

S-MATRIX CONSISTENCY / GAUGE THEORY + GRAVITY

Bootstrapping the Heterotic String

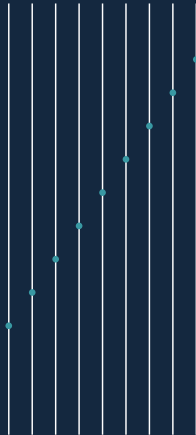
Tree-level completion of maximal SYM with gravity

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Work in progress ■ September 2026

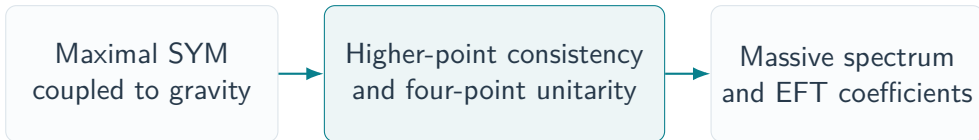


String amplitudes: from islands to uniqueness

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|--|-------------------------------------|---|
| <ul style="list-style-type: none"> • Aug 2020 Monodromy + positivity
Huang–Liu–Rodina–Wang [2008.02293]
Monodromy plane $\cap$ EFT-hedron:
shrinking open-superstring islands. • Oct 2023 Precision + flattening
Chiang–Huang–Weng [2310.10710]
Berman–Elvang–Herderschee [2310.10729]
Sharper monodromy-bootstrap isolation. • Jun 2024 Low-mass spectral input
Berman–Elvang [2406.03543]
Gap + string-valued scalar coupling ratio:
a small Veneziano island. • Jun 2025 Five-point splitting
Berman–Elvang–Figueiredo [2506.22538]
Hidden zeros + splitting; no gap-level IST:
shrinking string beta-function island. | <p>N</p> <p>N</p> <p>N</p> <p>N</p> | <ul style="list-style-type: none"> • Jan 2026 Six-point SUSY IR-only
Elvang–Herderschee–Morales [2601.11705]
IR/EFT input only: SUSY + factorization +
scalar parity + positivity. • May 2026 Analytic bootstrap A
Wan–Zhou [2605.11084]
Monodromy or splitting + moment positivity:
Veneziano uniquely fixed. • Jul 2026 Closed-string selection IR-only
Berman et al. [2607.14230]
IR/EFT input only: SUSY + factorization +
peculiar parity + positivity $\Rightarrow$ two corners. • Sep 2026 Six-point uniqueness A
Ilmo Sung [2609.29919]
Ward identities + low-order relation + positivity:
Veneziano fixed in a meromorphic class. |
|--|-------------------------------------|---|

How much of string theory is consistency?

Start with the massless theory, not a worldsheet.



Can the gauge and gravitational sectors *together* select the heterotic amplitude?

Recent progress

Open and closed strings from nonlinear consistency.

This work

Color-dressed constraints that couple the two sectors.

Analytic outlook

What actually turns a constraint surface into uniqueness?

The massless starting point

Sixteen supercharges

$$\mathcal{N} = 4 \text{ SYM} + \mathcal{N} = 4 \text{ supergravity}$$

Vector multiplet G	adjoint color
Gravitational multiplets $H^\pm$	color singlets
Cubic couplings	g, κ

Four-dimensional on-shell superspace organizes the Ward identities. The bootstrap keeps the *full color-dressed* amplitude.

What is being varied?

A tower of higher-derivative interactions and an unspecified massive completion.

What is being assumed?

Tree-level locality and factorization; supersymmetry; a restricted scalar-contact parity; dispersive positivity with suitable UV behavior.

The gravitational sector here is *half-maximal*, not the $\mathcal{N} = 8$ theory of the closed-string review.

