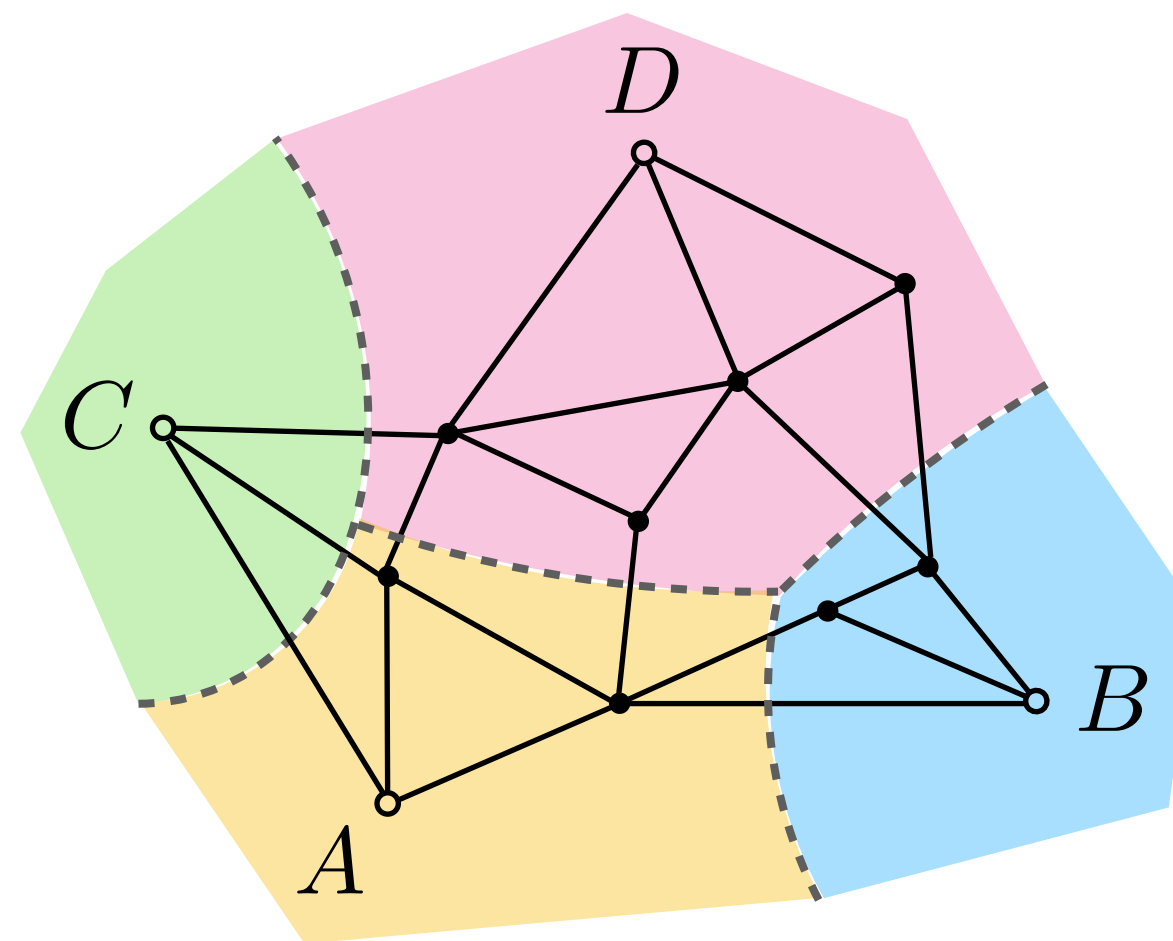


Multi-entropy in Random Tensor Networks

Simon Lin

UC Santa Cruz



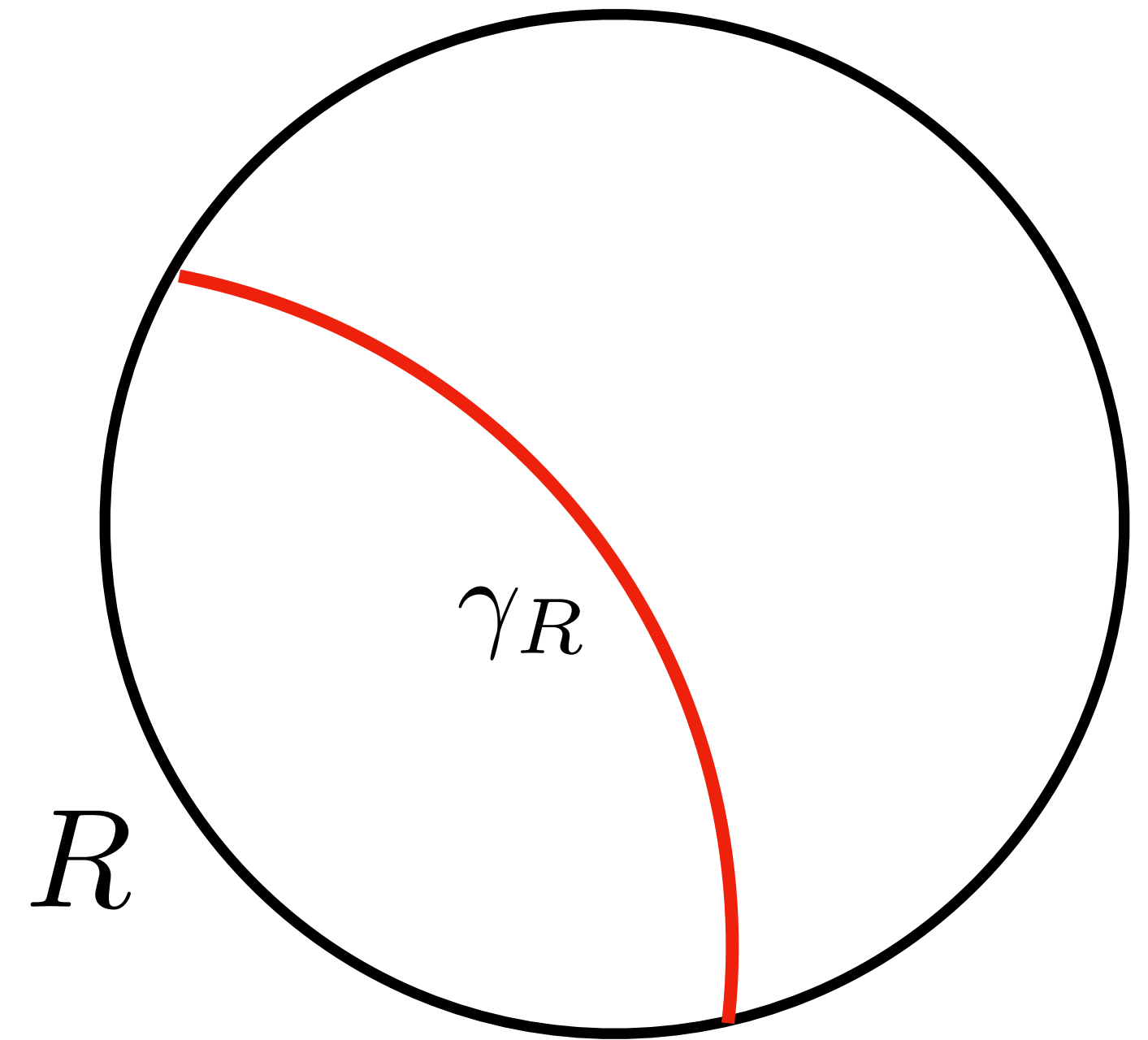
Taiwan String Workshop 2026, NTHU, Hsinchu, Taiwan



- based on: [2606.04470 \[hep-th\]](#), with [Ion Nechita](#) and [Miao Hu](#)
- See also [2412.07549](#) ([Iizuka-SL-Nishida](#)) and [2604.13261](#) ([Akella-Iizuka](#))

20 years since Ryu-Takayanagi [Ryu-Takayanagi '06]

- Boundary **entanglement entropy** = area of bulk **minimal surface**
- “Geometry from entanglement”
- A large body of our understanding of QG has grown out from this formula:
 - Generalized entropy and quantum error correction [Faulkner-Lewkowycz-Maldacena '13, Almheiri-Dong-Harlow '15, ...]
 - Bulk reconstruction and subregion-subregion duality [Dong-Harlow-Wall '16, Czech et al '12,...]
 - Page curve and replica wormholes [Almeiri-Engenhardt-Maldacena-Maxfield '19, Almheiri et al '20, Peninton et al '20...]



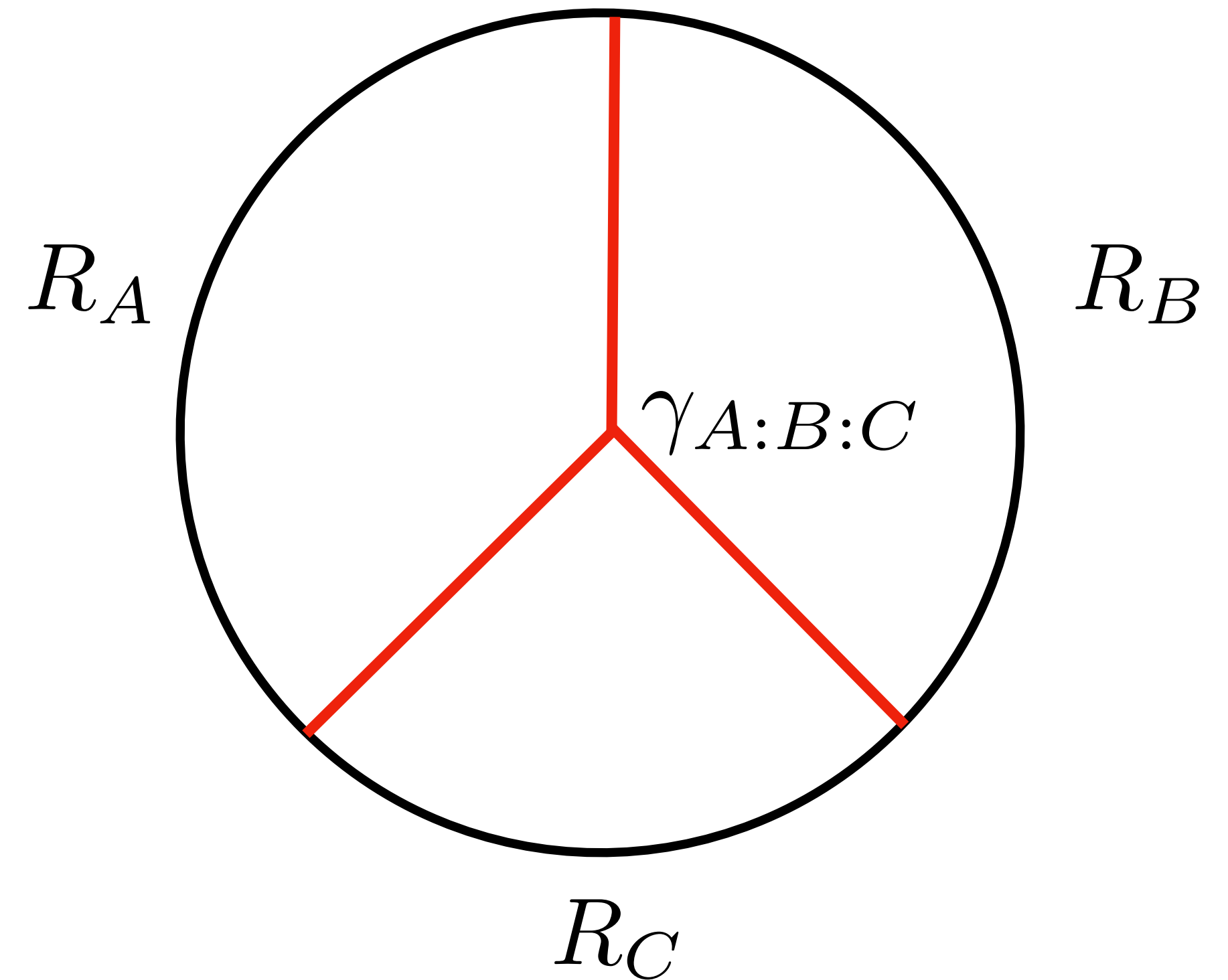
$$S(R) = \frac{A(\gamma_R)}{4G_N}$$

Many more!

