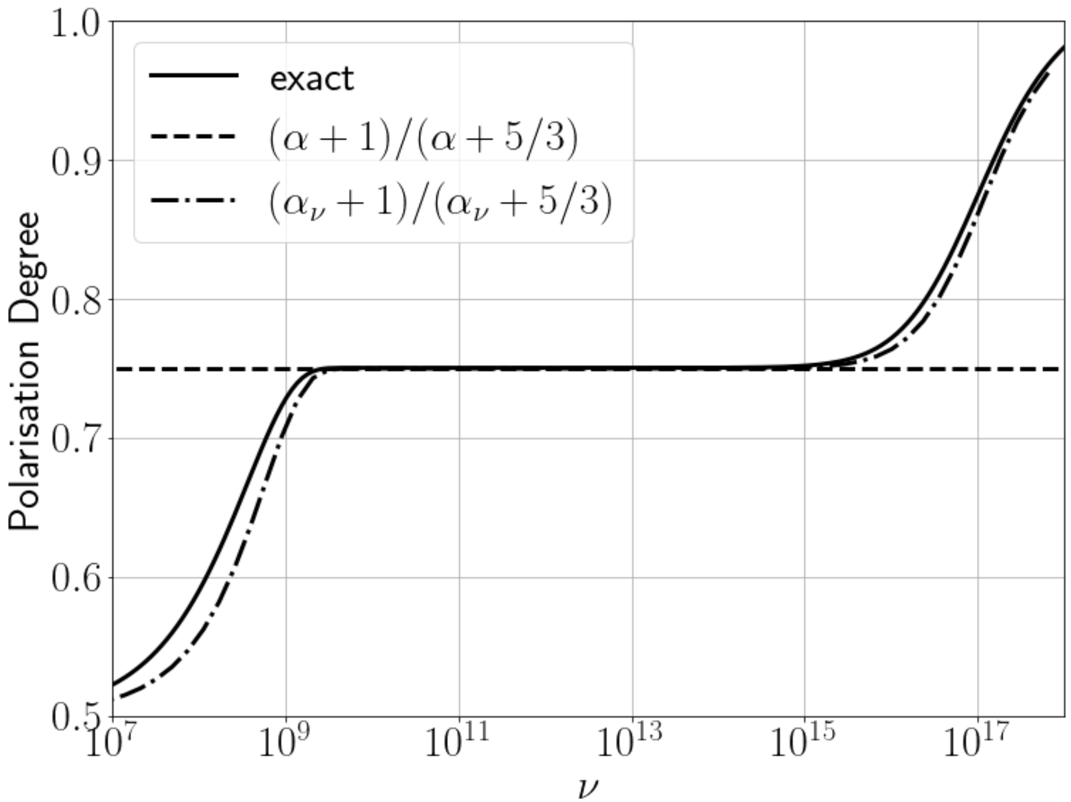
Polarisation degree: $$PD = \frac{\int_1^\infty \mathrm{d}\gamma \; n(\gamma) G(\frac{\nu}{\nu_c})}{\int_1^\infty \mathrm{d}\gamma \; n(\gamma) F(\frac{\nu}{\nu_c})} \approx \frac{\int_0^\infty \mathrm{d}\gamma \; n(\gamma) G(\frac{\nu}{\nu_c})}{\int_0^\infty \mathrm{d}\gamma \; n(\gamma) F(\frac{\nu}{\nu_c})} = \frac{p+1}{p+7/3} = \frac{\alpha+1}{\alpha+5/3}$$

Generalise the above formula to any $n(\gamma)$: $$PD \approx \frac{\alpha_\nu + 1}{\alpha_\nu + 5/3}, \qquad \alpha_\nu = -\frac{\mathrm{d}\log \varepsilon_I}{\mathrm{d}\log \nu}$$