Four-point data: two functions, two exchanges

$$\mathcal{A}_4 = \delta^{(8)}(\tilde{Q}) \frac{[34]^2}{\langle 12 \rangle^2} \mathcal{F}_4, \quad s + t + u = 0.$$

$$\begin{aligned} \mathcal{F}_4 = & \sum_{\text{3 orderings}} \text{tr}(1234) F^{\text{ST}}(s, t) \\ & + \sum_{\text{3 pairings}} \text{tr}(12) \text{tr}(34) F^{\text{DT}}(s, t). \end{aligned}$$

Single trace: gauge exchange

$$F^{\text{ST}} = -\frac{g^2}{st} + \sum_{k,q} a_{k,q} s^{k-q} t^q$$

$$a_{k,q} = a_{k,k-q}$$

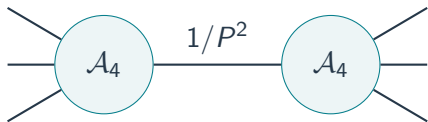
Double trace: gravitational exchange

$$F^{\text{DT}} = -\frac{\kappa^2}{s} + \sum_{\substack{k,q \\ q \text{ even}}} b_{k,q} s^{k-q} (t-u)^q$$

$$F^{\text{DT}}(s, t) = F^{\text{DT}}(s, u)$$

Two complementary consistency problems

Higher points: nonlinear equalities



$$\mathcal{A}_6 \sim \frac{\mathcal{A}_4 \mathcal{A}_4}{P^2} + \dots$$

Ward identities test whether residues can coexist with allowed contact terms.

Four points: positive spectral data

$$\rho_{J,R}(\mu^2) \geq 0$$

Dispersion relations map positive massive exchanges into EFT coefficients.

The intersection

A nonlinear consistency surface
 $\cap$ a dispersively allowed region.

Neither four-point supersymmetry nor positivity alone contains the whole story.

Elvang I: six-point consistency is nonlinear

Planar $\mathcal{N} = 4$ SYM EFT, with the scalar-parity restriction

$$F(s, t) = -\frac{g^2}{st} + a_{0,0} + a_{1,0}(s + t) + \cdots$$

Representative constraints

$$g^2 a_{2,0} = \frac{2}{5} a_{0,0}^2,$$

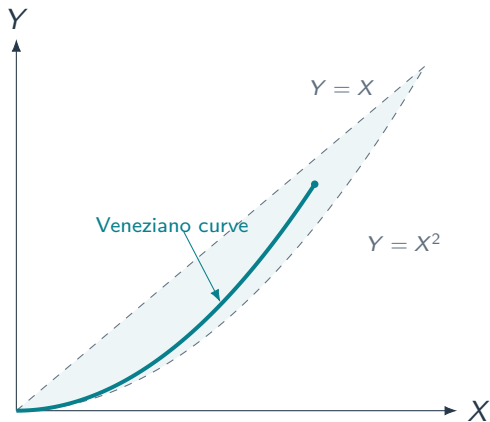
$$g^2 a_{2,1} = \frac{1}{10} a_{0,0}^2,$$

$$g^2 a_{3,1} = 2g^2 a_{3,0} - a_{1,0} a_{0,0}.$$

Four-point coefficients that looked independent become correlated by a six-point completion.

Factorization supplies the products; the Ward identities constrain them.

Elvang I: positivity drives convergence to Veneziano



Analytic reference region; not numerical bootstrap data

Gap units: $X = a_{1,0}/a_{0,0}$,
 $Y = a_{2,0}/a_{0,0}$.

$$0 \leq X^2 \leq Y \leq X \leq 1.$$

Adding the nonlinear constraints

At $k_{\max} = 7$, the reported width in X at fixed Y is below 5×10^{-4} .

$$Y = \frac{\zeta_4 \zeta_2}{\zeta_3^2} X^2$$

Numerical evidence for Veneziano, up to scale; not yet an analytic uniqueness theorem.

The extra input: “peculiar parity”

A generic scalar contact ansatz

$$\text{CT}_6 = P(s_{ij}) + \sum_I \epsilon_I Q_I(s_{ij}),$$

$$\epsilon_{ijkl} = 4i\epsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma.$$

The parity-odd terms can absorb otherwise nontrivial Ward constraints.

The restriction used here

$$\text{CT}_6 = P(s_{ij})$$

for the selected scalar components in each analyzed color sector.

In the planar gauge analysis, scalar-sector $SO(6) \rightarrow O(6)$ gives a useful interpretation.

This is an additional scalar-sector assumption. It is not a consequence of maximal supersymmetry, and is not ordinary parity imposed indiscriminately on all states.