A “higher-partite” RT formula [Gadde-Krishna-Sharma ‘22]

- Is there a higher-partite version for the minimal surface formula?
- Conjecture: boundary **multi-entropy** = area of bulk “**minimal soap film**”

Yes, but...



$$S^{(3)}(A : B : C) \stackrel{?}{=} \frac{A(\gamma_{A:B:C})}{4G_N}$$

Entanglement entropy

$$S(\rho) = -\operatorname{tr}(\rho_A \log \rho_A) = -\sum_i (\lambda_i \log \lambda_i)$$

- Bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- $S(\rho) > 0 \iff |\psi\rangle_{AB}$ is entangled
- Renyi entropy $S_n(\rho) = \frac{1}{1-n} \log \operatorname{tr}(\rho_A^n)$
- von Neumann limit $S(\rho) = \lim_{n \rightarrow 1} S_n(\rho)$

Replica trick and some notations

$$\rho_{ij} = A \underset{i}{\text{---}} \text{---} \underset{j}{\text{---}} A^* \quad \text{tr}(\rho^n) = \text{---} \text{---} \text{---} \text{---} \text{---} \text{---}$$

- Can be generalized into higher rank density matrices
- Contraction of density matrices

$$\rho_{AB} = \begin{matrix} A & \text{---} & \text{---} & A^* \\ B & \text{---} & \text{---} & B^* \end{matrix} \quad , \quad \text{tr}_B \rho_{AB} = \begin{matrix} A & \text{---} & \text{---} & A^* \\ & \text{---} & \text{---} & \end{matrix}$$

$$\text{tr}(\rho_A^n) = \text{tr}((\text{tr}_B \rho_{AB})^n) = \text{---} \text{---} \text{---} \text{---} \text{---} \text{---}$$

Replica trick and some notations

- Contraction patterns determined by permutations (twist operators) $g \in S_n$

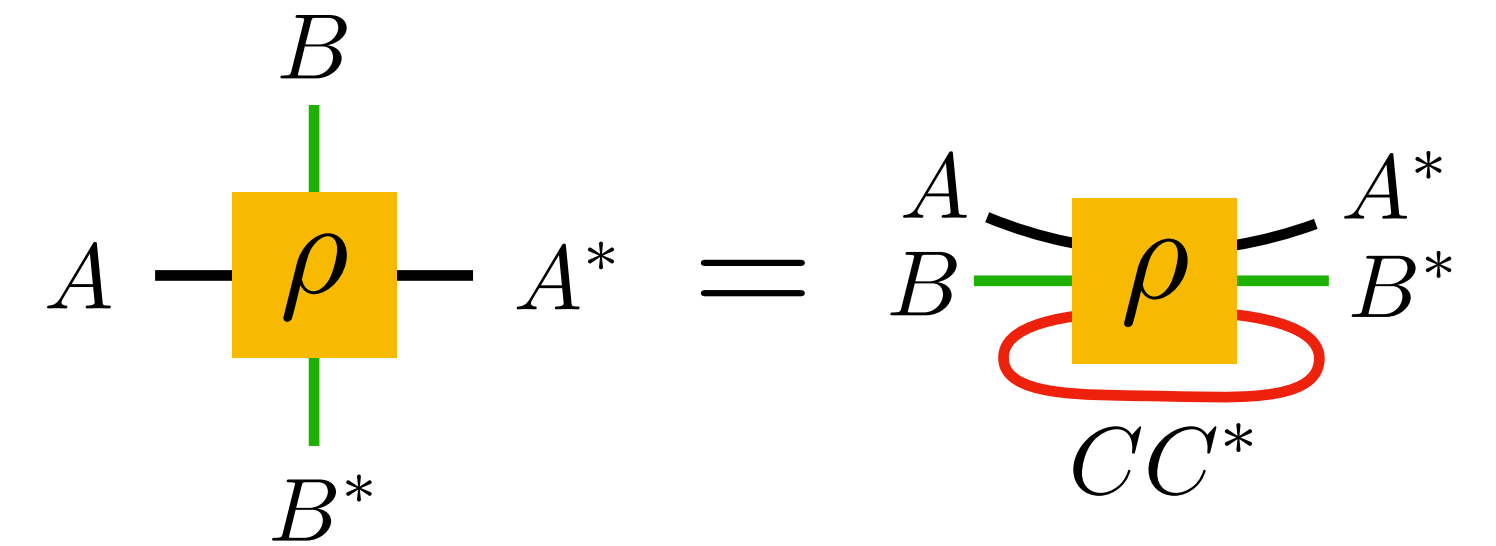
$\text{tr}(\rho_A^n) =$

 $\Rightarrow$
 $g_A = (12 \cdots n)$
 $g_B = \text{id}$

- Different trace multi-invariants $\leftrightarrow$ twist operators $\{g_i\}$
- Index i runs from 1 to the number of subsystems q

Multi-entropy [Gadde-Krishna-Sharma '22]

- q-partite pure state $|\psi\rangle_{ABC}$ (q=3 here)



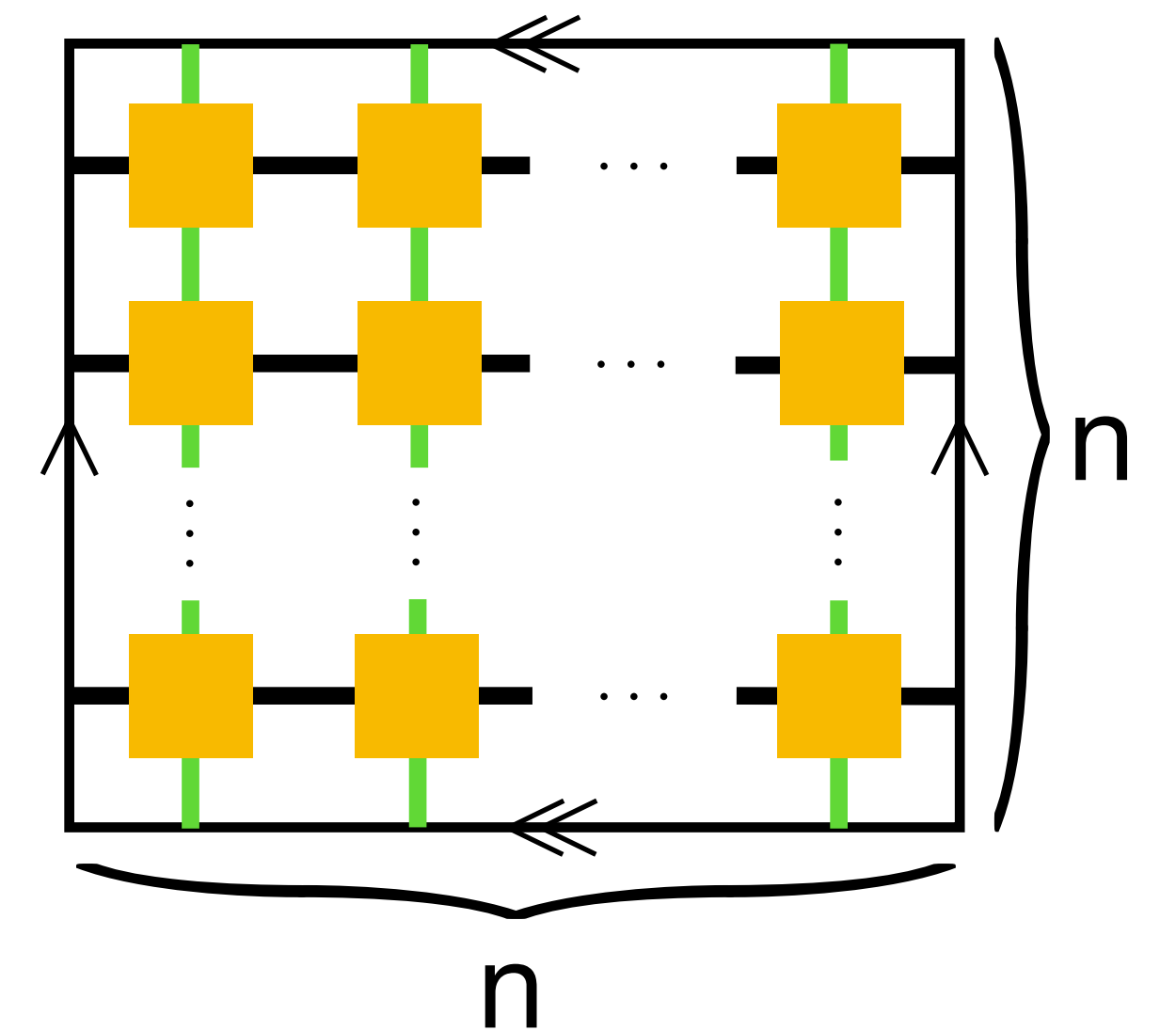
or (q-1)-partite mixed state $\rho_{AB} = \text{tr}_C(|\psi_{ABC}\rangle \langle \psi_{ABC}|)$

- (q-1)-dim symmetrical contraction

$$S_n^{(3)}(A : B : C) :=$$

- von-Neumann limit

$$S^{(q)}(A : B : \dots) := \lim_{n \rightarrow 1} S_n^{(q)}(A : B : \dots)$$



Multi-entropy [Gadde-Krishna-Sharma '22]

- twist operators: let $x = (x_1, x_2, \dots, x_{q-1})$ be the coordinates of the hypercube, then

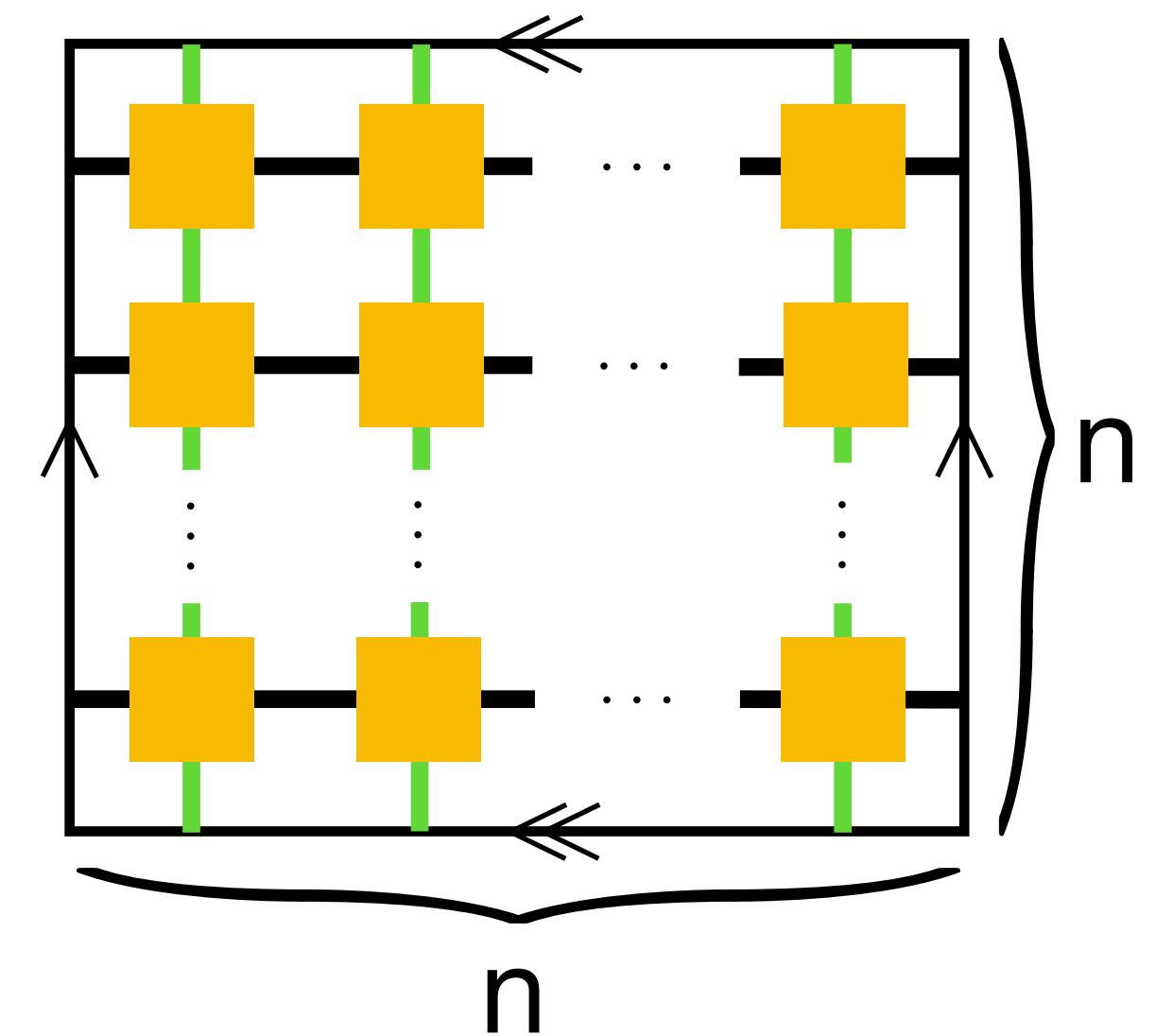
$$g_i(x) = (x_1, \dots, x_i + 1 \dots, x_{q-1})$$

and $g_q(x) = x$

- Replica group $\text{Sym}(\mathbb{Z}_n^{q-1}) = S_{n^{q-1}}$
- The set $\{g_i\}$ is “incompatible” in the sense that they are **pairwise distinct**, i.e.