Power-law distributions

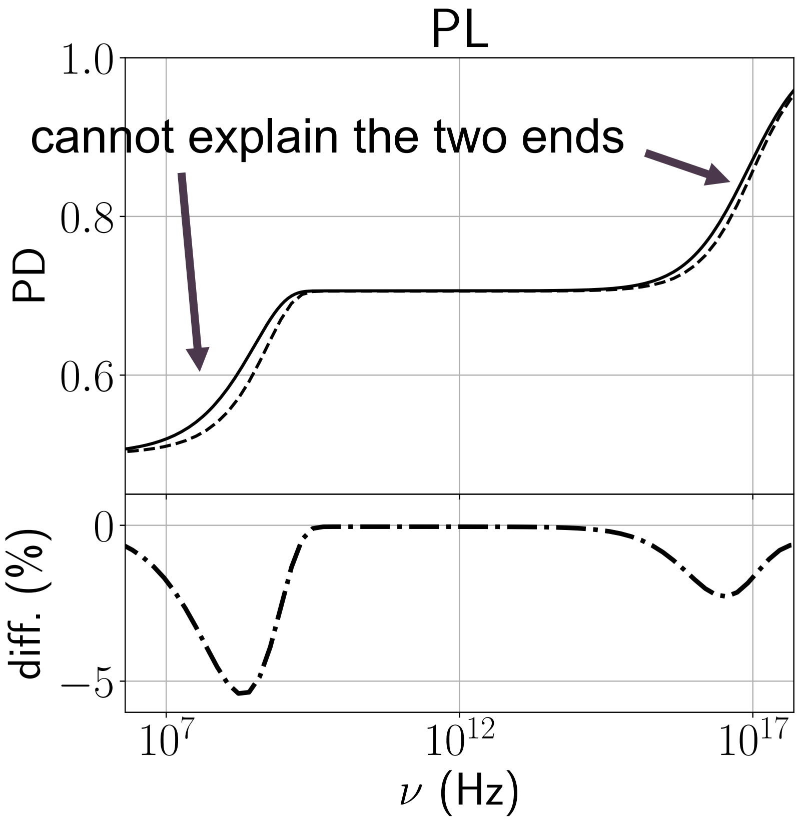
$n(\gamma) \propto \gamma^{-p}$ for $[\gamma_{\min}, \gamma_{\max}]$, where $p = 3$, thus $\alpha = (p-1)/2 = 1$



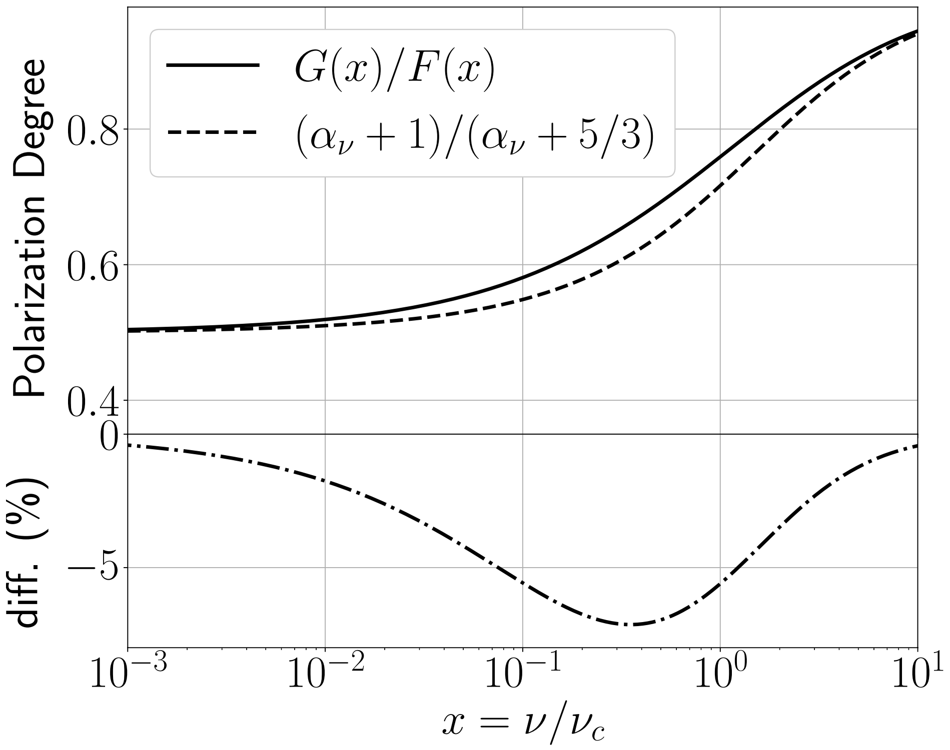


Why does it work?

- **Hypothesis:** any smooth distributions can be seen as a power law locally (wrong!)



cannot explain the mono-energy case



Mathematical attempts…

$$F(x) = x \int_x^\infty \mathrm{d}\eta \; K_{5/3}(\eta)$$

$$G(x) = x\, K_{2/3}(x)$$

mono-energy: $$PD = \frac{\int_1^\infty \mathrm{d}\gamma \; n(\gamma)\, G(\frac{\nu}{\nu_c})}{\int_1^\infty \mathrm{d}\gamma \; n(\gamma)\, F(\frac{\nu}{\nu_c})} = \frac{G(x)}{F(x)} = \frac{F''(x)}{F(x)} - \frac{1}{3x}\frac{F'(x)}{F(x)} + \frac{1}{3x^2}$$

The alternative formula: $$PD \approx \frac{\alpha_\nu + 1}{\alpha_\nu + 5/3} = \frac{F(x) - xF'(x)}{\frac{5}{3}F(x) - xF'(x)}$$

after suitable approximations?