Elvang II: gravity retains an infinite-spin alternative

$\mathcal{N} = 8$ supergravity EFT; $SU(4) \times SU(4)$ and selected scalar parity

$$F_{\text{grav}}(s, t, u) = \frac{\kappa^2}{stu} \mathcal{V}(s, t, u).$$

Virasoro–Shapiro

$$\mathcal{V}_{\text{VS}} = \prod_{v=s,t,u} \frac{\Gamma(1 - \alpha' v)}{\Gamma(1 + \alpha' v)}$$

Infinitely many mass levels; finitely many spins at each level.

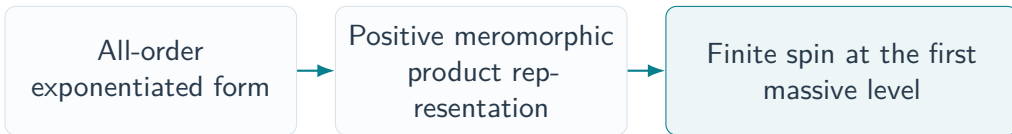
Infinite spin tower (IST)

$$\mathcal{V}_{\text{IST}} = \prod_{v=s,t,u} \frac{m^2 + v}{m^2 - v}$$

One massive level; a rational residue carrying infinitely many spins.

The numerical allowed domain is nonconvex, with VS and IST corners.

Elvang II: what selects the string spectrum?



$$\begin{aligned}
 &\text{Polynomial residue} \implies \text{pole cancellations} \\
 \implies & m_n^2 = n m_1^2 \implies \text{Virasoro-Shapiro.}
 \end{aligned}$$

Conditional analytic result: the exponentiated form was checked to finite order; its all-order extension is assumed in this argument.

The mixed question: does coupling gauge and gravity supply additional restrictions that distinguish heterotic strings from IST-type alternatives?

Color separates three kinds of nonlinear information

Six-point trace sector	Three-particle residue	Data constrained
$\text{tr}(T^6)$	$F^{\text{ST}} \times F^{\text{ST}}$	$a \times a$
$\text{tr}(T^2) \text{tr}(T^4)$	$F^{\text{ST}} \times F^{\text{DT}}$	$a \times b$
$\text{tr}(T^2) \text{tr}(T^2) \text{tr}(T^2)$	$F^{\text{DT}} \times F^{\text{DT}}$	$b \times b$

The $\text{tr}(T^3) \text{tr}(T^3)$ sector is not included in the present constraint system.

The new ingredient is cross-sector consistency, not two independent bootstraps.

How the six-point constraints are generated

Five independent NMHV basis functions

$$\mathcal{A}_6^{\text{NMHV}} = \sum_{I=1}^5 A_I X_I.$$

Eight scalar components in a trace-preserving orbit give an overdetermined projection:

$$\mathbf{Z} = \mathbf{C}\mathbf{A}, \quad w\mathbf{C} = 0 \Rightarrow w\mathbf{Z} = 0.$$

Sewing before elimination

$$\mathbf{Z} = \mathbf{Z}_{4 \times 4} + \mathbf{Z}_{3 \times 5} + \mathbf{CT}_6.$$

Match three- and two-particle residues; allow the permitted five-point freedom and parity-even six-point contacts.

Eliminate this higher-point freedom order by order.

A linear Ward problem becomes polynomial constraints on four-point Wilson coefficients.

The first mixed relation splits the problem

$$\kappa^2 a_{0,0} = 0$$

Open-string branch

$$\kappa = 0, \quad a_{0,0} \neq 0.$$

The mixed system sets $b_{k,q} = 0$ through the retained orders: return to the planar problem.

Gravitational / heterotic branch

$$\kappa \neq 0, \quad a_{0,0} = 0.$$

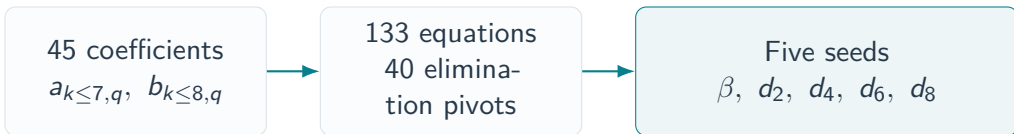
Gauge and gravitational contact data must turn on in a correlated way.

$$\kappa^2 a_{1,0} = 4g^2 b_{2,2}, \quad -\kappa^2 b_{1,0} = b_{0,0}^2.$$

The branch names do not themselves establish a string-theory UV completion.