$$g_i(x) \neq g_j(x) \quad \forall i \neq j$$

and they are “far away” from each other



Properties of $S_n^{(q)}$

- Reduces to (Renyi) entanglement entropy when $q = 2$

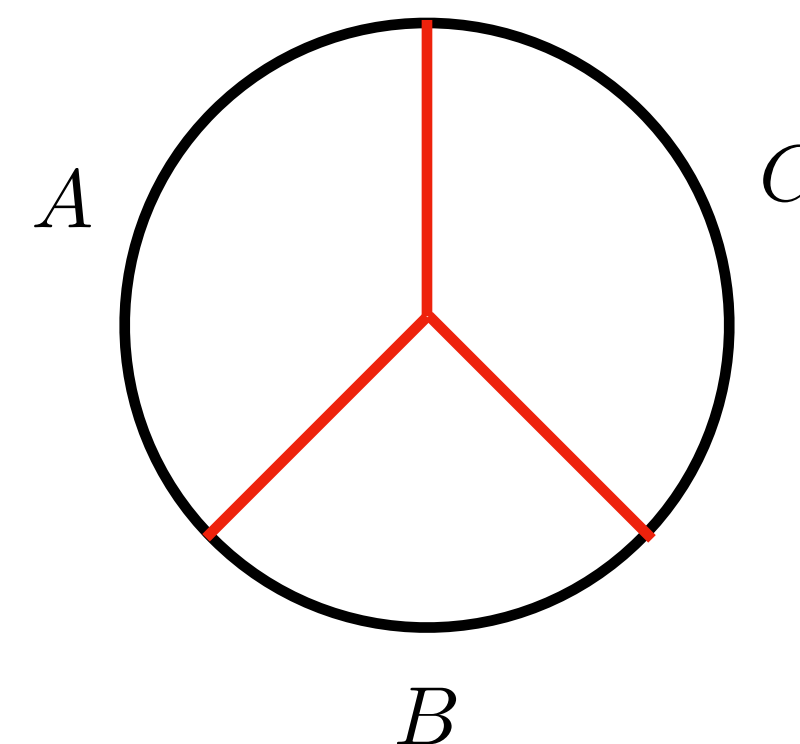
- Symmetric under permutation of subsystems

$$\text{GM}_n^{(3)}(A : B : C) = S_n^{(3)}(A : B : C) - \frac{1}{2}(S_n^{(2)}(A) + S_n^{(2)}(B) + S_n^{(2)}(C))$$

- Detects multi-partite entanglement through **genuine mutli-entropy**: $\text{GM}_n^{(q)} \neq 0 \implies |\psi\rangle$ is genuine multi-partite entangled [Iizuka-Nishida '25, Iizuka-SL-Nishida '25, Gadde '26]

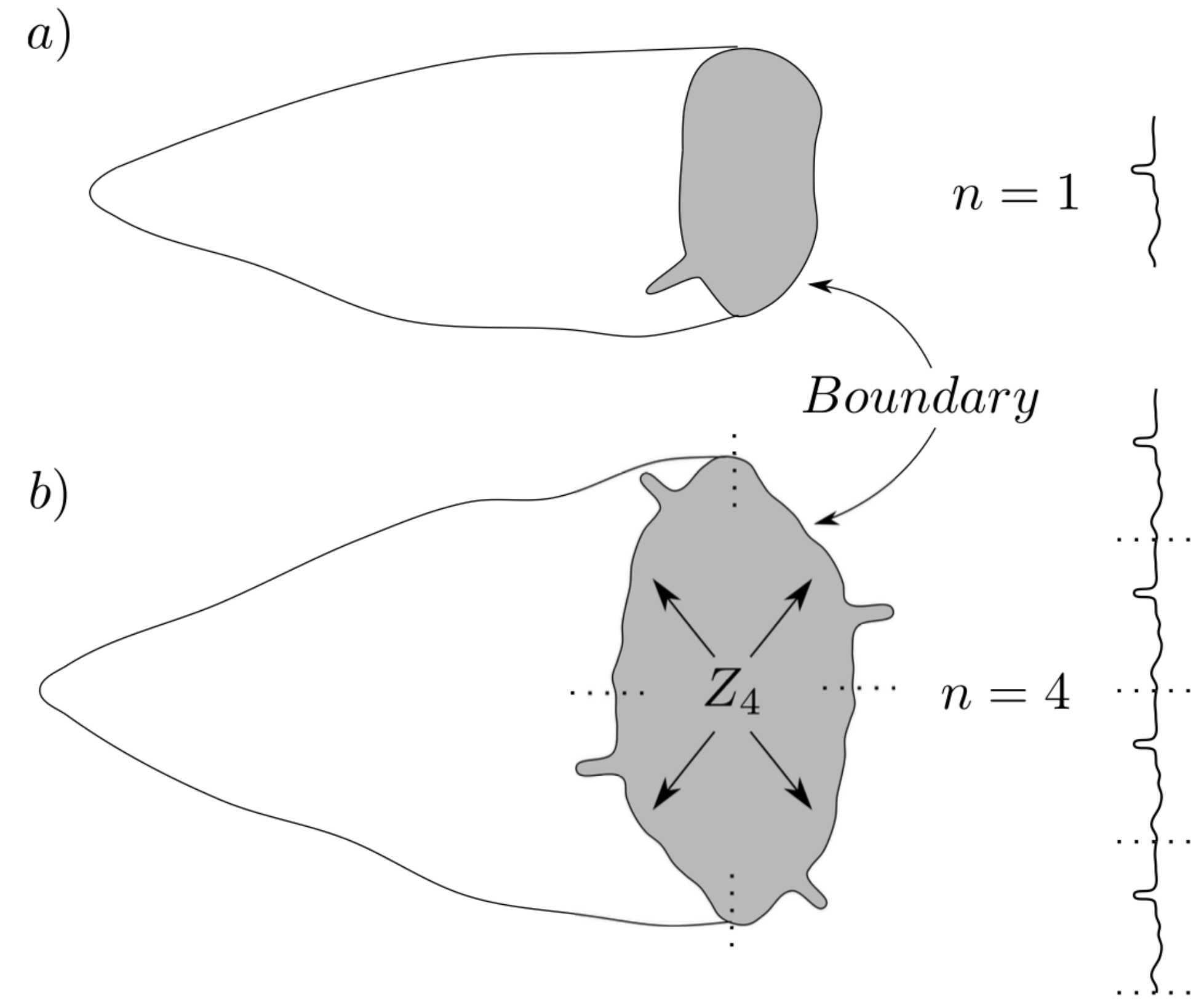
See also Kaberi Goswami's talk!

- Proposed holographic dual: multi-way cuts?
 - Problem: analytic continuation in n



A sketch of proof for RT [Lewkowycz-Maldacena '13]

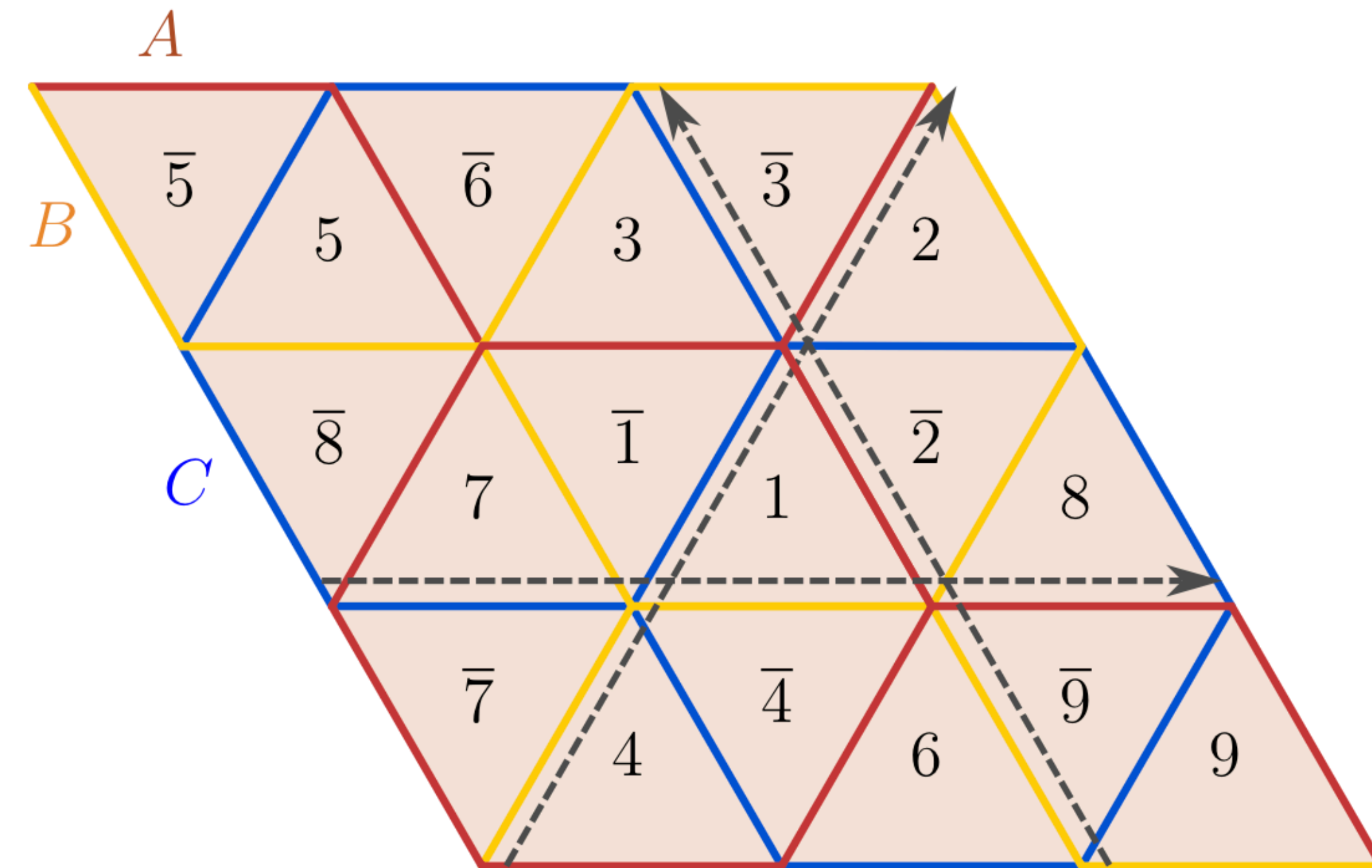
- Take ρ to be boundary CFT state (bipartite)
- Finding $\text{Tr } \rho^n$ is equivalent to calculating the GPI where BC at infinity is sourced by the boundary state
- Identify **replica-symmetric** saddles for integer n
- Take $n \rightarrow 1$: probe brane limit



taken from 1304.4926

What goes wrong for higher q ?

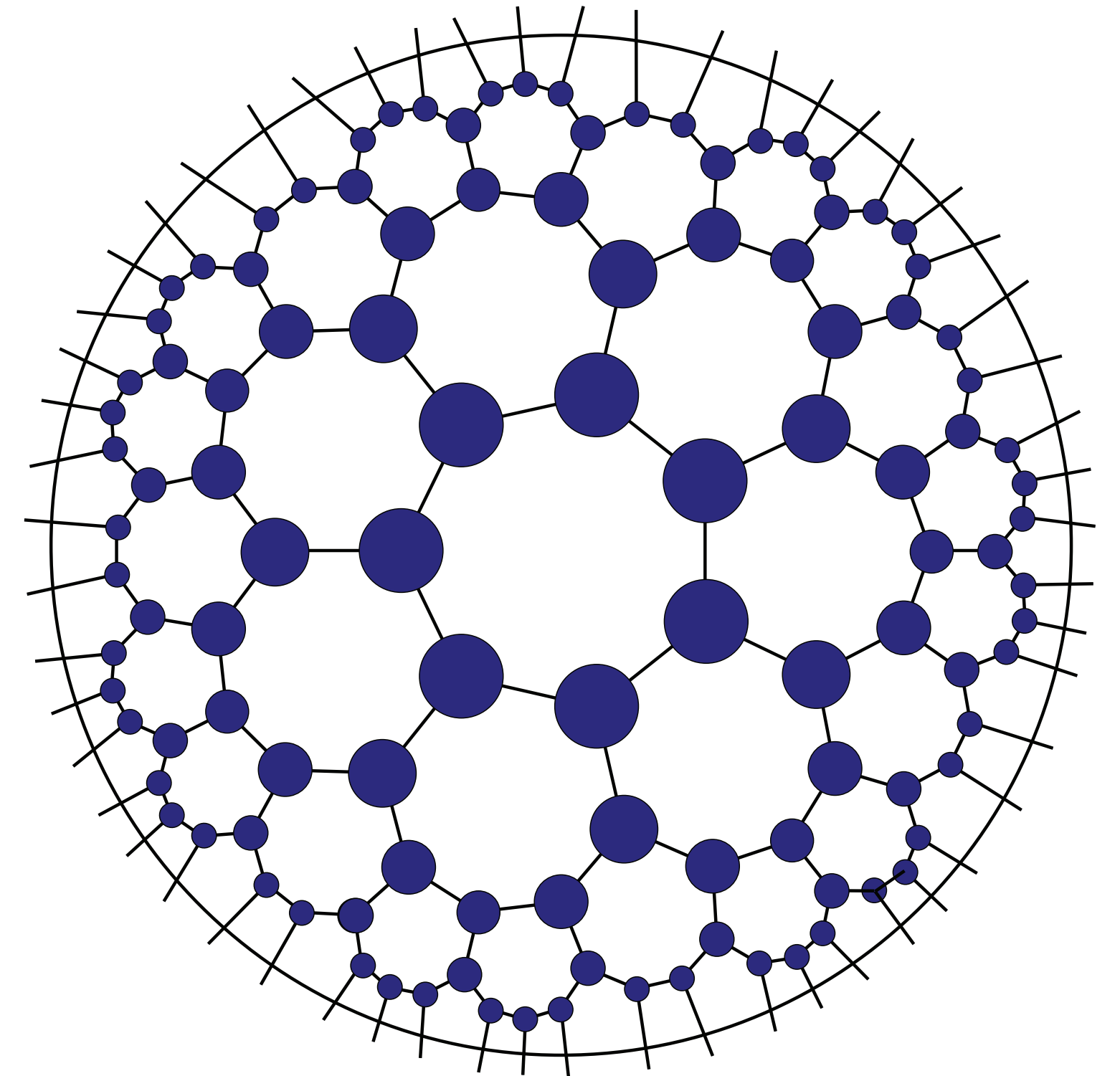
- **Replica symmetry breaking (RSB)** saddles for GPI when $n > 3$ [Penington-Walter-Witteveen '22, Gadde-Harper-Krishna '24]
- Freedom to fill in bulk with arbitrary spacetime is crucial for this construction
- Can we bypass RSB by working with “fixed-bulk-geometry” states instead?