If it works for mono-energy, it would not be surprising that it works for any smooth distributions

Any physical reasons?

- Why does the spectral index contain the information of polarisation?
- Would it be a mathematical coincidence?

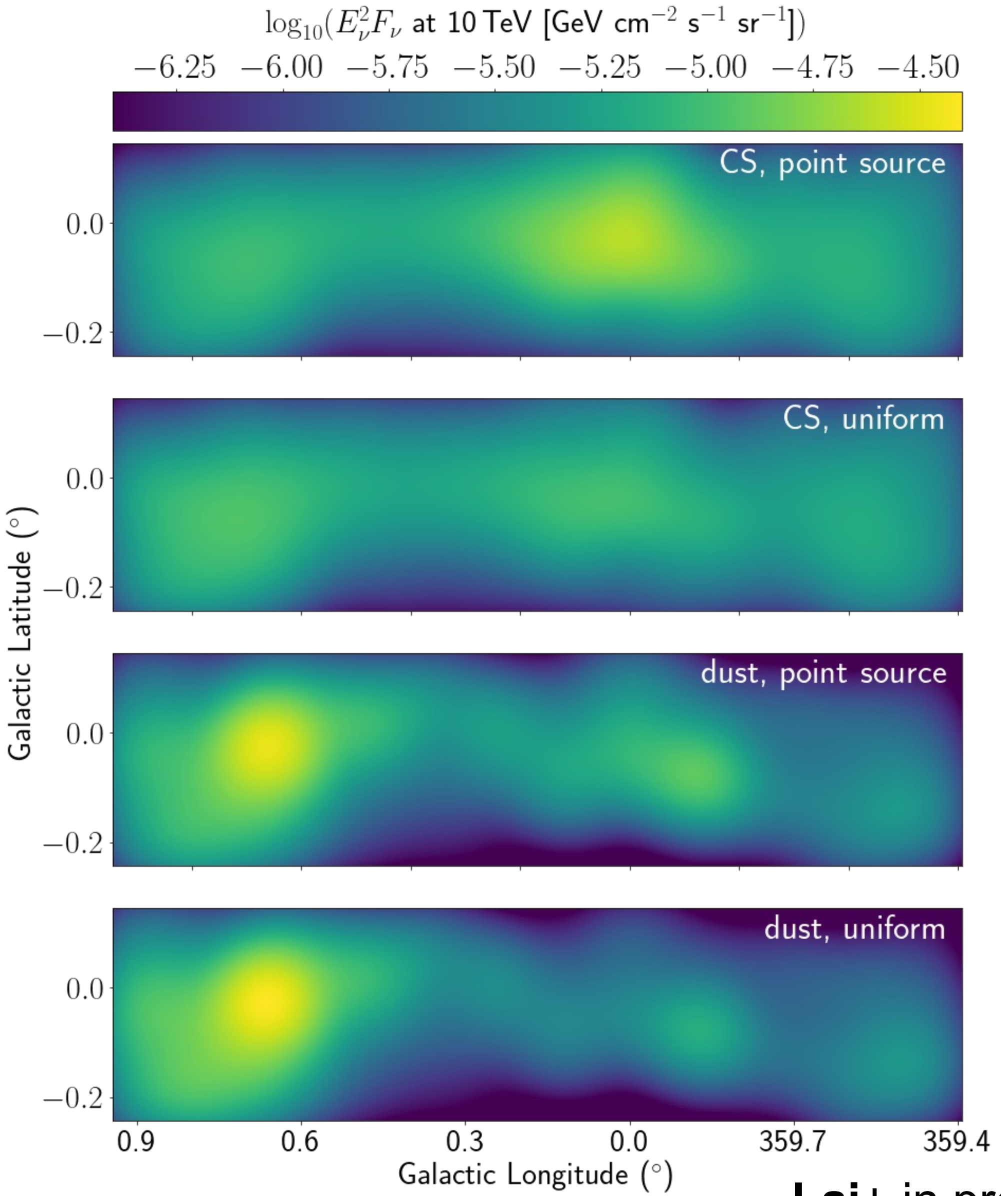
Applications in observational astronomy

- Electron energy distribution $n(\gamma)$ is not observable
- A reasonably good $n(\gamma)$ model requires good quality multi-wavelength measurement, which is often not available
- Spectral index can be easily measured
- A quick and reliable way to obtain the intrinsic polarisation degree

Summary

- We propose a new formula to estimate the polarisation degree $PD \approx \dfrac{\alpha_\nu + 1}{\alpha_\nu + 5/3}$
- This is useful in observational astronomy
- The reason why it works is unclear

neutrino images of the Galactic Centre



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