Main finite-order result: five free coefficients

$$\hat{a}_{k,q} = \frac{a_{k,q}}{g^2}, \quad \hat{b}_{k,q} = \frac{b_{k,q}}{\kappa^2}, \quad g\kappa \neq 0.$$



$$\beta = \hat{b}_{0,0}, \quad d_{2r} = \hat{b}_{2r,2r}.$$

$$\hat{a}_{1,0} = 4d_2,$$

$$\hat{a}_{3,0} = 16d_4,$$

$$\hat{a}_{4,1} = -8d_2^2,$$

$$\hat{b}_{1,0} = -\beta^2,$$

$$\hat{b}_{2,0} = \beta^3 - d_2,$$

$$\hat{b}_{3,2} = -\beta d_2.$$

Polynomial solution; no division by a seed; all retained equations vanish symbolically.

A common exponential organizes both trace sectors

Compact representation of the computed solution

$$F^{\text{ST}}(s, t) \doteq -\frac{g^2}{st} e^{E(s, t, u)}, \quad F^{\text{DT}}(s, t) \doteq -\frac{\kappa^2}{s(1 + \beta s)} e^{E(s, t, u)},$$

$$E(s, t, u) = \sum_{r \geq 1} \frac{4^r d_{2r}}{2r + 1} (s^{2r+1} + t^{2r+1} + u^{2r+1}).$$

Common dressing

The diagonal double-trace seeds fix all retained single-trace data.

Additional gravitational datum

β appears only in the double-trace prefactor, not in the single-trace solution.

Verified finite-order organization. The displayed infinite series suggests an all-order ansatz; it is not yet an all-order theorem.

Forward double-trace data form a geometric sequence

$$\sigma_k \equiv \sum_{q \text{ even}} \hat{b}_{k,q} = (-1)^k \beta^{k+1}, \quad 0 \leq k \leq 8.$$

$$H_{ij} = \sigma_{i+j} = \beta v_i v_j, \quad 0 \leq i, j \leq 4,$$

$$\mathbf{v} = (1, -\beta, \beta^2, -\beta^3, \beta^4)^\top.$$

$$E(s, 0, -s) = 0,$$

$$F^{\text{DT}}(s, 0) \doteq -\frac{\kappa^2}{s(1 + \beta s)}.$$

$\text{rank } H \leq 1$ is an exact algebraic statement about the retained coefficients.

These are trace-basis moments, not automatically positive spectral moments. Rank one does not imply one physical massive state.

The heterotic amplitude is a precise benchmark

$$V(s, t, u) = \prod_{v=s,t,u} \frac{\Gamma(1 - \alpha' v)}{\Gamma(1 + \alpha' v)},$$

$$F_{\text{het}}^{\text{ST}} = -\frac{g^2}{st} V, \quad F_{\text{het}}^{\text{DT}} = -\frac{\kappa^2}{s(1 + \alpha' s)} V.$$

Its location on the five-seed family

$$\beta = \alpha', \quad d_{2r} = 2^{1-2r} \zeta(2r+1) \alpha'^{2r+1}.$$

Using $x = \beta M_{\text{gap}}^2$, $B_{2r,2r} = d_{2r} M_{\text{gap}}^{4r+2}$ and $M_{\text{gap}}^2 = 1/\alpha'$:

$$(x, B_{2,2}, B_{4,4}, B_{6,6}, B_{8,8}) = \left(1, \frac{\zeta_3}{2}, \frac{\zeta_5}{8}, \frac{\zeta_7}{32}, \frac{\zeta_9}{128}\right).$$

All retained coefficient relations agree; this does not yet isolate the benchmark.