taken from 2211.16045

Random tensor network (RTN)

- Quantum states built from contracting Haar-random tensors
- An RT formula at large bond dimensions [\[Collins-Nechita-Zyczkowski '10, Hayden-Nezami-Qi-Thomas-Walter-Yang '16\]](#)
- Flat Renyi spectrum — models **fixed area states** in AdS/CFT [\[Akers-Rath '19, Dong-Marolf-Harow '19\]](#)
- Bulk geometry is fixed by the contraction pattern: The handlebody RSB saddle is not a valid solution for RTN states

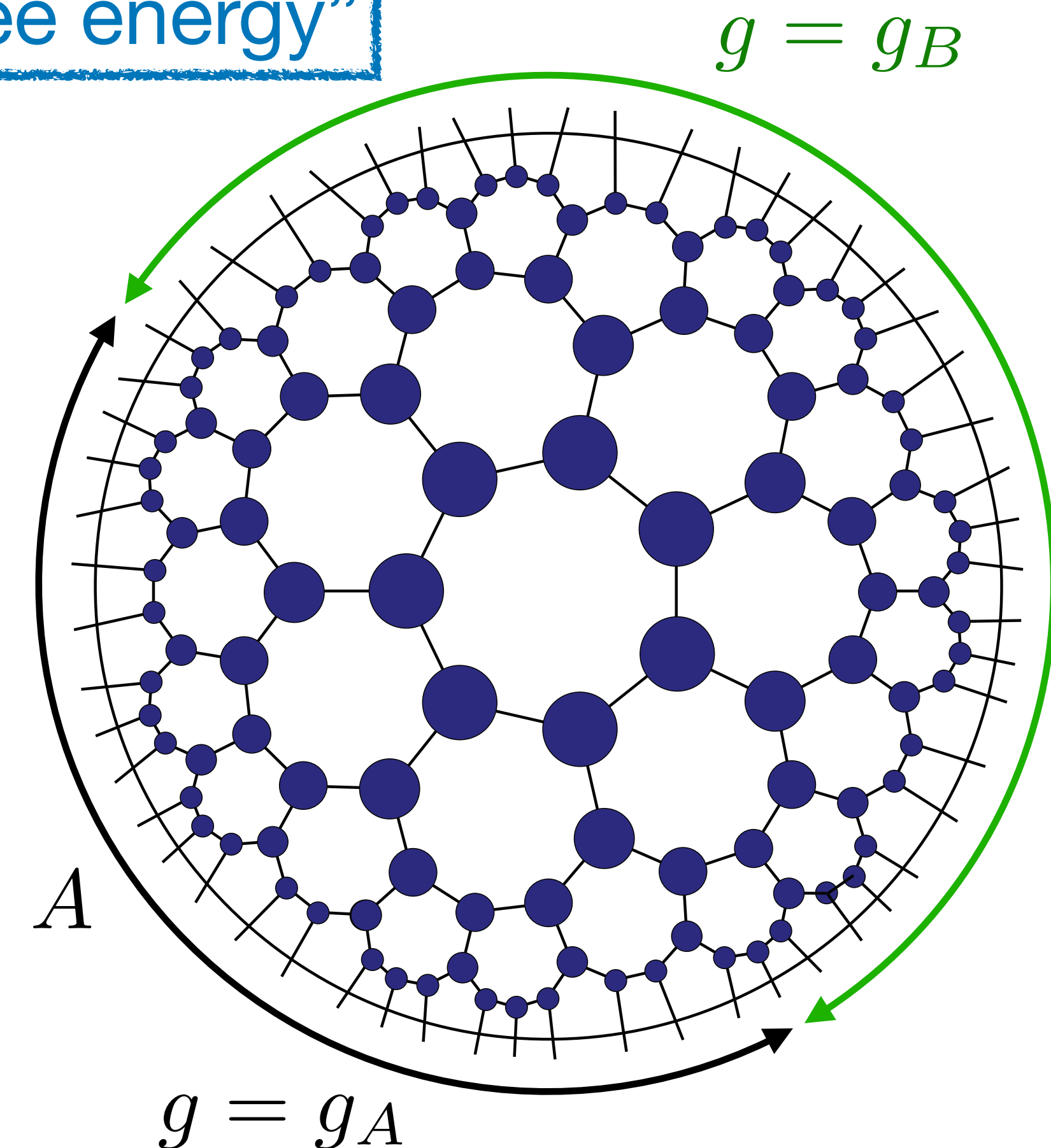


Replica tricks in RTN

- Haar averaging $\implies$ effective Ising model with “spins” taking value in the replica symmetry group S_n

$$\overline{\text{tr}(\rho_A^n)} \propto \sum_{g: V \rightarrow S_n} \exp \left[-\chi \sum_{(xy) \in E} d(g_x, g_y) \right]$$

“Free energy”



- $d(g, h) = n - \#(gh^{-1})$ is the **Cayley distance**
- boundary condition $g = g_{A/B}$ imposed on the dangling legs (boundary)
- summing over $S_n \leftrightarrow$ summing over different bulk topologies from gluing along common boundaries

The RT formula for RTN

- At large χ (low temperature) the lowest energy configuration is the “domain wall” solution
- All contribution to the free energy comes from the cut surface of r_A

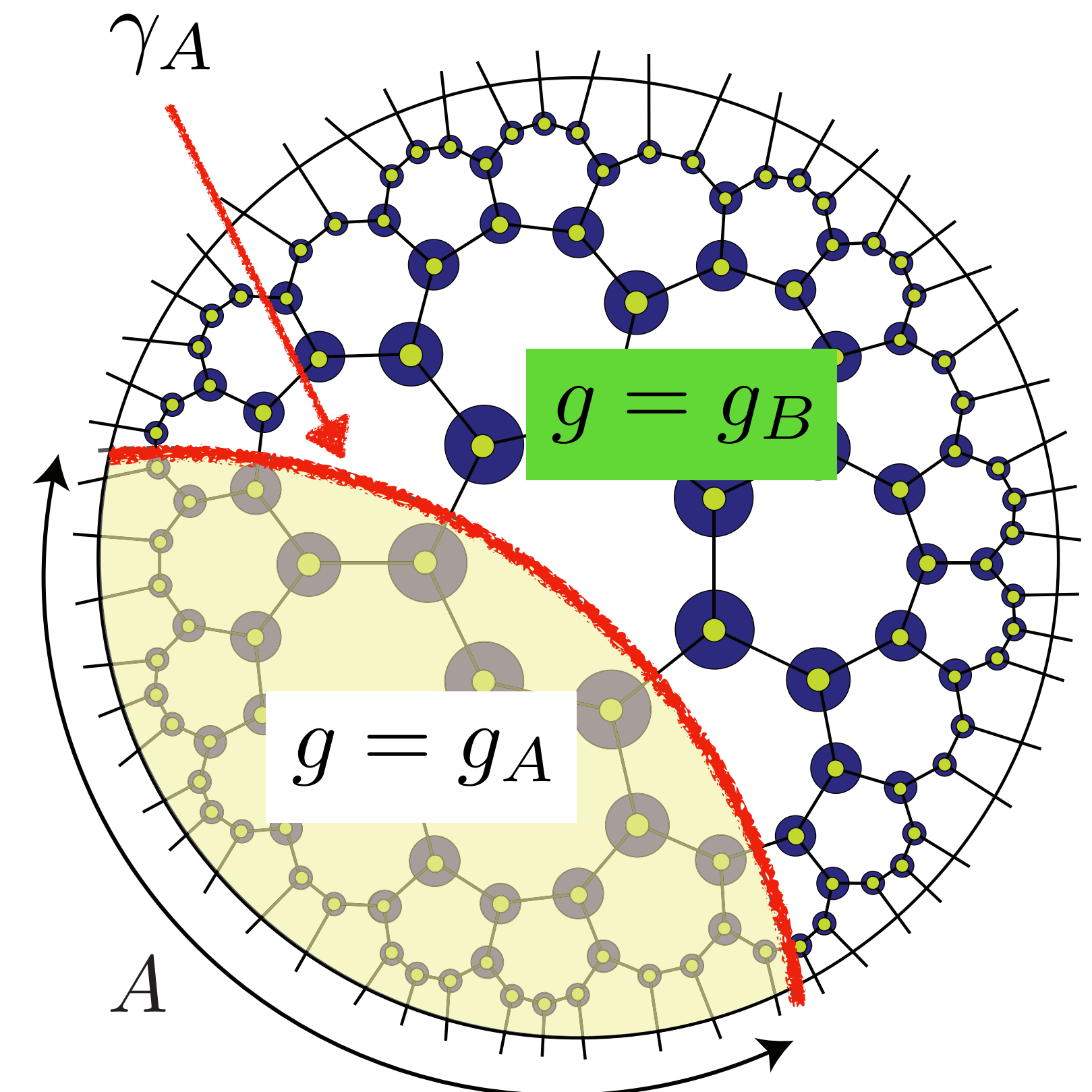
- The Renyi entropy is given by minimizing the area

$$S_n(A) = \min_{\gamma_A \sim A} \text{Area}(\gamma_A)$$

- $S_n(A)$ is the same for all Renyi index n : flat spectra

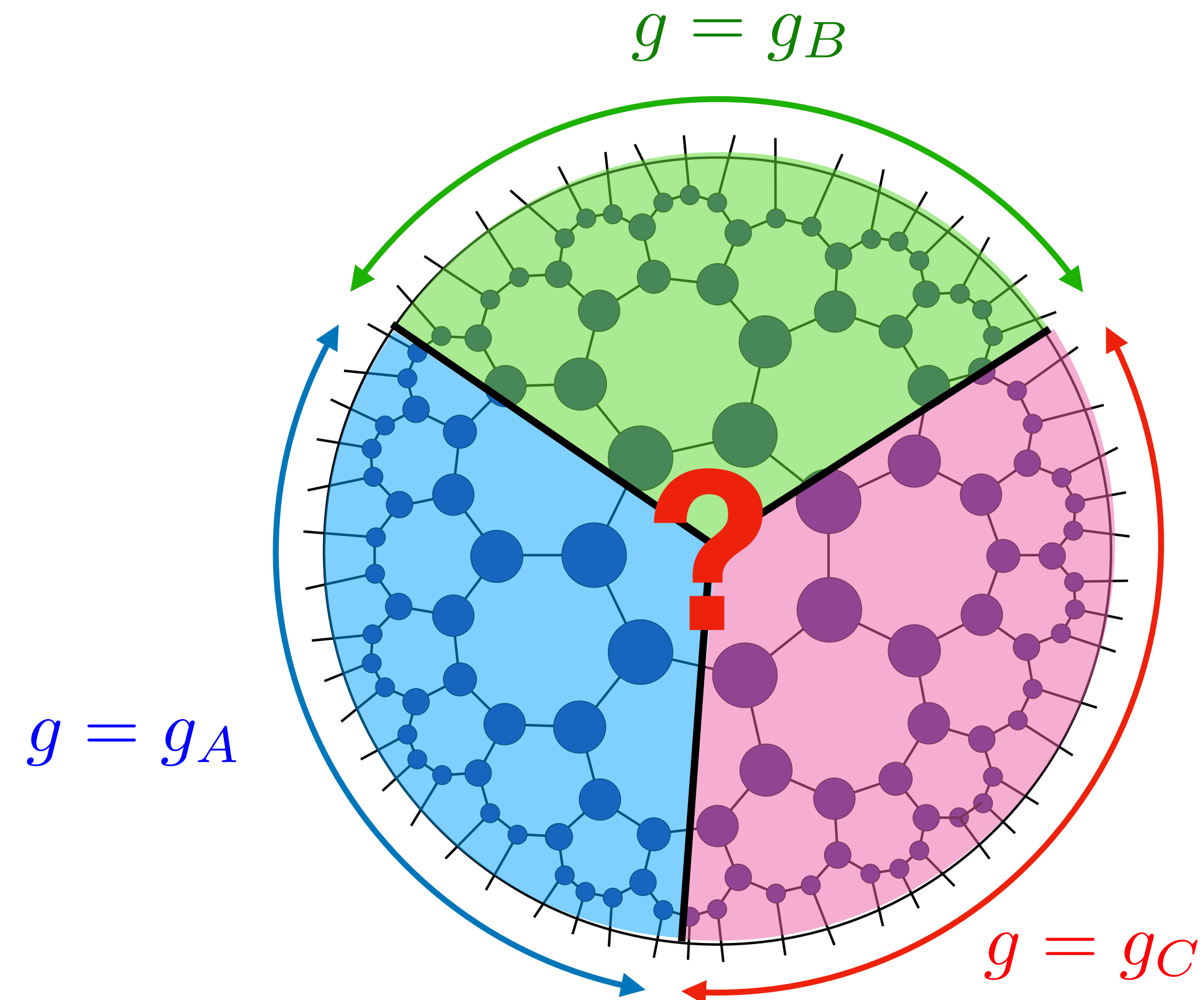
Can be proven rigorously using
max-flow/min-cut theorem

[Collins-Nechita-Zyczkowski '10]



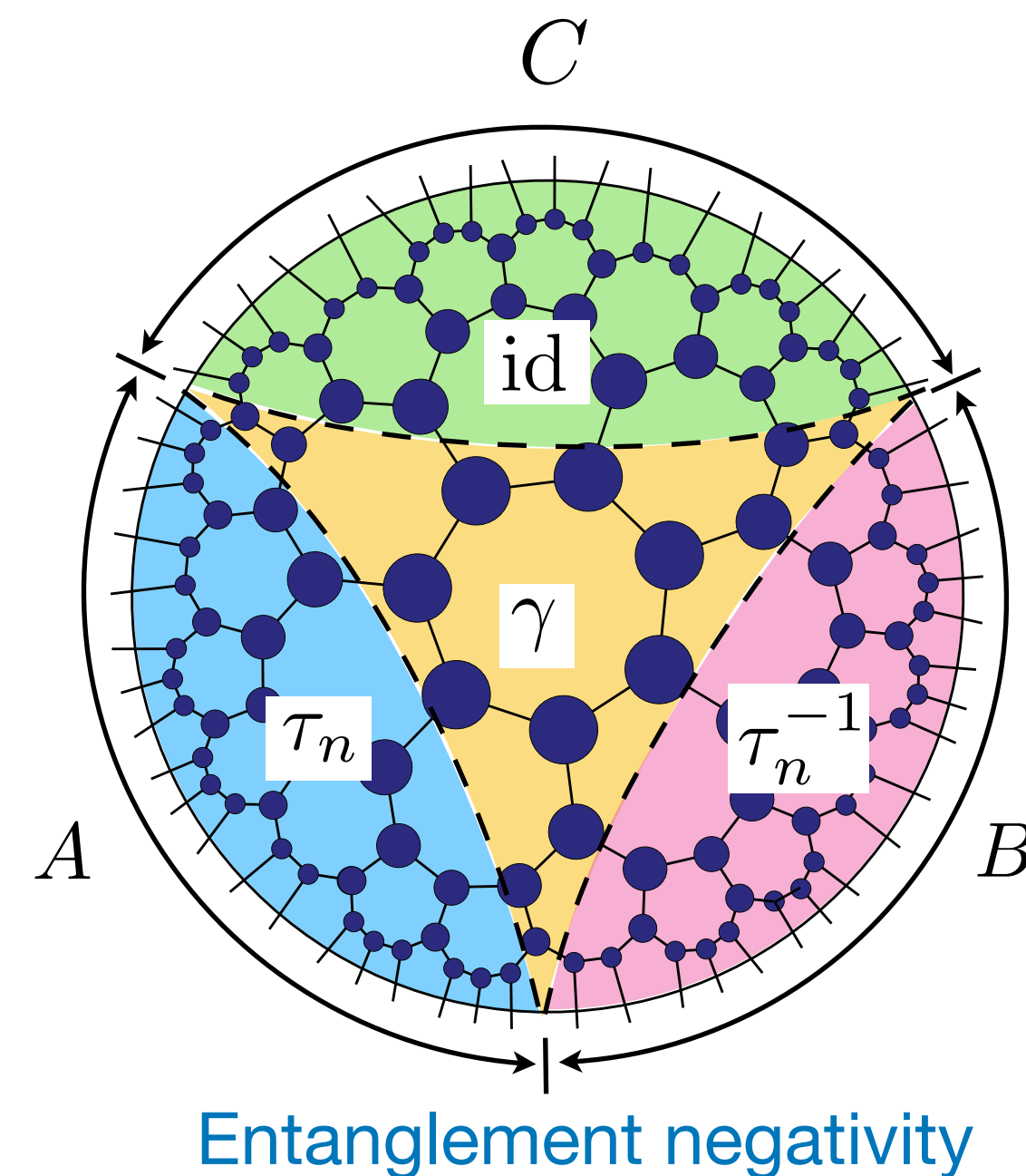
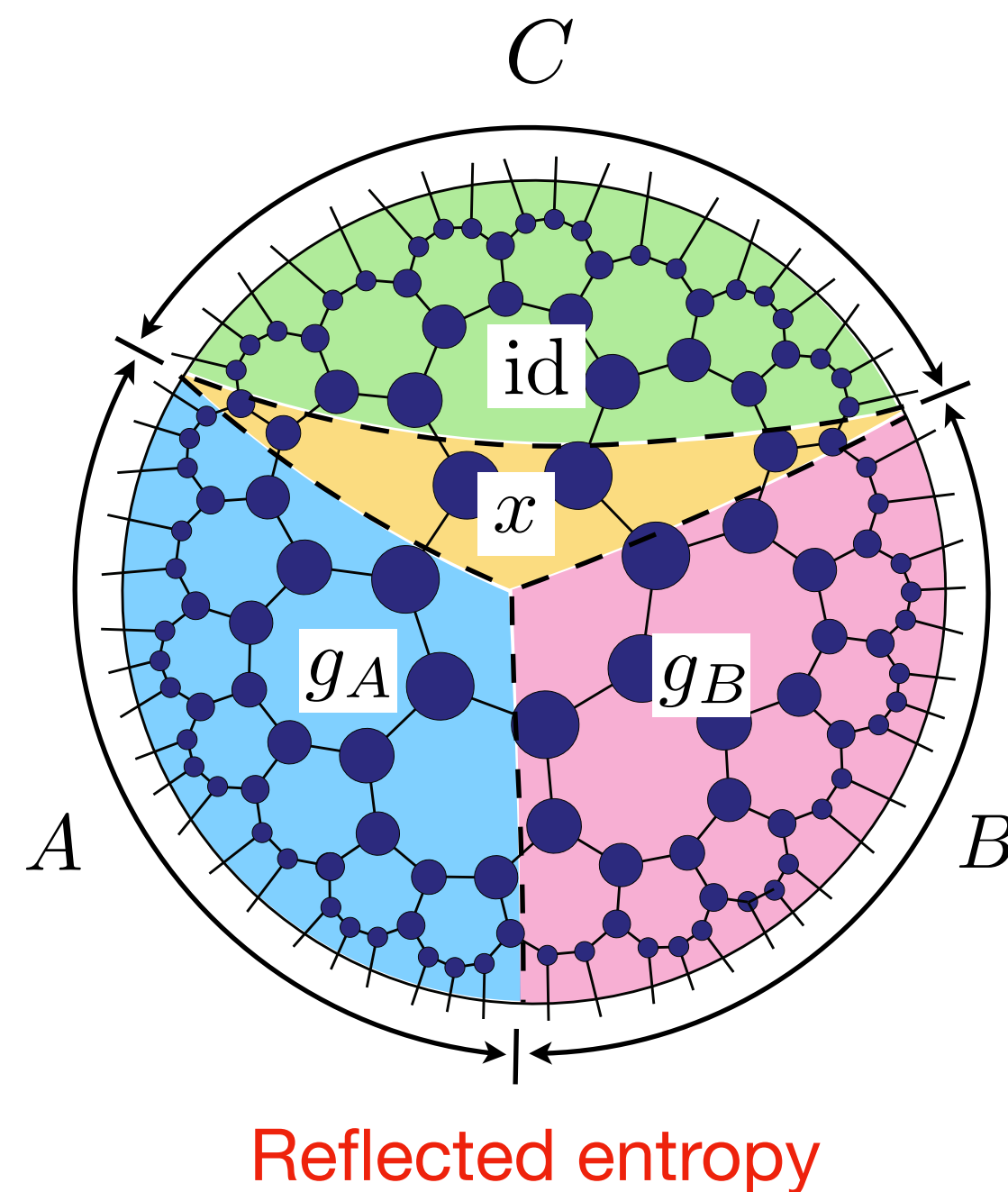
Multi-entropy in RTN

- Impose different boundary conditions according to twist operators $\{g_i\}$ of $S_n^{(q)}$
- Candidate saddle: domain wall $\leftrightarrow S_n^{(q)}$ determined by **minimal multi-way cut**
- However there might be other more dominant saddles known as **Replica Symmetry Breaking** (RSB) saddles