Finite products test what the constraints do not fix

Use first-mass units: $m^2 = 1$

$$W_q(s, t, u) = \prod_{n=1}^q \frac{(n+s)(n+t)(n+u)}{(n-s)(n-t)(n-u)}.$$

$$F_q^{\text{ST}} = -\frac{g^2}{st} W_q, \quad F_q^{\text{DT}} = -\frac{\kappa^2}{s(1+s)} W_q.$$

Same coefficient pattern

$$\zeta(k) \longrightarrow H_q^{(k)} = \sum_{n=1}^q \frac{1}{n^k}.$$

Different spectral structure

$q = 1$: a one-mass IST-type alternative.

$q \rightarrow \infty$: the heterotic gamma-function factor.

Algebraic test amplitudes are not automatically unitary in every color channel, or consistent at all multiplicities.

Positivity lives in irreducible color channels

EFT bookkeeping

$$\vec{T} = (\text{tr}(1234), \dots, \text{tr}(12) \text{tr}(34), \dots).$$

Single- and double-trace coefficients make locality and higher-point sewing transparent.

Unitarity bookkeeping

$$\text{Adj} \otimes \text{Adj} = \bigoplus_R R, \quad \vec{P}_s = M_s \vec{T}.$$

For generic $SO(n)$, six irreducible channels carry independent nonnegative spectral densities.

$$\text{Disc}_s \mathcal{F}_4 \sim \sum_{R,J} \rho_{J,R}(s) P_R G_J^{(D)} \left(1 + \frac{2t}{s} \right), \quad \rho_{J,R} \geq 0.$$

Impose positivity after the color projection, not coefficient by coefficient in the trace basis.

Turn the nonlinear surface into a sequence of scans

$$x = \frac{b_{0,0} M_{\text{gap}}^2}{\kappa^2}, \quad B_{k,q} = \frac{b_{k,q} M_{\text{gap}}^{2(k+1)}}{\kappa^2}, \quad A_{k,q} = \frac{a_{k,q} M_{\text{gap}}^{2(k+2)}}{g^2}.$$

Fix also $\widehat{g} = g^2/(\kappa^2 M_{\text{gap}}^2)$ and n .

Lower orders

At fixed x , the system through $a_{3,q}, b_{4,q}$ is affine in

$$B_{2,2}, B_{4,4}.$$

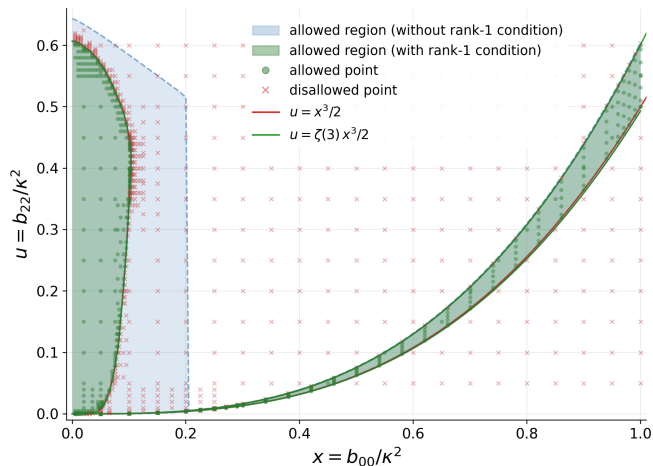
$$A_{4,1} = -8B_{2,2}^2 \quad \text{shows why fixing } x \text{ alone is insufficient.}$$

Full retained system

Fix x and $B_{2,2}$; all remaining relations are affine in

$$B_{4,4}, B_{6,6}, B_{8,8}.$$

A preliminary picture from the draft



PRELIMINARY

Figure notation:

$$u = B_{2,2}, \quad x = B_{0,0}.$$

The marked reference curves are

$$u = \frac{x^3}{2}, \quad u = \frac{\zeta_3 x^3}{2}.$$

The draft still requires a documented numerical setup and completed dispersive kernels. No certified extremality claim is attached here.

What is established, and what remains open?

Established in the finite system

An explicit algebraic solution

Five seeds determine all 45 retained coefficients; 133 symbolic checks pass.

A recognizable string pattern

Common odd-power exponential, forward geometric sequence, and exact heterotic benchmark agreement.

Not yet a uniqueness result

Completion of the calculation

Full Ward-matrix record, consistent dispersive normalization, IR subtraction and numerical certification.