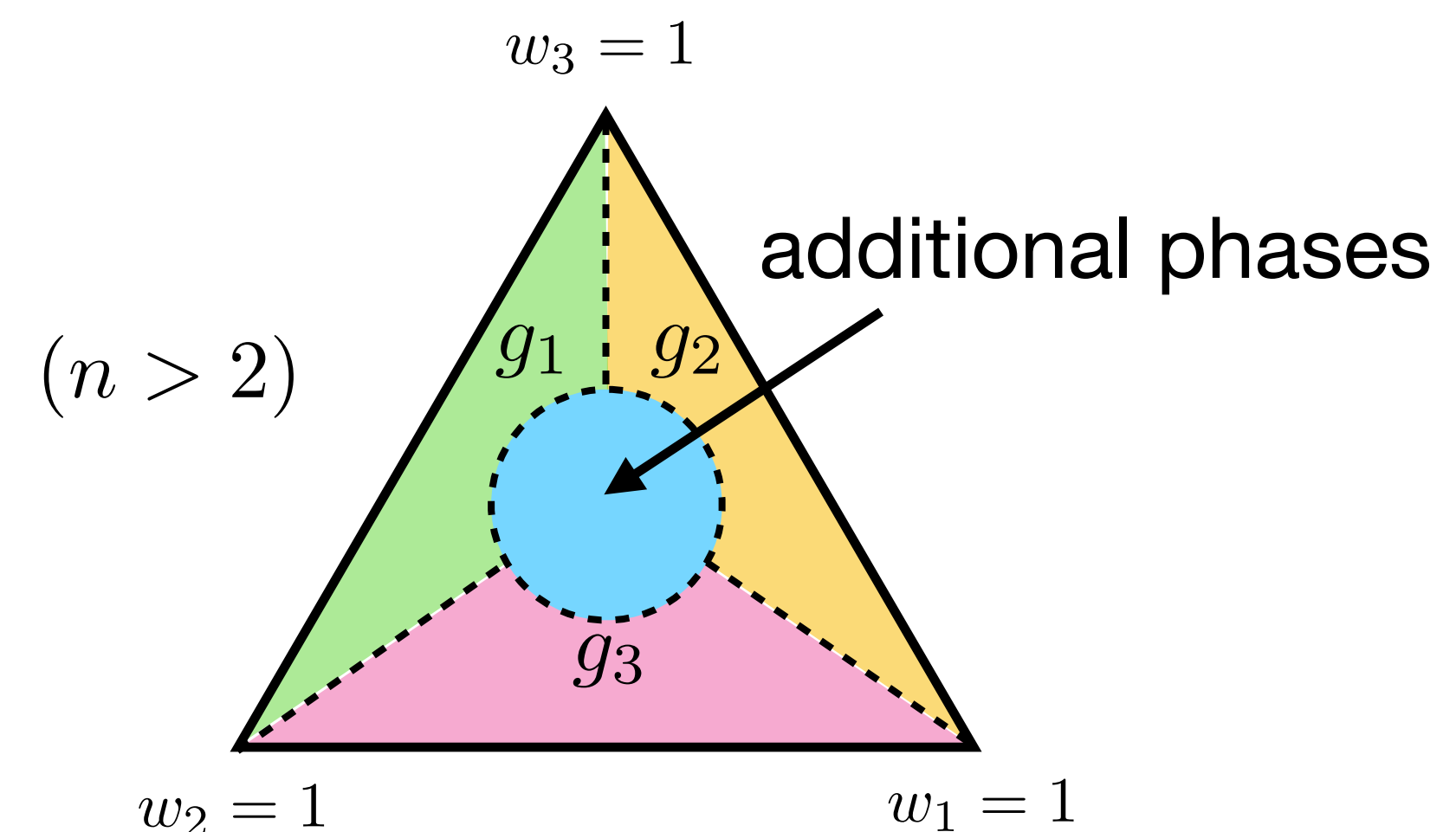
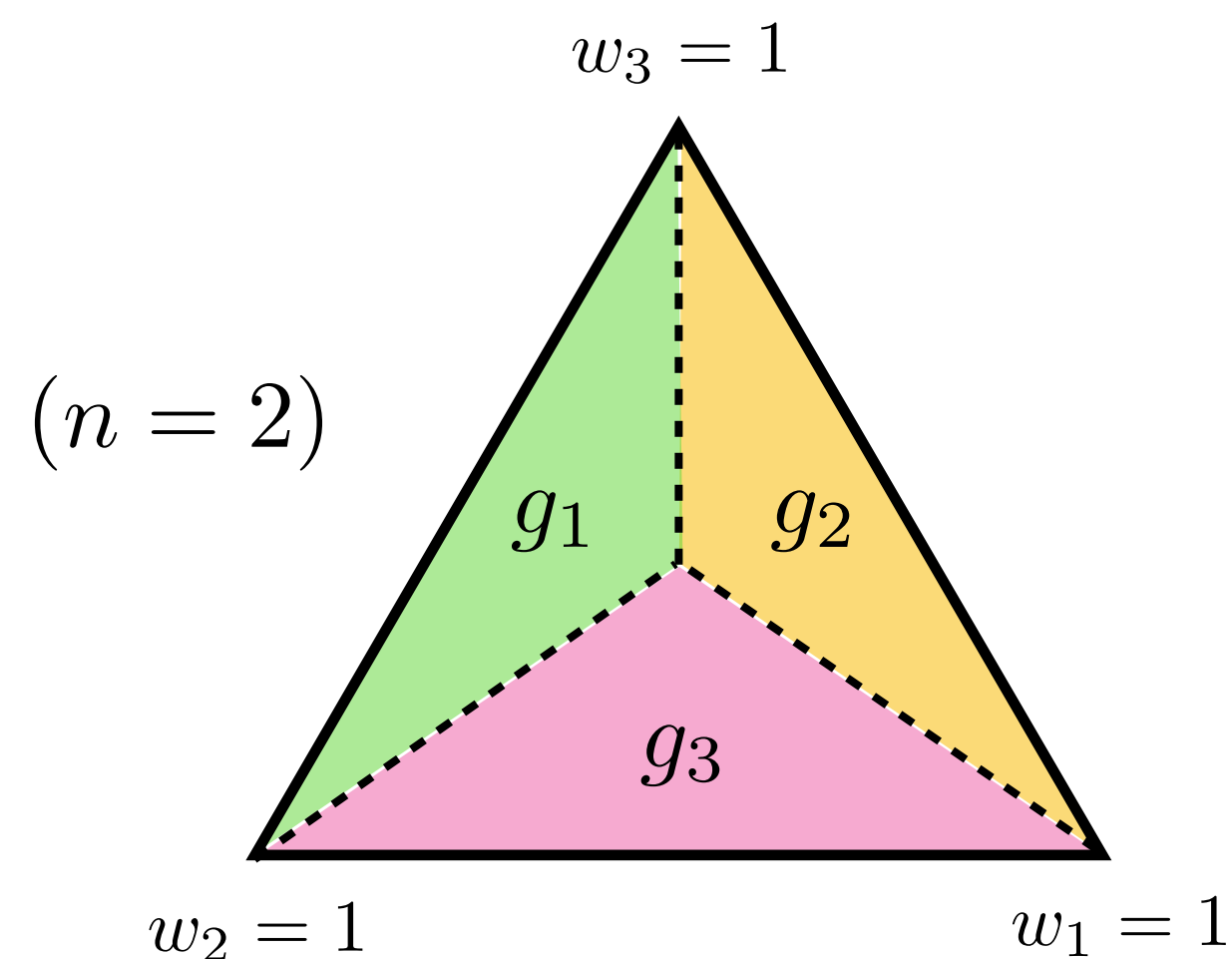
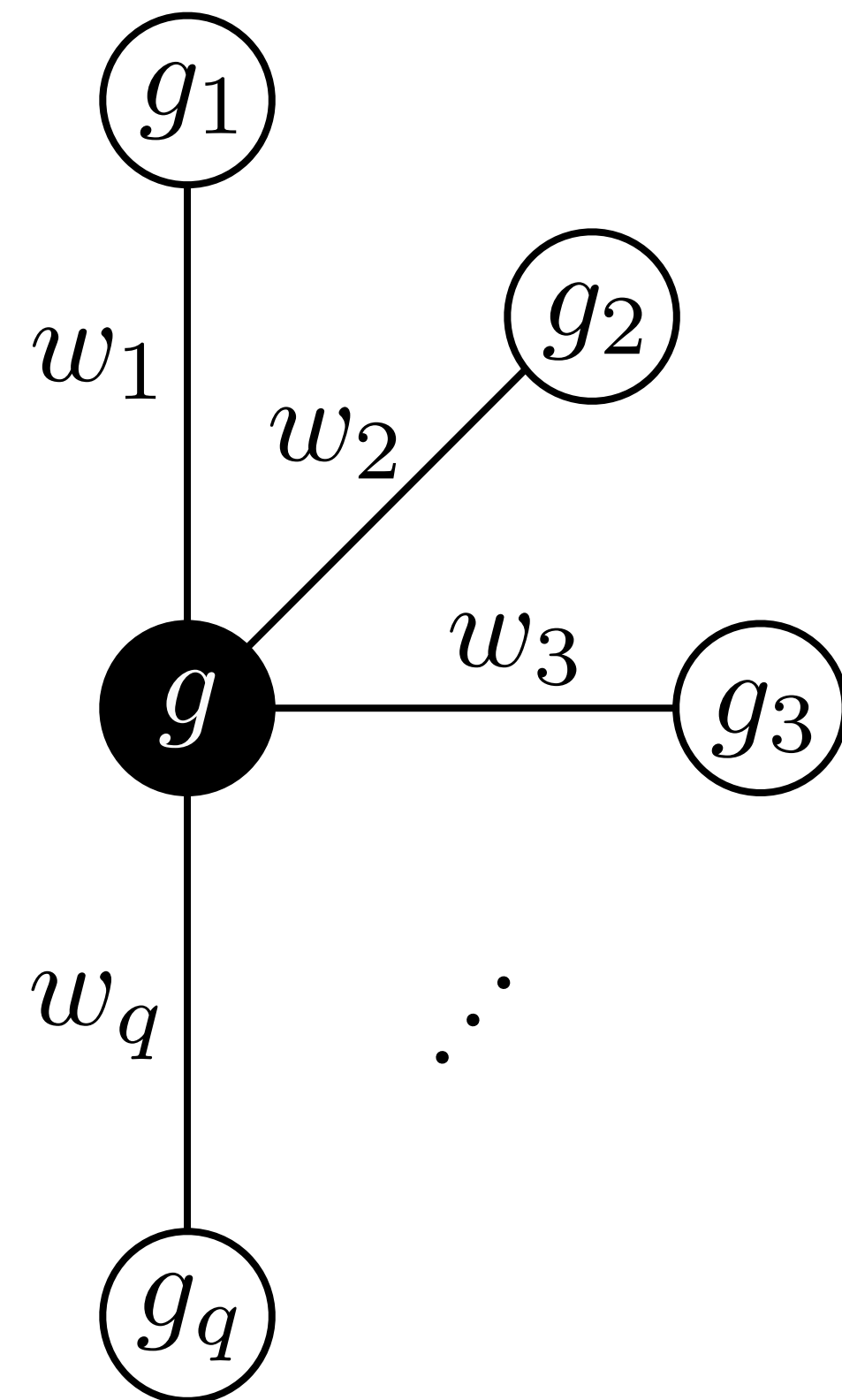
RSB saddles

- RSB saddles are known to exist for other multipartite entanglement measures such as **reflected entropy** and **entanglement negativity**
- A structure theorem [\[Akella-Iizuka '26\]](#) forbids RSB saddles of a certain type but not in general



Single random tensor [Hu-SL-Nechita '26]

- q -partite Haar-random state
- We proved:
 - No RSB for $n = 2$, any integer q
 - RSB saddle exists for all $(n, q) \geq (3, 3)$
- Example phase diagrams (e.g. $q = 3$):



Proof strategy

- Bound Cayley distance by the number of moved points in S_n : $d(g, e) \geq \frac{m(g)}{2}$
- Plug into the energy functional:

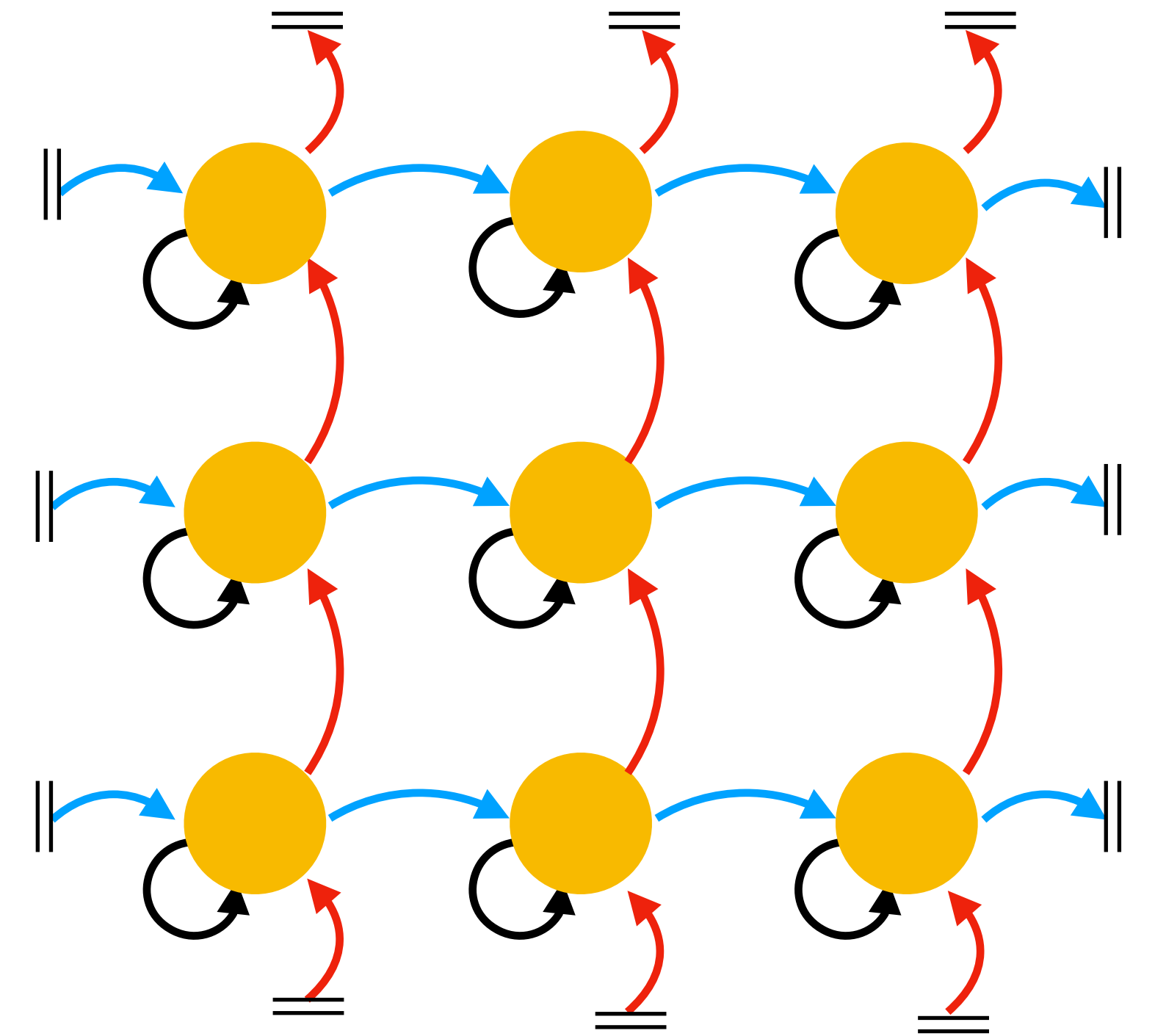
$$\sum_i d(g, g_i) \geq \frac{1}{2} \sum_i m(g^{-1}g_i) = \frac{1}{2} \sum_i |\{x \in X : g(x) \neq g_i(x)\}|$$

- g_i are pairwise distinct: Therefore $g(x)$ can only coincide with $g_i(x)$ for at most one i .

$$\sum_i d(g, g_i) \geq (q-1) \frac{|X|}{2} = (q-1)2^{q-2}$$

- Verify that $g = g_i$ saturates the bound

- Key property used: The set of g_i is **pairwise distinct** and $g_i g_j^{-1}$ is an **involution**
- This is equivalent to the statement that $g_i g_j^{-1}$ is a **fixed-point-free involution**
- For $n \geq 3$ the involution condition is no-longer true and the proof fails
- Indeed we will see there are RSB saddles when $n \geq 3$

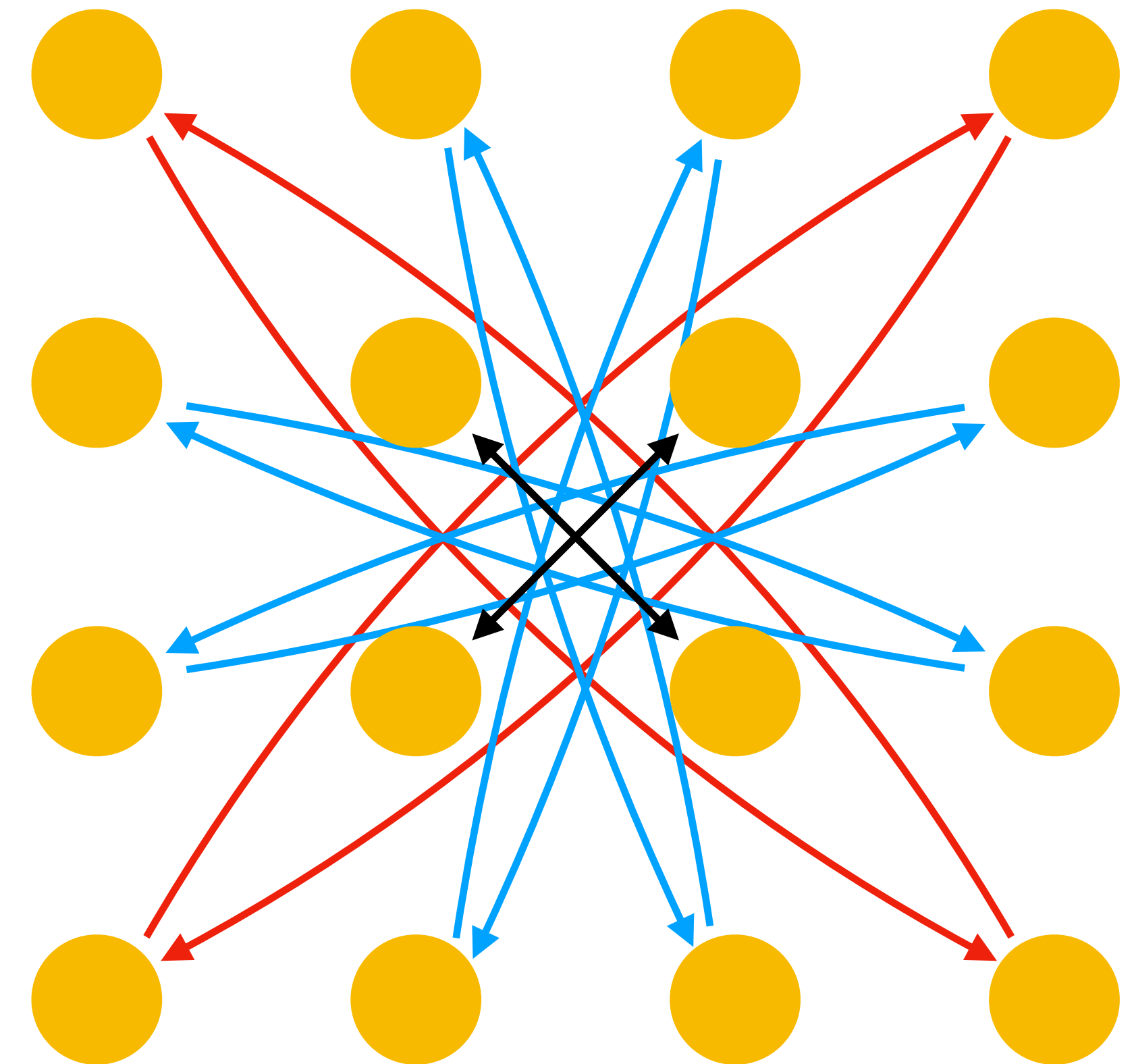


RSB solution for $n > 3$

- Consider the full reflector π where

$$\pi(i_1, \dots, i_q) = (-i_1, \dots, -i_q)$$

- More dominant* than the replica symmetric saddle for $(q, n) \geq (3, 3)$ for the equal bond-dimension case $w_1 = w_2 = \dots = w_q$
- Conjecture:** π is optimal for $(q, n) \geq (3, 3)$
 - Verified numerically up to $(q, n) = (4, 3)$ or $(2, 5)$

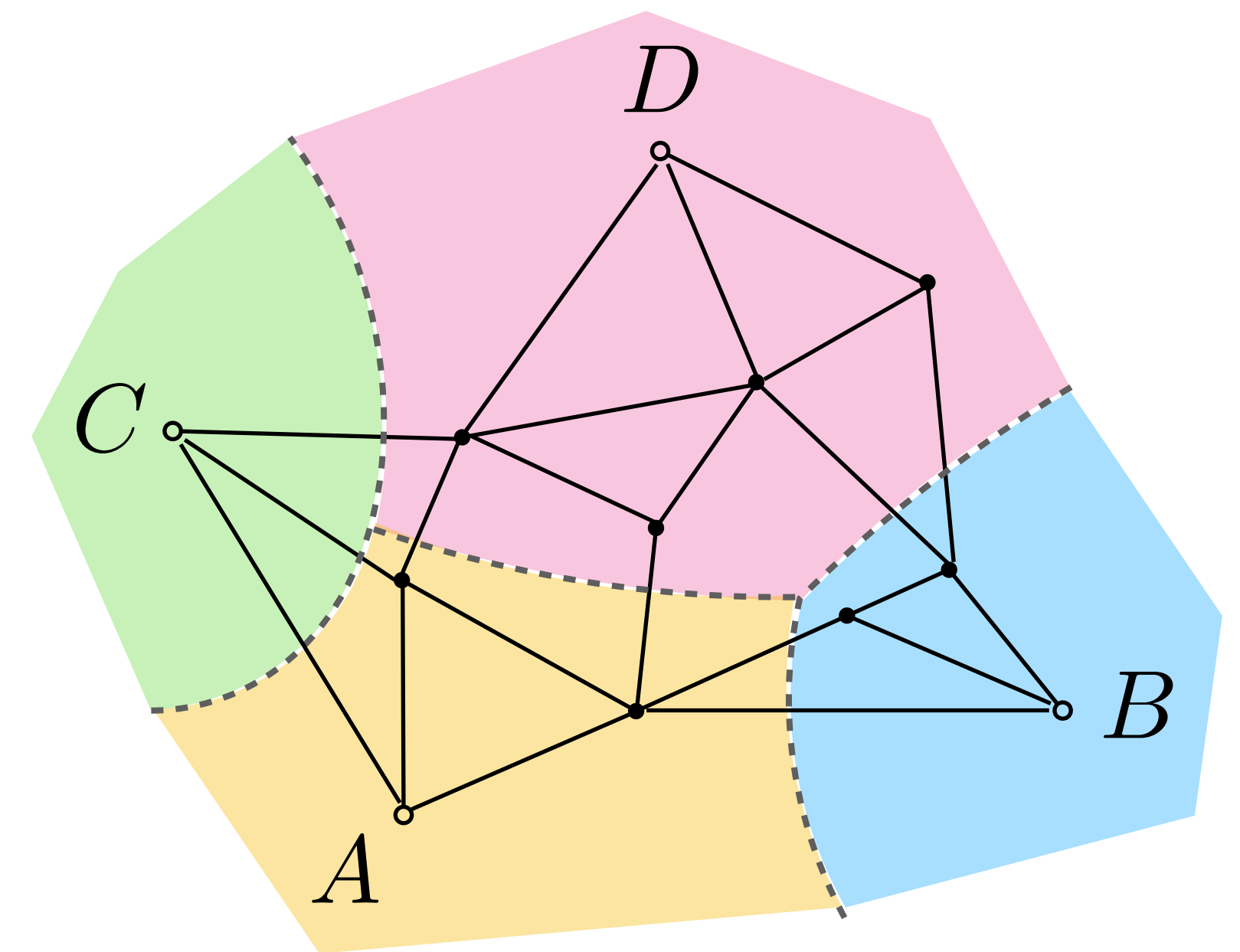


Generalization to RTN

- Same result generalizes to RTNs: dominant saddles are replica-symmetric for $n = 2$
- $n=2$ Renyi multi-entropy is given by the **minimal multi-way cut** across the network

$$S_2^{(q)}(R_1 : \cdots : R_q) = \frac{A(\gamma_{R_1 : \cdots : R_q})}{4G_N}$$

- Dominant saddle is unique if the minimal multi-way cut is unique
- Proof follows a similar strategy as in single tensor case

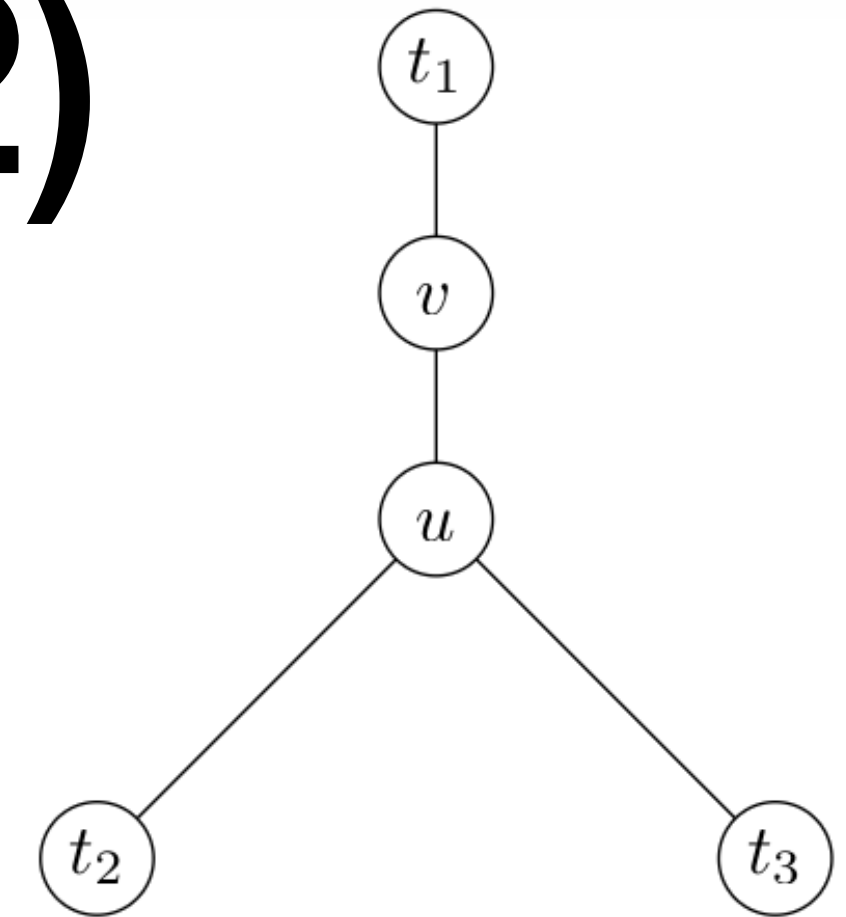


minimizer counting (n=2)

- If the minimal multiway cut is not unique, there might be more than one solutions
- Idea: assign each replica x a minimal partition λ_x . Construct $g_v(x) = x + e_{\lambda_x(v)}$ then g_v is optimal if and only if:

(V): for every vertex v the map $x \mapsto x + e_{\lambda_x(v)}$ is a bijection

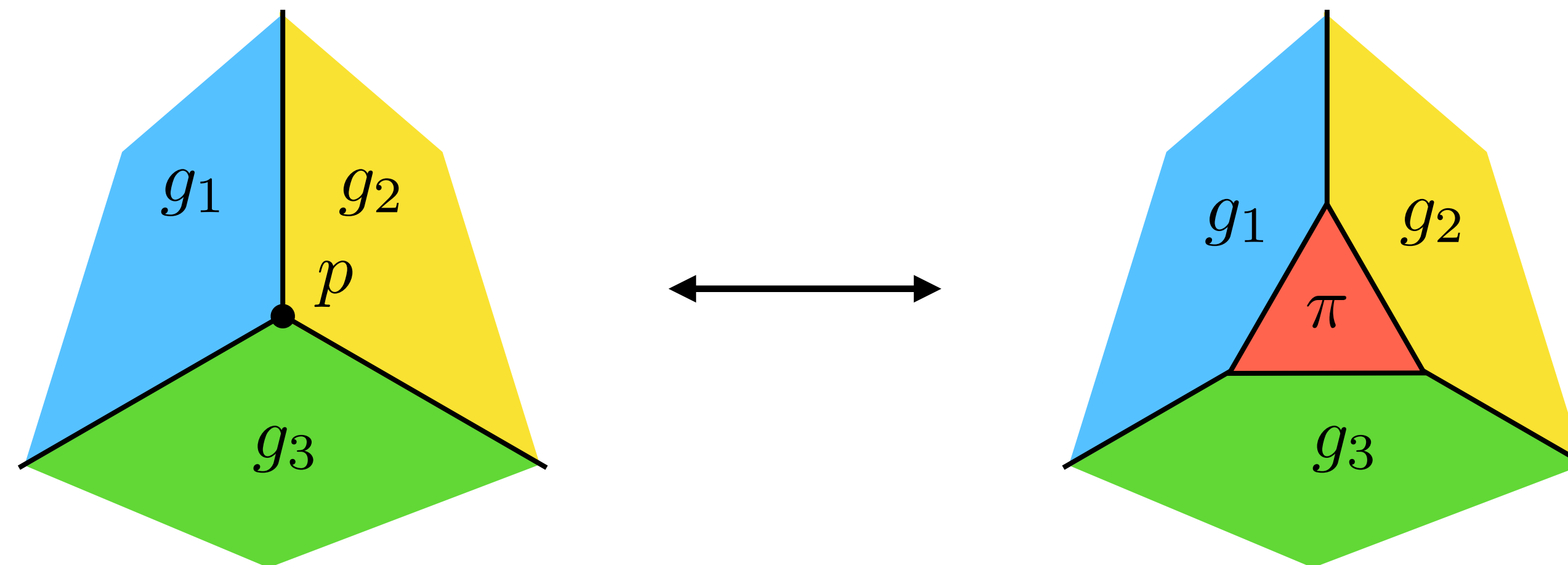
(E): for every edge (uv) , then $\lambda_{x+e_i+e_j}(u) = i, \lambda_{x+e_i+e_j}(v) = j$, where $i := \lambda_x(u), j := \lambda_x(v)$.