Completion of the argument

All-order constraints; physical selection of seeds; full spectral and higher-point consistency.

Sung: derive the splitting identity from six points

A complex factorization locus

$$p_1 = p_3, \quad s_{234} = s_{345} = 0,$$

with adjacent two-particle invariants nonzero. The Ward rows and their parity transforms coincide on the locus, but not transversely:

$$w - w^P = \tau\nu + O(\tau^2), \quad \mathbf{Z} = \tau^{-1}\mathbf{Z}_{-1} + O(1).$$

A zero isolates a residue

$$0 = [\tau^0]((w - w^P)\mathbf{Z}) = \nu\mathbf{Z}_{-1}.$$

Regular contact and five-point contributions drop out.

For $M = -stF/g^2$:

$$M(x, y)M(x + y, z) = M(x, y + z)M(y, z)$$

An all-order relation; splitting is a *consequence*, not an input.

Sung: a low-order equality fixes an infinite spectrum

Explicit ultraviolet assumptions

Simple planar meromorphic poles; a nonempty gapped spectrum; positive scalar-residue partial waves; exact unsubtracted regular forward dispersion.

$$M(s, t) = \frac{h(s)h(t)}{h(s+t)}.$$

Positivity implies $\mu_n \geq n\mu_1$; no initial finite-spin restriction.

Spectral saturation

$$\frac{\sum_n \mu_n^{-4}}{(\sum_n \mu_n^{-2})^2} \geq \frac{\zeta_4}{\zeta_2^2} = \frac{2}{5}.$$

$$\frac{g^2 a_{2,0}}{a_{0,0}^2} = \frac{2}{5} \implies \mu_n = n\mu_1.$$

$$F(s, t) = -\frac{g^2}{st} \frac{\Gamma(1 - s/\mu_1)\Gamma(1 - t/\mu_1)}{\Gamma(1 - (s+t)/\mu_1)}$$

Wan–Zhou: two analytic routes from dispersive moments

Positive measures on a compact interval

$$(\Lambda^2)^p a_{p+q,q} = \int_0^1 z^p d\mu_q(z), \quad d\mu_q \geq 0.$$

Route 1: monodromy input

Monodromy fixes a subset of EFT moments;
Hausdorff determinacy fixes the rest.

$$\text{moments} \implies \text{Veneziano}$$

Route 2: five-point splitting input

Splitting and cyclicity give the same cocycle equation,
followed by a product representation and spectral
selection.

$$\text{splitting} \implies B = \frac{h(s)h(t)}{h(s+t)}$$

The analytic machinery is complementary; the origin of the functional equation is different.

Different inputs, different scopes

Approach	Higher-point / algebraic input	Selection mechanism
Elvang I	Scalar parity + six-point SUSY	Numerical positivity convergence
Elvang II	Scalar parity + all-order ansatz for analytic argument	Finite-spin residue condition selects VS
Sung	Derives the cocycle from six-point Ward identities	Low-order equality saturates a spectral bound
Wan–Zhou	Monodromy <i>or</i> five-point splitting	Moment determinacy / product constraints
This draft	Mixed and triple-trace Ward constraints	Color-channel positivity: work in progress

A theorem about the planar four-point Veneziano function does not directly classify the coupled heterotic problem.

What would an analytic heterotic proof require?

First: derive, rather than resum

Find all-order mixed/triple-trace Ward identities that force a common dressing:

$$-\frac{st}{g^2} F^{\text{ST}} \stackrel{?}{=} -\frac{s(1+\beta s)}{\kappa^2} F^{\text{DT}}.$$

Then: use the right positive data

Constrain the dressing and β with positive *irreducible-color* spectral measures, not trace signs.

Identify what excludes IST alternatives and fixes the odd-power seeds.

The key question is whether cross-sector consistency supplies a new saturating identity.

Gauge theory and gravity constrain one another

**Four-point freedom becomes
six-point correlation.**

$$a \times a \oplus a \times b \oplus b \times b$$

New result

An explicit five-seed solution through $a_{7,q}$ and $b_{8,q}$.

String structure

A common exponential, a forward rank-one pattern, and heterotic agreement.

Next question

Can all-order Ward identities and color positivity select the string?

Thank you