$g(u)$	$g(v)$
g_1	g_1
g_2	g_1
g_2	(00 10 11 01)
g_2	(00 01 11 10)
g_2	g_2
g_3	g_1
g_3	(00 10)
g_3	(01 11)
g_3	g_3

RSB in isometric RTN

- For RTN modeling metric spaces, can construct local RSB solutions from the reflector π
- If the metric is flat, RSB saddles take over when $n \geq 6$
- For hyperbolic spaces, the size of the π patch is controlled by the local curvature

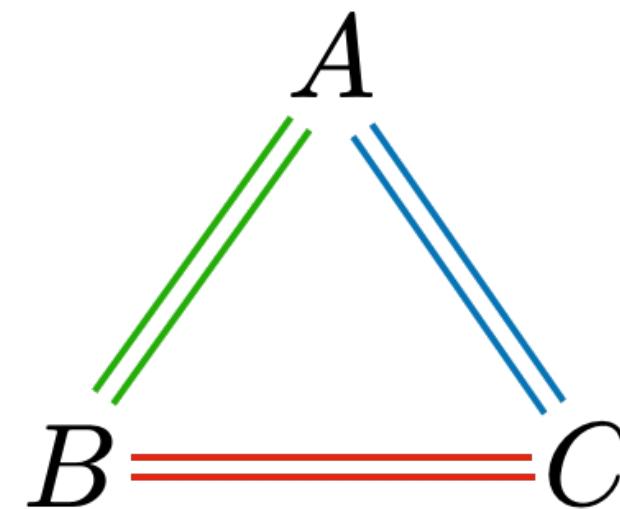


Genuine Multi-partite entanglement in RTN

- Our result can be used to show that RTN states have large genuine multi-partite entanglement
- Genuine q-partite entangled: cannot be written as products of genuine k-partite entangled states for $k < q$

- example: the W state $|W\rangle = \frac{1}{3}(|001\rangle + |010\rangle + |100\rangle)$

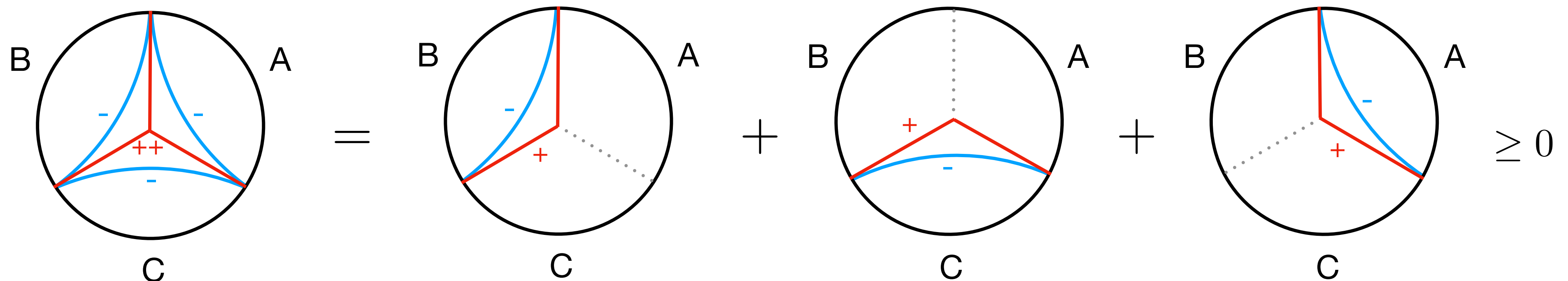
- non-example: the triangle state



Genuine Multi-partite entanglement in RTN

- **Genuine multi-entropy** $\text{GM}_n^{(q)}$ [Iizuka-Nishida '26, Iizuka-**SL**-Nishida '26]
 - Linear combinations of $S_k^{(q)}$ of $k \leq n$

$$\text{Ex. } \text{GM}_n^{(3)}(A : B : C) = S_n^{(3)}(A : B : C) - \frac{1}{2}(S_n^{(2)}(A) + S_n^{(2)}(B) + S_n^{(2)}(C))$$
 - Detects genuine multipartite entanglement:
$$\text{GM}_n^{(q)} \neq 0 \Rightarrow |\psi\rangle \text{ is genuine } q\text{-partite entangled}$$
- Since $n=2$ case is dual to the multi-way cut, can prove $\text{GM}_2^{(q)} \neq 0$ geometrically



Genuine Multipartite entanglement in holography

- For actual AdS/CFT states, we now have one more piece of evidence that the multiway cut conjecture is valid at least for $n = 2$ (and false for $n \geq 3$)
- There is a version of minimal surface dual without the need to continue $n \rightarrow 1$. However the dual picture is a bulk with backreacted cosmic branes [\[Dong '16\]](#)
- We still expect that the same result $\text{GM}_2^{(q)} \neq 0$ after replacing the probe brane after allowing backreaction
 - This is supported by direct CFT calculations, see [\[2401.04236 Harper-Takayanagi-Tsuda\]](#)

Building multi-invariants with multiway-cut property

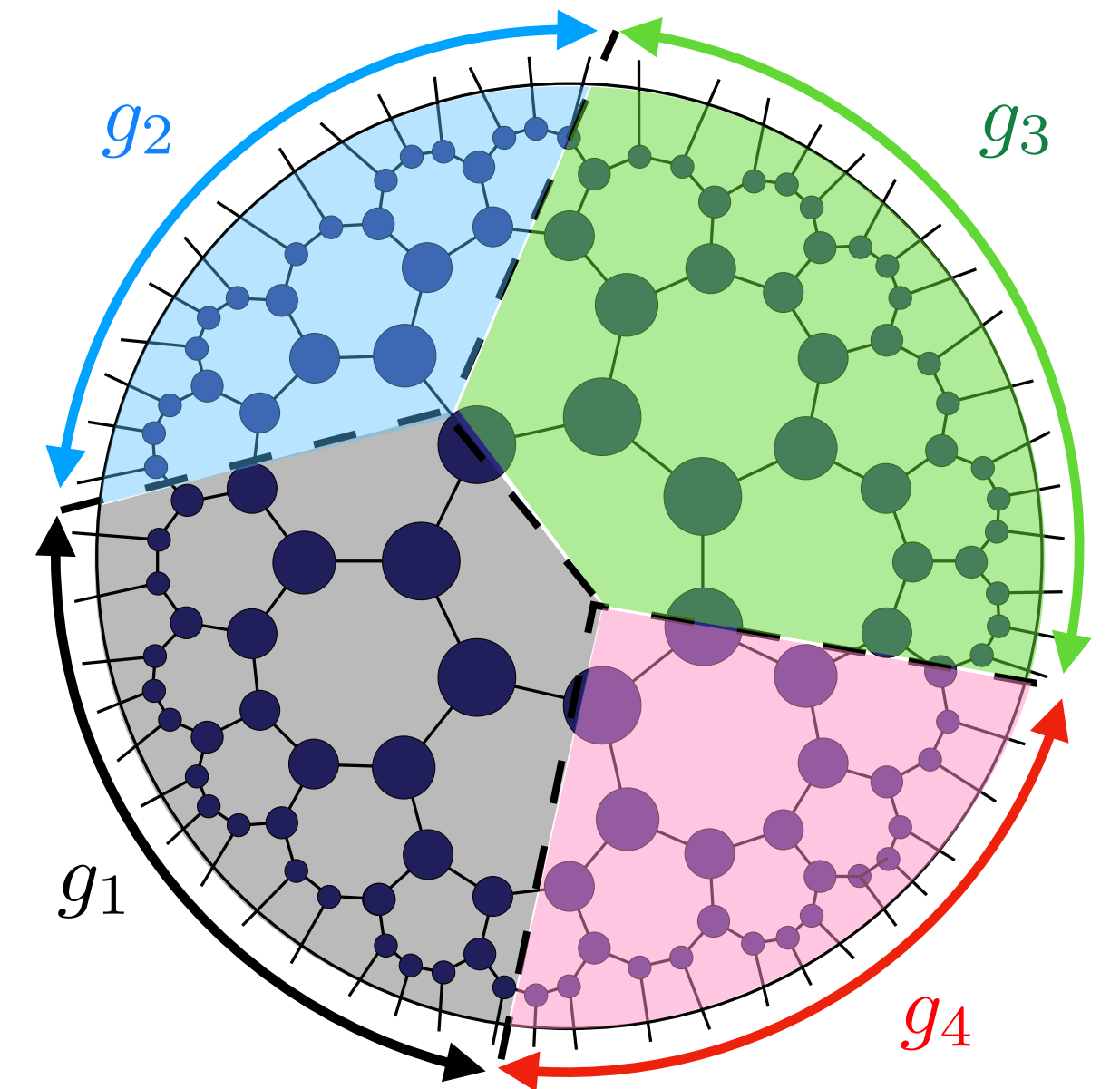
- Our proof also applies for any multi trace invariants as long as its twist operators $\{g_i\}$ satisfies the **fixed-point-free involution** property

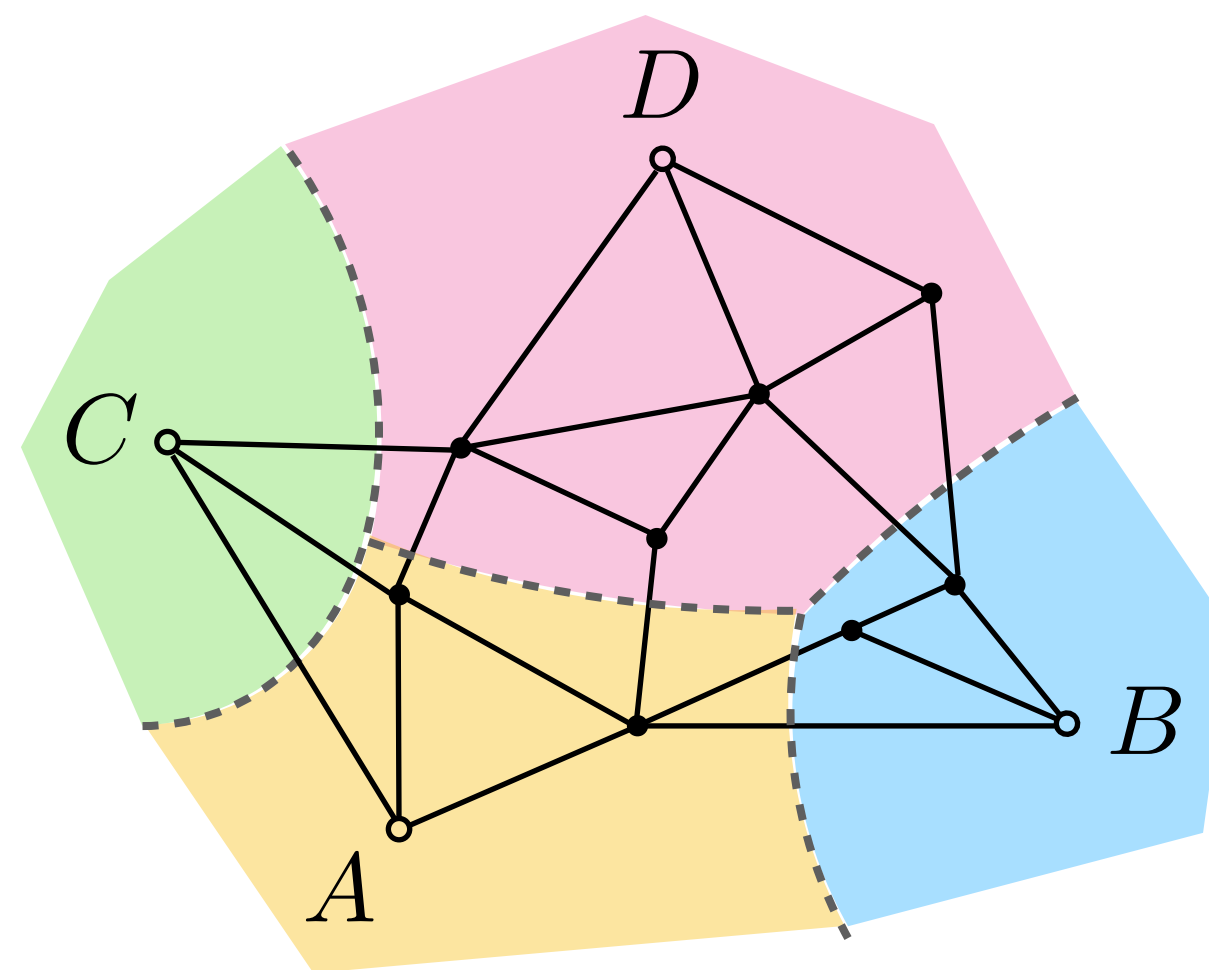
$$g_i g_j^{-1}(x) \neq x, \quad (g_i g_j^{-1})^2 = \text{id}$$

- An example: Consider the length-2 hypercube $\mathbb{Z}_2^{q-1}$. The set of all translations $g_y(x) = x + y$, $y \in \mathbb{Z}_2^{q-1}$ is a size 2^{q-1} family and satisfies the property
- The multi-entropy twist operators correspond to a subfamily of size q .

Summary

- Multi-entropy $S^{(q)}$ is conjectured to be dual to the area of minimal multiway-cut in holographic states
- To investigate this claim, we study the Renyi multi-entropy $S_n^{(q)}$ in RTNs, a holographic tensor network model
 - For $n = 2$, we proved that $S_2^{(q)}$ is dual to the area of minimal multi-way cuts for RTN states
 - For $n \geq 3$, we give explicit counter-examples for the conjecture: evidence of replica symmetry breaking
- The claim that holographic CFT states have large genuine multipartite entanglement is still likely true despite the failure of the conjecture





Thank you